Dependently typed algebraic theories

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- This talk will oscillate between syntactic definitions and category theory, so do interrupt when necessary! ☺
- The work on $\mathbb{S}$-contextual categories is joint with Peter LeFanu Lumsdaine.
- The work on opetopic nerve functors is joint with Cédric Ho Thanh.

Outline

1. Introduction (algebraic theories revisited)
2. Dependent type signatures as lfd-categories
3. Contextual categories as monoids in collections
4. Interesting example: opetopic nerve functors
5. Perspective: theories of higher algebraic structures

Introduction

Algebraic theories

Many algebraic structures can be specified by **algebraic** (or "Lawvere") **theories**. Such a theory consists of:

1. A set S of sorts,
2. a set Σ of S-sorted (finitary) function symbols, each written

$$x{:}A_1, \ldots, x_n{:}A_n \vdash f : A \qquad (A_1, \ldots, A_n, A \in S),$$

3. Equations between S-sorted terms over Σ, each written

$$x{:}A_1, \ldots, x_n{:}A_n \vdash t = u : A.$$

Example

The theory of **associative monoids**:

1. set of sorts $\{M\}$,
2. function symbols

$$\vdash 1 : M \qquad\qquad x, y{:}M \vdash \circ : M,$$

3. equations

$$x{:}M \vdash 1 \circ x = x : M \qquad\qquad x{:}M \vdash x \circ 1 = x : M$$

$$x, y, z{:}M \vdash (x \circ y) \circ z = x \circ (y \circ z) : M.$$

also of groups, rings, unicoloured operads,. . .

Generalising algebraic theories

Some algebraic structures cannot be described by algebraic theories.
E.g. small categories, coloured operads, small groupoids,
n-categories, simplicial sets . . .

Two well-known generalisations of algebraic theories are :

- Essentially algebraic theories
- Generalised algebraic theories (à la Cartmell)

Example

The essentially algebraic theory of categories :

1. sorts are $\{O, H\}$,
2. (partial) function symbols are

$$f{:}H \vdash \mathrm{s} : O \qquad f{:}H \vdash \mathrm{t} : O$$

$$x{:}O \vdash 1 : H \qquad f, g{:}H, \mathrm{s}g = \mathrm{t}f{:}O \vdash \circ : H$$

3. equations are

$$x{:}O \vdash \mathrm{s}(1(x)) = x : O \qquad x{:}O \vdash \mathrm{t}(1(x)) = x : O$$

$$f{:}H \vdash 1(\mathrm{t}(f)) \circ f = f : H \qquad f{:}H \vdash f \circ 1(\mathrm{s}(f)) = f : H$$

$$\ldots \vdash \mathrm{s}(g \circ f) = \mathrm{s}(f) : H \qquad \ldots \vdash \mathrm{t}(g \circ f) = \mathrm{t}(g) : H$$

$$\ldots \vdash h \circ (g \circ f) = (h \circ g) \circ f : H.$$

Example

The *generalised* algebraic theory of categories :

1. sorts are

$$\vdash O \qquad\qquad x, y{:}O \vdash H(x, y)$$

2. function symbols are

$$x{:}O \vdash 1 : H(x, x) \quad \ldots, f{:}H(x, y), g{:}H(y, z) \vdash \circ : H(x, z)$$

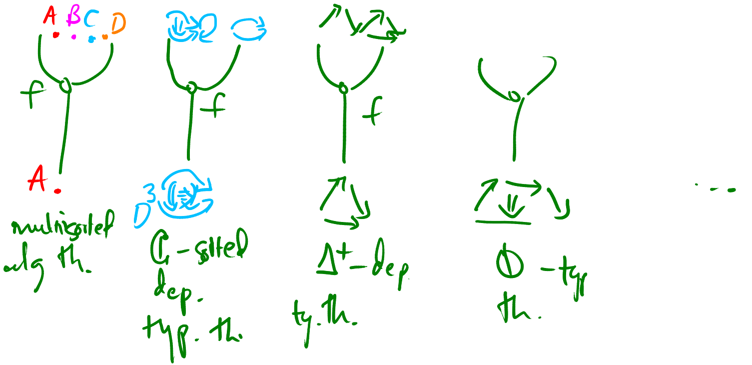
3. equations are

$$\ldots \vdash 1 \circ f = f : H(x, y) \qquad\qquad \ldots \vdash f \circ 1 = f : H(x, y)$$

$$\ldots \vdash h \circ (g \circ f) = (h \circ g) \circ f : H(x_1, x_4).$$

- I will present a strict subclass of generalised algebraic theories (**dependently typed algebraic theories**).
- But which nicely generalise multisorted algebraic theories from a category-theoretic point of view.
- And there are many interesting examples of these!

The guiding intuition is that the contexts of variables of these theories are *finite cell complexes* (whose generating cells are of some kind of shape, and of finite dimension).



Algebraic theories as finite-product categories

Every S-sorted algebraic theory $\mathbb{T}$ has a **syntactic category** $\mathsf{C}_{\mathbb{T}}$ of contexts.

The objects of $\mathsf{C}_{\mathbb{T}}$ are finite contexts of S-sorted variables (e.g. $x{:}A, y{:}B$ where $A, B \in S$).

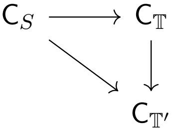
A morphism $\Gamma \to (x_1{:}A_1, \ldots, x_n{:}A_n)$ is a list $(t_i)_{1 \leq i \leq n}$ of terms of $\mathbb{T}$, sorted as $\Gamma \vdash t_i : A_i$.

Composition of morphisms is defined by substitution.

The syntactic category C_S of the S-sorted algebraic theory with *no* function symbols (and no equations) is a finite-product completion of S.

For every $\mathbb{T}$, there is a unique identity-on-objects, finite-product preserving functor $\mathsf{C}_S \to \mathsf{C}_{\mathbb{T}}$.

A morphism $\mathbb{T} \to \mathbb{T}'$ is simply a functor $\mathsf{C}_{\mathbb{T}} \to \mathsf{C}_{\mathbb{T}'}$ making the triangle commute. Call this category $\mathcal{T}\mathrm{h}_S$.



A **model** of $\mathbb{T}$ is a finite-product preserving functor $\mathsf{C}_{\mathbb{T}} \to \mathcal{S}\mathrm{et}$. Morphisms of models are natural transformations.

The category of models of C_S is just $\mathcal{S}\mathrm{et}_{/S}$.

So $\mathsf{C}_{\mathbb{T}} \to \mathcal{S}\mathrm{et}$ is a model of $\mathbb{T}$ iff the composite $\mathsf{C}_S \to \mathsf{C}_{\mathbb{T}} \to \mathcal{S}\mathrm{et}$ is in $\mathcal{S}\mathrm{et}_{/S}$.

Multisorted algebraic theories as monoids in collections

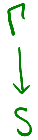
Let $S \in \mathcal{S}\mathrm{et}$ be "a set of sorts".

The over-category $\mathcal{F}\mathrm{in}_{/S}$ is the category of f.p. objects of $\mathcal{S}\mathrm{et}_{/S}$.

(N.B. $\mathcal{F}\mathrm{in}$ is the category of finite sets.)

$\mathcal{F}\mathrm{in}_{/S}$ is also the free completion of S under finite coproducts.

(Objects of $\mathcal{F}\mathrm{in}_{/S}$ are exactly finite contexts of S-sorted variables.)



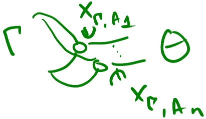
Definition

The category of S-**sorted cartesian collections** is the presheaf category $\mathcal{C}\mathrm{oll}_S \coloneqq [\mathcal{F}\mathrm{in}_{/S}, \mathcal{S}\mathrm{et}_{/S}] = [\mathcal{F}\mathrm{in}_{/S} \times S, \mathcal{S}\mathrm{et}]$.

Every presheaf $X \in \mathcal{C}\mathrm{oll}_S$ is a signature* of S-sorted function symbols. (For every finite context $\Gamma \in \mathcal{F}\mathrm{in}_{/S}$ and every sort $A \in S$, $X_{(\Gamma, A)}$ is its set of function symbols $\Gamma \vdash f : A$.)

Collections can be composed: for $X, Y \in \mathcal{C}\mathrm{oll}_S$,

$$(Y \circ X)_{(\Gamma,A)} := \int^{\Theta \in \mathcal{F}\mathrm{in}_{/S}} Y_{(\Theta,A)} \times \mathcal{S}\mathrm{et}_{/S}\big(\Theta, X_{(\Gamma,-)}\big).$$



$- \circ -$ is a monoidal product on $\mathcal{C}\mathrm{oll}_S$, with unit $i \colon \mathcal{F}\mathrm{in}_{/S} \hookrightarrow \mathcal{S}\mathrm{et}_{/S}$.

Left Kan extension along $i\colon \mathcal{F}\mathrm{in}_{/S} \hookrightarrow \mathcal{S}\mathrm{et}_{/S}$ is a full monoidal embedding $\mathcal{C}\mathrm{oll}_S \hookrightarrow \left[\mathcal{S}\mathrm{et}_{/S}, \mathcal{S}\mathrm{et}_{/S}\right]$ whose image is the category of **finitary** endofunctors on $\mathcal{S}\mathrm{et}_{/S}$.

$$\begin{array}{ccc} \mathcal{F}\mathrm{in}_{/S} & \xrightarrow{\;X\;} & \mathcal{S}\mathrm{et}_{/S} \\ {\scriptstyle i}\big\downarrow & \cong & \nearrow_{\mathrm{Lan}_i X} \\ \mathcal{S}\mathrm{et}_{/S} & & \end{array}$$

So $\mathcal{M}\mathrm{on}(\mathcal{C}\mathrm{oll}_S, \circ, i) \subset \mathcal{M}\mathrm{nd}(\mathcal{S}\mathrm{et}_{/S})$ is the full subcategory of finitary monads on $\mathcal{S}\mathrm{et}_{/S}$.

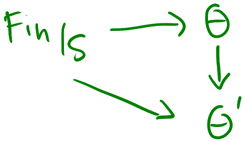
For T a monad on $\mathcal{S}\mathrm{et}_{/S}$, we have a commutative square

$$\begin{array}{ccc} \mathcal{F}\mathrm{in}_{/S} & \xrightarrow{l_T \text{ i.o.}} & \Theta_T \\ \big\downarrow & & \big\downarrow \\ \mathcal{S}\mathrm{et}_{/S} & \xrightarrow{F_T} & T\text{-}\mathcal{A}\mathrm{lg}. \end{array}$$

T.F.A.E.

1. T is finitary.
2. l_T preserves finite coproducts.
3. $(l_T)^*(l_T)_!$ restricts to $\mathcal{S}\mathrm{et}_{/S}$. ($l_T$ is Lawvere theory with arities $\mathrm{Fin}_{/S} \hookrightarrow \mathrm{Set}_{/S}$)

An S-**sorted Lawvere theory** is an identity-on-objects finite-coproduct preserving functor $\mathcal{F}\mathrm{in}_{/S} \to \Theta$. A morphism is just a functor $\Theta \to \Theta'$ making the triangle commute. Call this category $\mathcal{L}\mathrm{aw}_S$.



We have

$$\mathcal{M}\mathrm{on}(\mathcal{C}\mathrm{oll}_S, \circ, i) \simeq \mathcal{F}\mathrm{in}\mathcal{M}\mathrm{nd}(\mathcal{S}\mathrm{et}_{/S}) \simeq \mathcal{L}\mathrm{aw}_S \simeq \mathcal{T}\mathrm{h}_S.$$

We could play the same game by taking $\mathbb{S}$ to be a small category, and replacing

1. $\mathcal{S}\mathrm{et}_{/S}$ and $\mathcal{F}\mathrm{in}_{/S}$ with $\widehat{\mathbb{S}}$ and $\widehat{\mathbb{S}}_{\mathrm{fp}}$ resp.,
2. finite coproducts with finite colimits,
3. $\mathcal{L}\mathrm{aw}_S$ with $\mathcal{L}\mathrm{aw}_{\mathbb{S}}$.

But the syntactic counterpart of $\mathcal{T}\mathrm{h}_S$ is not straightforward.

I will show that when $\mathbb{S}$ is a small category of "shapes" that corresponds to a dependent type signature, there is a $\mathcal{T}\mathrm{h}_{\mathbb{S}}$ that is just as elegant.

Dependent type signatures as locally finite direct categories

Dependent type signatures

Begin with a syntactic definition.

A **(dependent) type signature** S is a graded set $S = \coprod_{n \in \mathbb{N}} S_n$, such that each S_j is a set of *type declarations* over the signature $S_{<j} = \coprod_{i<j} S_i$.

A **type declaration** over a signature S is a pair (Γ, A) where Γ is a (Martin-Löf) context typed by S and A is a (fresh) type symbol.

Examples

- $S = \emptyset$
- $S =$

$$\{\ \vdash \mathrm{Ob}\ \} = S_1$$

$$\{\ x, y{:}\mathrm{Ob} \vdash \mathrm{Hom}(x, y)\ \} = S_2$$

- $S =$

$$\{\ \vdash C\ \} = S_1$$

$$S_2 = \{\ x_1{:}C, \ldots, x_k{:}C, y{:}C \vdash Op(x_1, \ldots, x_k, y) \qquad \forall k \in \mathbb{N}\ \}$$

Every signature S has a **syntactic category** whose objects are contexts Γ typed by S and whose morphisms are context morphisms $\Gamma \to \Delta$.

$(x_1, \ldots x_k)$

For a signature S, let $\mathbb{C}_S$ be the full subcategory of its category of contexts on the contexts $(\Gamma, x{:}A)$ for all (Γ, A) in S.

Then $\mathbb{C}_S^{\mathrm{op}}$ is a *locally finite direct* category.

Lfd categories

A **direct** category is a small category C such that the relation

$$c < d \qquad \Leftrightarrow \qquad \exists \text{ a non-identity arrow } c \to d$$

on the objects of C is well-founded (i.e. no infinite chains $\ldots < c_0$).

If C is direct and $X \in \widehat{C}$, then C/X is direct.

A category C is **locally finite** if each of its slice categories is finite, i.e. in every cartesian square in $\mathcal{C}\mathrm{at}$ of the form

$$\begin{array}{ccc} A & \longrightarrow & C^{\to} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{t}} \\ \star & \longrightarrow & C \end{array} \qquad A \text{ is finite.}$$

If C is locally finite and $X \in \widehat{C}$, then C/X is locally finite.

(Recall:

$$\begin{array}{ccc} A^{\to} & \xrightarrow{p^{\to}} & B^{\to} \\ {\scriptstyle \mathrm{t}}\downarrow & & \downarrow{\scriptstyle \mathrm{t}} \\ A & \xrightarrow{p} & C \end{array} \qquad \text{is cartesian iff } p \text{ is a disc. fibration.)}$$

Categorical definition of type signatures

$\mathbb{S} \in \mathcal{C}\mathrm{at}$ is **lfd** if it is locally finite and direct.
So if $X \in \widehat{\mathbb{S}}$, then $\mathbb{S}/X$ is lfd.

For every lfd category $\mathbb{S}$, there is a unique type signature S such that $\mathbb{S} \cong \mathbb{C}_S$. $S = \{ (\partial c, c) \mid c \in \mathbb{S} \}$

$\mathbb{S}$ is lfd iff $\mathbb{S}^{\mathrm{op}}$ is *simple* (or "one-way finitely branching") in the sense of Makkai[Mak95]. (FOLDS paper)

Examples of dependent type signatures/lfd categories

0. Any set S, seen as a discrete category.
1. The category $\{s, t : 0 \rightrightarrows 1\}$.
2. The category $\mathbb{G}$ of globes.
3. The category Δ^+ of semi-simplices.
4. The category $\mathrm{elt}_{\mathrm{pl}}$ of planar corollas/elementary trees.
5. The category Ω_{pl} of planar trees.
6. The category $\mathbb{O}$ of opetopes.
7. The category of globular pasting schemes. $\Theta_0 \subset \widehat{\mathbb{G}}$

Contextual categories as monoids in collections

Contextual categories (Cartmell, PhD thesis)

A *contextual category* C is the data of:

1. a small category C,
2. a grading of objects as $\mathrm{ob}(\mathsf{C}) = \coprod_{n\in\mathbb{N}} \mathsf{C}_n$,
3. an object $1 \in \mathsf{C}_0$,
4. "parent" functions $\mathrm{ft}(-)_n \colon \mathsf{C}_{n+1} \to \mathsf{C}_n$ (we will usually suppress the subscript),
5. for each $\Gamma \in \mathsf{C}_{n+1}$, a map $\mathsf{p}_\Gamma \colon \Gamma \to \mathrm{ft}(\Gamma)$,
6. for each $\Gamma \in \mathsf{C}_{n+1}$ and $f \colon \Gamma' \to \mathrm{ft}(\Gamma)$, an object $f^*\Gamma$ together with a connecting map $f.\Gamma \colon f^*\Gamma \to \Gamma$;

such that:

- 1 is the unique object in C_0;
- 1 is a terminal object in C;
- for each $\Gamma \in \mathsf{C}_{n+1}$, and $f \colon \Gamma' \to \mathrm{ft}(\Gamma)$, we have $\mathrm{ft}(f^*\Gamma) = \Gamma'$, and the square

$$\begin{array}{ccc} f^*\Gamma & \xrightarrow{f.\Gamma} & \Gamma \\ {\scriptstyle \mathsf{p}_{f^*\Gamma}}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{p}_\Gamma} \\ \Gamma' & \xrightarrow{f} & \mathrm{ft}(\Gamma) \end{array}$$

  is cartesian (the *canonical pullback* of Γ along f); and
- canonical pullbacks are strictly functorial.

The category of contextual categories is denoted $\mathcal{C}\mathrm{xlCat}$.

The syntactic category of a Martin-Löf type theory is a contextual category.

The initial $\mathbb{S}$-contextual category

Let $\mathbb{S}$ be an lfd category.

For $c \in \mathbb{S}$, let $\mathbb{S}^-_{/c} \subset \mathbb{S}_{/c}$ be obtained by removing the identity morphism $1_c \colon c \to c$.
Let the **boundary** of c be the subrepresentable presheaf $\partial c \hookrightarrow c$ that is the colimit of $\mathbb{S}^-_{/c} \to \mathbb{S} \hookrightarrow \widehat{\mathbb{S}}$.

Since $\mathbb{S}$ is locally finite, every ∂c is finitely presentable.

We define the set $I_{\mathbb{S}} := \{\partial c \hookrightarrow c \mid c \in \mathbb{S}\}$.

The category $\mathcal{C}\mathrm{ell}(\mathbb{S})$ of *finite $I_{\mathbb{S}}$-cell complexes* is defined as:

- objects are finite sequences $\emptyset \to X_1 \to \ldots \to X_n$ in $\widehat{\mathbb{S}}$, along with for each morphism $X_i \to X_{i+1}$, a choice of pushout square

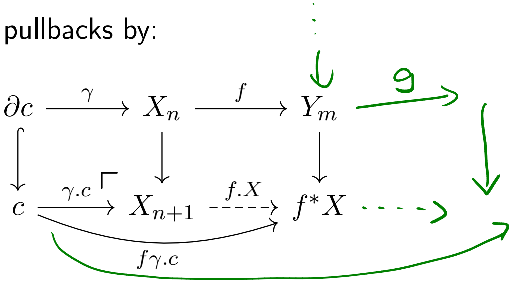
$$\begin{array}{ccc} \partial c & \xrightarrow{\gamma} & X_i \\ \big\downarrow & & \big\downarrow \mathsf{p}_{X_{i+1}} \\ c & \xrightarrow{\gamma.c} & X_{i+1}, \end{array}$$

- its morphisms are defined by $\mathcal{C}\mathrm{ell}(\mathbb{S})(\emptyset \to \ldots X, \emptyset \to \ldots Y) := \widehat{\mathbb{S}}(X, Y)$.

The image of the fully faithful forgetful functor $\mathcal{C}\mathrm{ell}(\mathbb{S}) \to \widehat{\mathbb{S}}$ is $\widehat{\mathbb{S}}_{\mathrm{fp}}$. (Since $\mathbb{S}$ is lfd, $I_{\mathbb{S}}$ is a f.p. cellular model for $\widehat{\mathbb{S}}$.)

$\mathcal{C}\mathrm{ell}(\mathbb{S})^{\mathrm{op}}$ has the structure of a contextual category.

- Grade the objects $\emptyset \to X_1 \to \dots X_n$ by length.
- Define canonical pullbacks by:



Rmq: the struc. of a ctxl cat does not transfer across equiv. of categories.

Let D be a contextual category. Then $f \colon \mathbb{S}^{\mathrm{op}} \to \mathsf{D}$ is a **contextual functor** if for all c in $\mathbb{S}$,

1. fc is in D_{k_c}, where $k_c = \sum_{d \in \mathbb{S}} |\mathbb{S}(d, c)| = |\mathrm{ob}(\mathbb{S}_{/c})|$,
2. $\mathrm{ft}(fc)$ is a limit of $(\mathbb{S}^{-}_{/c})^{\mathrm{op}} \to \mathbb{S}^{\mathrm{op}} \xrightarrow{f} \mathsf{D}$, and $\mathsf{p}_{fc} \colon fc \to \mathrm{ft}(fc)$ is the canonical morphism of limits given by $\mathbb{S}^{-}_{/c} \subset \mathbb{S}_{/c}$.

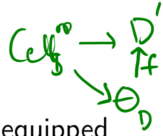
The category $\mathcal{C}\mathrm{xlCat}_{\mathbb{S}^{\mathrm{op}}/}$ of contextual functors under $\mathbb{S}^{\mathrm{op}}$ has as morphisms those of $\mathcal{C}\mathrm{xlCat}$ making the triangle commute.

The inclusion* $\mathbb{S}^{\mathrm{op}} \subset \mathcal{C}\mathrm{ell}(\mathbb{S})^{\mathrm{op}}$ is the initial object in $\mathcal{C}\mathrm{xlCat}_{\mathbb{S}^{\mathrm{op}}/}$.

That is, $\mathcal{C}\mathrm{xlCat}_{\mathbb{S}^{\mathrm{op}}/} \simeq \mathcal{C}\mathrm{xlCat}_{\mathcal{C}\mathrm{ell}(\mathbb{S})^{\mathrm{op}}/}$.

(coslice)

An $\mathbb{S}$-**contextual category** is a morphism of contextual categories $\mathcal{C}\mathrm{ell}(\mathbb{S})^{\mathrm{op}} \to \mathsf{D}$ such that in its (i.o.,ff) factorisation $\mathcal{C}\mathrm{ell}(\mathbb{S})^{\mathrm{op}} \xrightarrow{l_{\mathsf{D}}} \Theta_{\mathsf{D}} \xhookrightarrow{k_{\mathsf{D}}} \mathsf{D}$:

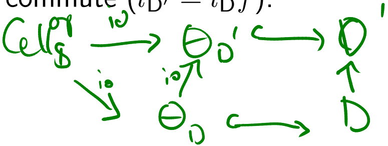


- $k_{\mathsf{D}} \colon \Theta_{\mathsf{D}} \hookrightarrow \mathsf{D}$ is initial among all $g \colon \mathcal{C}\mathrm{ell}(\mathbb{S})^{\mathrm{op}} \to \mathsf{D}'$ equipped with a functor $\Theta_{\mathsf{D}} \to \mathsf{D}'$ such that $g = \mathcal{C}\mathrm{ell}(\mathbb{S})^{\mathrm{op}} \to \Theta_{\mathsf{D}} \to \mathsf{D}'$.

The category $\mathcal{C}\mathrm{xlCat}_{\mathbb{S}}$ of $\mathbb{S}$-contextual categories is the full subcategory of $\mathcal{C}\mathrm{xlCat}_{\mathcal{C}\mathrm{ell}(\mathbb{S})^{\mathrm{op}}/}$ on the $\mathbb{S}$-contextual categories.

Proposition

A morphism of $\mathbb{S}$-contextual categories is just a functor $f \colon \Theta_{\mathsf{D}} \to \Theta_{\mathsf{D}'}$ making the triangle commute ($l_{\mathsf{D}'} = l_{\mathsf{D}} f$).



$$\begin{array}{ccccc} \mathrm{Cell}_S^{op} & \longrightarrow & D & \xrightarrow{f_{!D}} & D' \\ & {\scriptstyle i.o.}\searrow & \uparrow & \nearrow & \\ & & \Theta_D & & \end{array}$$

$\mathrm{Cell}_{\delta}^{op}$ $\vdash c$ $\{ c \in C_1 \}$

$- \vdash$ $(x : c, y : c) \in C_2$

$$S_C \xrightarrow[\text{nonwide}]{\text{nonfull}} C \qquad Q \, ?$$

Final picture

Playing the same game as for algebraic theories, we have

$$\mathcal{M}\mathrm{on}(\mathcal{C}\mathrm{oll}_{\mathbb{S}}, \circ, i) \simeq \mathcal{F}\mathrm{in}\mathcal{M}\mathrm{nd}(\widehat{\mathbb{S}}) \simeq \mathcal{L}\mathrm{aw}_{\mathbb{S}} \simeq \mathcal{C}\mathrm{xl}\mathcal{C}\mathrm{at}_{\mathbb{S}}.$$

(Rmq: $\mathrm{Coll}_{\mathbb{S}}^{op} \longrightarrow D$ in CxlCat preserves finite limits)

Opetopic theories

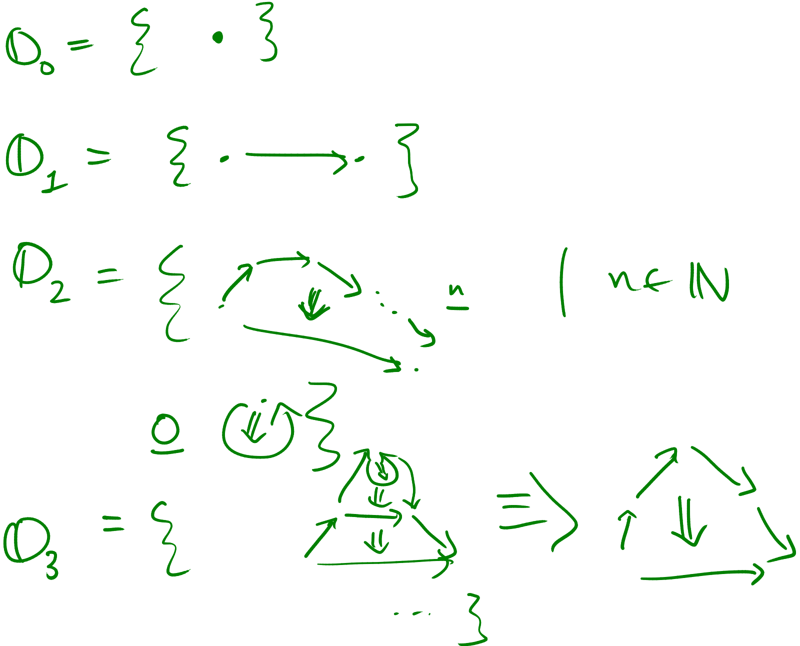
Opetopes

I won't recall the category $\mathbb{O}$ of opetopes (this is a little involved).

But for those who know it, it is easy to see that it is an lfd category.

$\mathbb{O}$ contains (as full subcats) the category $\{\mathrm{s}, \mathrm{t} \colon 0 \rightrightarrows 1\}$ as well as the category $\mathrm{elt}_{\mathrm{pl}}$ of planar corollas/elementary trees.

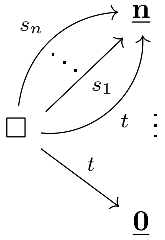
Pictures



Let $\mathbb{O}_{m,n}$ be the full subcategory of $\mathbb{O}$ of objects of dimensions $m \leq i \leq n$. Then

$$\mathbb{O}_{0,1} = \{ \ \diamond \ \overset{s}{\underset{t}{\rightrightarrows}} \ \square \ \}$$

$$\mathbb{O}_{1,2} =$$



So we have $\mathcal{G}\mathrm{ph} = \widehat{\mathbb{O}_{0,1}}$ (directed graphs) and $\mathcal{C}\mathrm{oll} = \widehat{\mathbb{O}_{1,2}}$ (coloured planar collections).

The category $\widehat{\mathbb{O}_{1,3}}$ is the category of *coloured combinatorial patterns* of Loday [Lod12].

↪ slides

We have monadic adjunctions $\widehat{\mathbb{O}_{0,1}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{C}\mathrm{at}$ (small categories) and $\widehat{\mathbb{O}_{1,2}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{O}\mathrm{pd}$ (planar coloured $\mathcal{S}\mathrm{et}$-operads).

We also have a monadic adjunction $\widehat{\mathbb{O}_{1,3}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{C}\mathrm{omb}$ (planar coloured combinads [Lod12]).

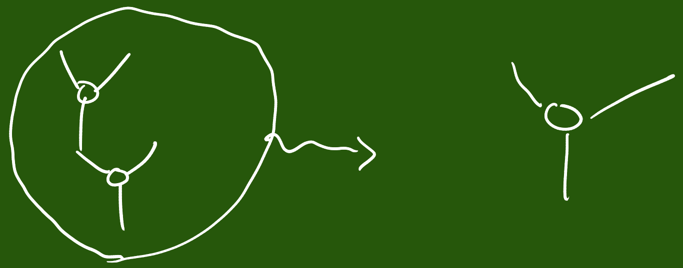
Opetopic nerve theorem

Theorem [HTLS19]

$\mathcal{C}\mathrm{at}, \mathcal{O}\mathrm{pd}, \mathcal{C}\mathrm{omb}$ are *reflective* subcategories of $\widehat{\mathbb{O}}$. The idempotent monad is finitary in each case.

There are "idempotent" $\mathbb{O}$-typed theories for Cat, Opd, Comb.

$\hat{\mathcal{O}}$: $\forall\, \omega \in \mathcal{O}$. $\Lambda[\omega] \hookrightarrow \omega$



$$\hat{\mathcal{O}}\left[\{\Lambda[\omega] \hookrightarrow \omega \mid \omega \in \mathcal{O}_{\geq 2}\}^{-1}\right] = \mathrm{Cat}$$

Pictures

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