Simplicial Moore paths are polynomial

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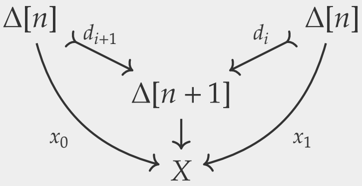
Today

1. Definition: Simplicial Moore path functor M
2. Theorem: M is a polynomial functor
3. Why this is nice

I. Definition of M

Intuition

A path segment between n-simplices x_0, x_1 in some simplicial set X is an $n+1$-simplex in X such that:



An n-dimensional *traversal* is a finite sequence

$$\theta : \{1, \ldots , l\} \to [n] \times \{+, -\}$$

to be thought of as a path:

$$\begin{array}{ccccccccc} \Delta[n] & & & & \Delta[n] & & & & \Delta[n] \\ & \overset{d^{n+1}_{\theta^-(1)}}{\searrow} & & \overset{d^{n+1}_{\theta^+(1)}}{\swarrow} & & \overset{d^{n+1}_{\theta^-(2)}}{\searrow} & & \overset{d^{n+1}_{\theta^+(k)}}{\swarrow} & \\ & & \Delta[n+1] & & & & \Delta[n+1] \cdots \Delta[n+1] & & \end{array}$$

where $\theta^\pm(k) = i + \frac{1\mp1}{2}$ if $\theta(k) = (i, +)$ and $\theta^\pm(k) = i + \frac{1\pm1}{2}$ if $\theta(k) = (i, -)$.
The colimit $\widehat{\theta}$ of this diagram is the *geometric realization* of the traversal.

Traversals form a simplicial set

For θ an n-dimensional traversal, and $\alpha : [m] \to [n]$ an arrow between simplices, the action of α on θ is given by 'pullback':

$$\begin{array}{ccc} \{1, \ldots, l\} & \xrightarrow{\theta \cdot \alpha} & [m] \times \{+, -\} \\ {\scriptstyle \widehat{\alpha}} \downarrow & & \downarrow {\scriptstyle \alpha \times \{+,-\}} \\ \{1, \ldots, k\} & \xrightarrow{\theta} & [n] \times \{+, -\} \end{array}$$

where $\widehat{\alpha}$ is order-preserving, and the restriction of $\theta \cdot \alpha$ to every fibre $\widehat{\alpha}^{-1}(i)$ is order-reversing if $\theta(i) = (x, +)$ for some x and order-preserving in case $\theta(i) = (x, -)$.

The *simplicial set* of traversals is denoted $\mathbb{T}_0$.

Lemma (Van den Berg - Garner)

The simplicial action induces morphisms between the geometric realizations

$$\widehat{\theta \cdot \alpha} \to \widehat{\theta}$$

in such a way that taking the geometric realization is a functor

$$\widehat{(-)} : \mathbb{T} \to \text{SSets}$$

where $\mathbb{T}$ is the category of elements of $\mathbb{T}_0$.

Definition (Van den Berg - Garner / Berger)

The simplicial Moore path functor $M : \mathsf{SSets} \to \mathsf{SSets}$ is defined by:

$$MX(n) = \sum_{\theta \in \mathbb{T}_0} \mathrm{Hom}(\widehat{\theta}, X)$$

where the simplicial action is induced by precomposition with the induced morphism.

Alternatively, $MX = \sum_{\mathbb{T}_0 \to 1} \widehat{M}X$ where $\widehat{M}$ is right adjoint to the left Kan extension of geometric realization along Yoneda:

$$\widehat{(-)} : \mathsf{SSets}/\mathbb{T}_0 \cong [\mathbb{T}^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{SSets}$$

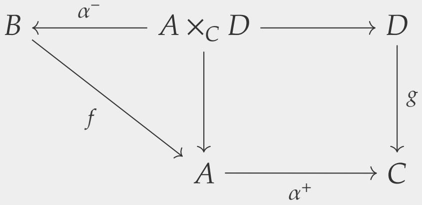
II. M is a polynomial functor

Polynomials

A *polynomial* in a l.c.c. category $\mathcal{E}$ is a morphism $f : B \to A$ in $\mathcal{E}$. A *morphism of polynomials*

$$\alpha : (f : B \to A) \to (g : D \to C)$$

is a pair of arrows α^+, α^- making up a commutative diagram:



Reasoning about polynomials

A morphism between polynomials $f : B \to A$ and $g : D \to C$ can be construed as a map $\alpha^+ : A \to C$ together with a family of maps

$$\left(\alpha_a^- : D_{\alpha^+(a)} \to B_a\right)_{a \in A}.$$

With this notation, it is easy to infer what composition should be.

Polynomial Functors

There is a full and faithful functor

$$P : \mathsf{Poly}(\mathcal{E}) \to \mathsf{FEnd}(\mathcal{E})$$

from the category of polynomials into the category of 'fibred endofunctors' $\mathcal{E} \to \mathcal{E}$ and fibred natural transformations between them. It is given by:

$$P(f) = \mathcal{E} \xrightarrow{B^*} \mathcal{E}/B \xrightarrow{\Pi_f} \mathcal{E}/A \xrightarrow{\Sigma_A} \mathcal{E}$$

A thing of this form is what one usually calls a *polynomial functor*.

Reference: Abbott, Altenkirch, Ghani. "Categories of Containers" (2003).

Back to traversals

There is a canonical retraction

$$r_\theta : \widehat{\theta} \to \Delta[n]$$

of every traversal, with sections

$$s_\theta : \Delta[n] \to \widehat{\theta} \quad t_\theta : \Delta[n] \to \widehat{\theta}$$

called *source* and *target*. In fact, these form natural transformations:

$$r : \widehat{(-)} \Rightarrow y \circ U \quad s : y \circ U \Rightarrow \widehat{(-)} \quad t : y \circ U \Rightarrow \widehat{(-)}$$

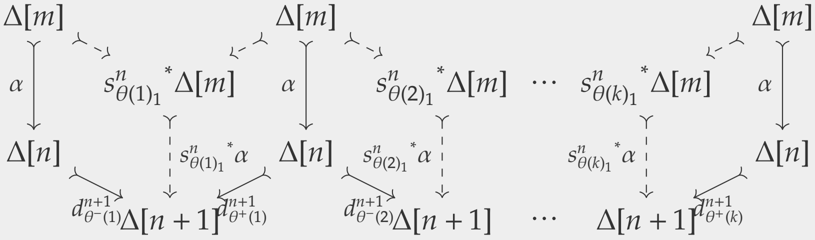
where $U : \mathbb{T} \to \Delta$ is the forgetful functor and y the Yoneda embedding.

Proposition

The natural transformation r is *cartesian* (and hence also s, t). That is, the following diagram is always a pullback:

$$
\begin{array}{ccc}
\widehat{\theta \cdot \alpha} & \xrightarrow{\ \hat{\alpha}\ } & \hat{\theta} \\
{\scriptstyle r_{\theta\cdot\alpha}}\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow{\scriptstyle r_{\theta}} \\
\Delta[m] & \xrightarrow{\ \alpha\ } & \Delta[n]
\end{array}
$$

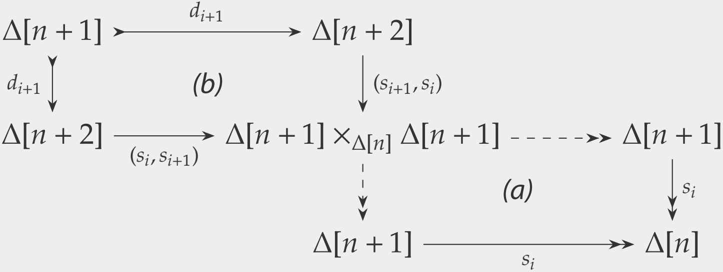
This is easy for monomorphic α, and many others, because **almost all simplicial identities are pullbacks** . So the action then looks like this:



The one exception being the identity $s_i s_i = s_i s_{i+1}$

Dealing with the one exception

Consider the following commutative diagram in SSets, where the square *(a)* is a pullback. Then the square *(b)* is a pushout.



Proposition

The natural transformation r is *cartesian* (and hence also s, t). That is, the following diagram is always a pullback:

$$\begin{array}{ccc} \widehat{\theta \cdot \alpha} & \xrightarrow{\widehat{\alpha}} & \widehat{\theta} \\ {\scriptstyle r_{\theta\cdot\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle r_{\theta}} \\ \Delta[m] & \xrightarrow{\alpha} & \Delta[n] \end{array}$$

By taking colimits, the natural transformations $r : \widehat{(-)} \Rightarrow y \circ U$ and $s, t : y \circ U \Rightarrow \widehat{(-)}$ induce a new retraction:

$$r_1 : \mathbb{T}_1 \to \mathbb{T}_0 \text{ with sections } s_1, t_1 : \mathbb{T}_0 \to \mathbb{T}_1$$

where $\mathbb{T}_0 \cong \operatorname{colim} y \circ U$ is a categorical fact and $\mathbb{T}_1 \cong \operatorname{colim} \widehat{(-)} = \hat{1}$.

We wonder about the pullback:

$$\begin{array}{ccc} ? & \longrightarrow & \mathbb{T}_1 \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow {\scriptstyle r_1} \\ \Delta[n] & \xrightarrow{\theta} & \mathbb{T}_0 \end{array}$$

Van Kampen colimits

Suppose $G : \mathbb{D} \to \widehat{\mathbb{C}}$ is a functor into some category of presheaves, and let $\iota : G \Rightarrow \operatorname{colim} G$ be the colimiting cocone.

The colimit given by ι is called *Van Kampen* if the adjunction $\operatorname{colim} \dashv \iota^*$, where

$$\operatorname{colim} : [\mathbb{D}, \widehat{\mathbb{C}}] \Downarrow_{\mathsf{cart}} G \to \widehat{\mathbb{C}} / \operatorname{colim} G$$

where the domain is the category of *cartesian* natural transformations, is an equivalence of categories.

A lemma on Van Kampen colimits

A colimit is *Van Kampen* if and only whenever (1) is a commutative diagram with a cartesian natural transformation $\alpha : F \Rightarrow G$, the statements (i), (ii) are equivalent.

1. The square is a pullback square
2. The top arrow is a colimiting cocone

$$\begin{array}{ccc} F & \Longrightarrow & \operatorname{colim} F \\ {\scriptstyle \alpha}\Big\Downarrow & & \Big\downarrow \\ G & \underset{\iota}{\Longrightarrow} & \operatorname{colim} G \end{array} \tag{1}$$

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1. The square is a pullback square
2. The top arrow is a colimiting cocone

$$\begin{array}{ccc} \widehat{(-)} & \Longrightarrow & \mathbb{T}_1 \\ {\scriptstyle r}\Downarrow & & \downarrow \\ y \circ U & \underset{\iota}{\Longrightarrow} & \mathbb{T}_0 \end{array} \qquad (2)$$

A lemma on Van Kampen colimits

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1. The square is a pullback square
2. The top arrow is a colimiting cocone

$$\begin{array}{ccc} \widehat{\theta} & \longrightarrow & \mathbb{T}_1 \\ {\scriptstyle r_\theta}\downarrow & & \downarrow{\scriptstyle r} \\ \Delta[n] & \xrightarrow[\theta]{} & \mathbb{T}_0 \end{array} \tag{3}$$

Lemma

Let $\iota : y \circ U \Rightarrow P$ be the colimiting cocone of the colimit of the category of elements of a presheaf P.

Then

$$\mathrm{colim} \dashv i^* : \widehat{\mathbb{C}}/P \to [\mathcal{E}(P), \widehat{\mathbb{C}}] \Downarrow_{\mathsf{cart}} y \circ U$$

is an equivalence of categories, i.e. the colimit is Van Kampen.

Not as easy as you might think!

Moore paths are polynomial

Since geometric realization is given by pullback, this is also true of its left Kan extension.

$$\begin{array}{ccc} \widehat{X} & \longrightarrow & \mathbb{T}_1 \\ \downarrow & & \downarrow r \\ X & \longrightarrow & \mathbb{T}_0 \end{array} \tag{4}$$

Where $X \to \mathbb{T}_0$ is a presheaf on $\mathbb{T}$. Hence, the Moore path functor, defined in terms of the right adjoint $\widehat{M}$ to left Kan extension, becomes:

$$MX = \sum_{\mathbb{T}_0 \to 1} \widehat{M}X = \sum_{\mathbb{T}_0 \to 1} \prod_{r:\mathbb{T}_1 \to \mathbb{T}_0} r^* X$$

So, it is a polynomial functor.

III. Why this is nice

Moore Structure

Let $\mathcal{E}$ be a category with finite limits. A *Moore structure* on $\mathcal{E}$ is an endofunctor $M : \mathcal{E} \to \mathcal{E}$ with:

$$MX \times_{(t,s)} MX \xrightarrow{\mu_X} MX \underset{s_X}{\overset{t_X}{\rightrightarrows}} X, \quad X \xrightarrow{r_X} MX$$

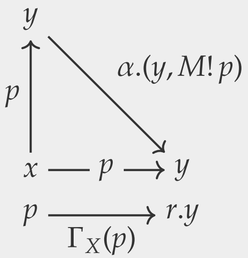
- zero $r : 1 \Rightarrow X$
- source $s : M \Rightarrow 1$
- target $t : M \Rightarrow 1$
- concatenation $\mu : M \times_{t,s} M \Rightarrow M$

We view $M1$ as the object of path lengths …

Moore Structure (cont.)

…and a *connection* or *path contraction*:

$$\Gamma : M \Rightarrow MM \quad \text{s.t. } (M, s, \Gamma) \text{ is a comonad, looking like}$$



where

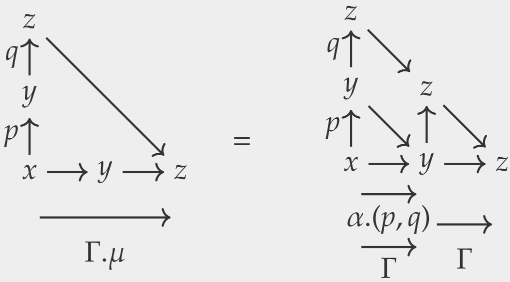
$$\alpha : X \times M1 \to M(X \times 1) \cong M(X)$$

takes the constant path of a fixed length …

Axioms for Moore Structure

- Many! Some weaker, some stronger than the original paper (Van den Berg-Garner 2012)
- A major milestone was the following *distributive law*

$$\Gamma.\mu = \mu.(M\mu.\nu_X.(\Gamma.p_1, \theta_{MX}.\alpha_{MX,1}.(p_2, M!.p_1)), \Gamma.p_2)$$



Theorem (Van den Berg, Faber)

A category with (symmetric) Moore structure admits an algebraic weak factorisation system of naive fibrations and hyperdeformation retracts.

A suitable category with (symmetric) Moore structure and a *dominance* admits a notion of 'effective fibration' stable under pushforward in a constructive metatheory.

For the simplicial Moore path functor, the effective fibrations are equivalent to Kan fibrations in the classical setting.

But how to prove the axioms, e.g. the distributive law, for simplicial Moore paths?

Polynomial comonads

The category of polynomial functors for some category $\mathcal{E}$ carries a monoidal structure given by functor composition: $F \otimes G = F \circ G$.

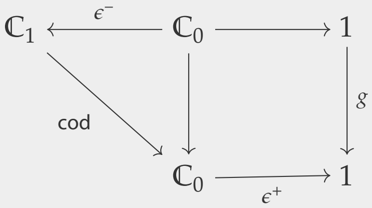
A comonoid for this monoidal structure is called a *polynomial comonad*.

Recall that we would like to give M the structure of a comonad:

$$\Gamma : M \Rightarrow M \otimes M \quad s : M \Rightarrow 1$$

A helpful fact

Every internal category $\mathbb{C}_1, \mathbb{C}_0$ in $\mathcal{E}$ determines a polynomial comonad. The counit is induced by the codomain map and the identity:



Similarly, one can figure out a comultiplication

$$(\delta^+, \delta^-)\colon\ \mathrm{cod} \to \mathrm{cod} \otimes \mathrm{cod}\,.$$

Conversely, every polynomial comonad is induced by an internal category.

Reference: Abbott, Altenkirch, Ghani. "Categories of Containers" (2003).

The breakfast realization

The polynomial $r : \mathbb{T}_1 \to \mathbb{T}_0$ is an internal poset. The object $\mathbb{T}_1$ is the simplicial set of traversals together with a 'position' along its path, and r (the codomain map) is the forgetful map.

The domain map $\mathrm{dom} : \mathbb{T}_1 \to \mathbb{T}_0$ takes the 'tail' of the traversal from that position onwards.

This yields a very simple description of the polynomial comonad M. Checking the axioms for Moore structure (e.g. the distributive law!) is now much easier. And coalgebras are precisely internal presheaves on $\mathbb{T}_0$.

To summarise

- The simplicial Moore path functor M provides a constructive approach to Kan fibrations.
- The object is combinatorially difficult.
- By representing M as a polynomial functor, we have been able to give it a much simpler description.

The full paper:

Van den Berg, Faber, “Effective Kan fibrations in Simplicial Sets” (2020)

`arxiv.org/abs/2009.12670`