Induced and higher-dimensional stable independence

Michael Lieberman
(Joint with Jiří Rosický and Sebastien Vasey)

Brno University of Technology
Department of Mathematics

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We consider a few recent results concerning stable independence on abstract categories. Chiefly:

- We show that, under certain conditions, stable independence can be induced from a subcategory. Building on recent model-theoretic work, this gives stable independence in many important categories of groups and modules.
- Time (probably not) permitting, we also consider the phenomenon of *excellence*; that is, stable independence in arbitrary finite dimensions.

Nearly all of the results considered here are joint with Jiří Rosický and Sebastien Vasey, spanning several papers: [LRV1], [LRV2], and [LRV3].

One of the central features of model theory—classical or abstract—is that it is most successful when all morphisms are *monomorphisms*.

Example

Consider T_{ab}, the theory of abelian groups. The category of models of T_{ab}, $\mathbf{Mod}(T_{ab})$, is a wide subcategory of $\mathbf{Ab}$ but is decidedly non-full: however we construe it, its morphisms will certainly be monos.

This restriction is genuinely costly:

Fact

The category $\mathbf{Ab}$ has pushouts; $\mathbf{Mod}(T_{ab})$ does not, as the induced maps will not be monos!

Basic problem: given a category $\mathcal{K}$ and family of $\mathcal{K}$-morphisms $\mathcal{M}$, how much is lost in passing to $\mathcal{K}_{\mathcal{M}}$, the subcategory of $\mathcal{K}$ whose morphisms are precisely those in $\mathcal{M}$?

For the moment, we consider **very** nice $\mathcal{K}$, namely $\mathcal{K}$ locally presentable.

In general, passing to $\mathcal{K}_{\mathcal{M}}$ expels us from the paradise of locally presentable categories, leaving us with, if we are lucky, accessibility. That is, we may lose some colimits, including pushouts.

Fact

Say a category $\mathcal{C}$ is accessible with all morphisms mono (...). If $\mathcal{C}$ has pushouts, it is small.

So if we engineer $\mathcal{K}_{\mathcal{M}}$ to be nice, we lose pushouts. Such is life.

We take the view, perhaps controversially, that stable (*nonforking*) independence is best understood as a stand-in for the vanished pushouts.

This is extremely ahistorical...

Version 1: Fix a theory T, monster model $\mathfrak{C}$. We say the type of a tuple $\bar{a} \in \mathfrak{C}$ over a model B does not fork over $C \subseteq B$ if the type over C has the same complexity, i.e. Morley rank. Notation:

$$\bar{a} \overset{(\mathfrak{C})}{\underset{C}{\perp}} B$$

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Version 2: Again, given a theory T and monster model $\mathfrak{C}$, we say

$$A \underset{C}{\overset{(\mathfrak{C})}{\downarrow}} B$$

if the type of any $\bar{a} \in A$ over B does not fork over C. One can think of this as a kind of independence relation: A is independent from B over C.

One can think of $\downarrow$ as an abstract ternary relation, and axiomatize stable (or *simple*) independence directly.

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Version 3: In abstract model theory, we can only work over models, and may not have a monster model. We end up with $\mathop{\smile\!\!\!\!\!\!|}\,$ as a quaternary relation

$$M_1 \underset{M_0}{\overset{M_3}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$$

axiomatized as before, [BGKV]. In particular, we are picking out a family of diagrams of strong embeddings of the form

$$\begin{array}{ccc} M_1 & \to & M_3 \\ \uparrow & \mathop{\smile\!\!\!\!\!\!|} & \uparrow \\ M_0 & \to & M_2 \end{array}$$

Idea: Do this in an arbitrary category $\mathcal{K}$.

Definition

An independence notion $\downarrow$ on $\mathcal{K}$ is a family of commutative squares in $\mathcal{K}$ (suitably closed). We say that $\downarrow$ is **weakly stable** *if it satisfies*

1. *Existence: Any span $M_1 \leftarrow M_0 \rightarrow M_2$ can be completed to an independent square.*
2. *Uniqueness: there is only one independent square for each span, up to equivalence.*
3. *Transitivity: horizontal and vertical compositions of independent squares are independent.*

Fact

If $\downarrow$ is weakly stable, these squares satisfy the usual cancellation property of pushouts.

We must impose a locality condition—accessibility now appears.

Consider the category $\mathcal{K}_{\downarrow}$:

- Objects: Morphisms $f : M \to N$ in $\mathcal{K}$.
- Morphisms: A morphism from $f : M \to N$ to $f' : M' \to N'$ is a $\mathop{\smile}\limits^{|}$-independent square

$$\begin{array}{ccc} M' & \longrightarrow & N' \\ \uparrow & \mathop{\smile}\limits^{|} & \uparrow \\ M & \longrightarrow & N \end{array}$$

Definition

1. *We say that* $\mathop{\smile}\limits^{|}$ *is* λ-**continuous** *if* $\mathcal{K}_{\downarrow}$ *is closed under* λ*-directed colimits.*
2. *We say that* $\mathop{\smile}\limits^{|}$ *is* λ-**accessible** *if* $\mathcal{K}_{\downarrow}$ *is* λ*-accessible.*
3. *We say* $\mathop{\smile}\limits^{|}$ *is* λ-**stable** *if it is weakly stable and* λ*-accessible.*

Returning to the basic framework, i.e. $\mathcal{K}$ a category, $\mathcal{M}$ a class of morphisms, there is a natural candidate for stable independence:

Definition

We say a square

$$\begin{array}{ccc} M_1 & \longrightarrow & M_3 \\ \uparrow & & \uparrow \\ M_0 & \longrightarrow & M_2 \end{array}$$

*in $\mathcal{K}$ is $\mathcal{M}$-**effective** if*

1. *all morphisms are in $\mathcal{M}$,*
2. *the pushout of $M_1 \leftarrow M_0 \rightarrow M_2$ exists, and*
3. *the induced map from the pushout to M_3 is in $\mathcal{M}$.*

If $\mathcal{M} = \{\text{regular monos}\}$, these are the *effective unions* of Barr.

To force these squares to form a nice independence relation, we need a few additional properties:

Definition

Let $\mathcal{K}$ be a category.

1. *We say that $\mathcal{M}$ is* **coherent** *if whenever $gf \in \mathcal{M}$ and $g \in \mathcal{M}$, $f \in \mathcal{M}$.*
2. *We say that $\mathcal{M}$ is a* **coclan** *if pushouts of morphisms in $\mathcal{M}$ exist, and $\mathcal{M}$ is closed under pushouts.*
3. *We say $\mathcal{M}$ is* **almost nice** *if it is a coherent coclan, and* **nice** *if, in addition, it is closed under retracts.*

Proposition

If $\mathcal{M}$ is almost nice, the $\mathcal{M}$-effective squares give a weakly stable independence notion on $\mathcal{K}_{\mathcal{M}}$.

We return to our special case: $\mathcal{K}$ is a locally presentable category, and $\mathcal{M}$ a family of morphisms.

Note

As we said, if $\mathcal{M}$ is almost nice, $\mathcal{K}_{\mathcal{M}}$ has a weakly stable independence notion, given by the $\mathcal{M}$-effective squares.

When is this notion $\aleph_0$-continuous ($\mathcal{K}_{\mathcal{M},\downarrow}$ closed under directed colimits)? When is it $\aleph_0$-accessible ($\mathcal{K}_{\mathcal{M},\downarrow}$ finitely accessible) and therefore stable?

While existence and uniqueness seem to be the thornier issues in model theory, it is these properties that are most problematic here.

In this special case, though, there is an easy (and rather surprising!) answer.

Theorem

Let $\mathcal{K}$ be locally presentable, $\mathcal{M}$ nice and $\aleph_0$-continuous. The following are equivalent:

1. *$\mathcal{K}_{\mathcal{M}}$ has a stable independence notion.*
2. *$\mathcal{M}$-effective squares form a stable independence notion on $\mathcal{K}_{\mathcal{M}}$.*
3. *$\mathcal{M}$ is cofibrantly generated (and accessible).*

Note

If $\mathcal{M}$ is the left class of a coherent WFS, it is nice and $\aleph_0$-continuous. So this seems interesting...

This proof, in [LRV2], makes use of the canonicity of stable independence on any category with *directed bounds*. The proof of that fact, in turn, is interesting in its own right.

We need a little model-theoretic background.

Definition

An *abstract elementary class* (or *AEC*) in a fixed finitary signature Σ consists of a nonempty class of Σ-structures $\mathcal{K}$ and a designated family of embeddings between them, $\mathcal{M}$, with

- $\mathcal{M}$ coherent in $\mathbf{Emb}(\Sigma)$.
- $\mathcal{M}$ closed under directed colimits in $\mathbf{Emb}(\Sigma)$.
- $\mathcal{K}_{\mathcal{M}}$ accessible: Löwenheim-Skolem-Tarski.

This is too fancy, really: AECs are perfectly concrete, and a straightforward generalization of elementary classes.

Definition

A μ-AEC is just as above, but the signature may be μ-ary, and and $\mathcal{M}$ need only be closed under μ-directed colimits.

Note

Accessible categories with all morphisms monomorphisms and μ-AECs are exactly the same thing.

In place of syntactic types, we use *orbital* (or *Galois*) types:

Definition

Given $f_i : M_0 \to M_i$ and $a_i \in UM_i$, $i = 1, 2$, we say that a_1 and a_2 have the same Galois type over M_0 if there are $g_i : M_i \to N$ such that the following diagram commutes

$$\begin{array}{ccc} M_1 & \xrightarrow{g_1} & N \\ {\scriptstyle f_1}\uparrow & & \uparrow{\scriptstyle g_2} \\ M_0 & \xrightarrow[f_2]{} & M_2 \end{array}$$

and $Ug_1(a_1) = Ug_2(a_2)$.

Amalgamation guarantees this is an equivalence relation.

A standard way of measuring the complexity of a first-order theory or (μ-)AEC is *stability*, in the sense of counting types:

Definition

We say that a (μ-)AEC $\mathcal{K}$ is λ-Galois-stable if for any $M \in \mathcal{K}$, $|UM| = \lambda$, there are at most λ Galois types over M in $\mathcal{K}$. We say that $\mathcal{K}$ is *Galois stable* if it is λ-Galois-stable for some λ.

How does this relate to stable independence?

Theorem

Let $\mathcal{K}$ be a μ-AEC with directed bounds. If $\mathcal{K}$ has a stable independence relation, $\mathcal{K}$ is Galois stable on a proper class of cardinals.

Galois stability is strictly weaker—stable independence implies amalgamation, for one thing.

Let us assume that $\mathcal{K}$ is an accessible category with all morphisms monomorphisms (hence, morally speaking, a μ-AEC).

There are a number of model-theoretic conditions on $\mathcal{K}$—the failure of the order property plus a few large cardinals, or Galois stability—that ensure stable independence on a subcategory of $\mathcal{K}$.

It is natural, then, to ask: if some subcategory $\mathcal{L}$ of $\mathcal{K}$ has stable independence, under what conditions on

- the category $\mathcal{K}$, and
- the embedding $\mathcal{L} \hookrightarrow \mathcal{K}$

can we infer the existence of stable independence on $\mathcal{K}$?

Conditions on $\mathcal{K}$

We will not be able to manufacture stable independence on $\mathcal{K}$ out of whole cloth.

In particular, $\mathcal{K}$ must have an independence notion that is

1. weakly stable, and
2. $\aleph_0$-continuous.

The latter property implies only that $\mathcal{K}_\downarrow$ is closed under directed colimits—lacking the local character that would come with, say, $\aleph_0$-accessibility.

So we assume a lot, but much less than stability.

Conditions on $\mathcal{L} \hookrightarrow \mathcal{K}$

We will require that $\mathcal{L}$ is a cofinal subcategory, in a slightly weaker sense than usual:

Definition

We say that a functor $F : \mathcal{L} \to \mathcal{K}$ is *cofinal* if for any $K \in \mathcal{K}$ and any finite sequence $(FL_i \xrightarrow{f_i} K)_{i \in I}$, there is $L \in \mathcal{L}$, $K \xrightarrow{g} FL$, and $(FL_i \xrightarrow{Fg_i} FL)_{i \in I}$ such that $Fg_i = f_i \circ g$ for all $i \in I$. A subcategory is cofinal if the inclusion is cofinal.

- If $\mathcal{L}$ is a full subcategory of $\mathcal{K}$ and for every $K \in \mathcal{K}$ there is a morphism $f : M \to L$ with $L \in \mathcal{L}$, then $\mathcal{L}$ is a cofinal subcategory of $\mathcal{K}$ in the above sense.
- The category of λ-saturated models of a μ-AEC is a cofinal subcategory.

Conditions on $\mathcal{L} \hookrightarrow \mathcal{K}$

In addition to cofinality, we will need to ensure that the embedding $\mathcal{L} \hookrightarrow \mathcal{K}$ plays well with the accessible structure on $\mathcal{K}$:

Definition

We say that a subcategory $\mathcal{L}$ of a category $\mathcal{K}$ is *accessibly embedded* if

- $\mathcal{L}$ is a full subcategory, and
- $\mathcal{L}$ is closed under λ-directed colimits in $\mathcal{K}$ for some λ.

This is, somewhat confusingly, different from requiring that the embedding $\mathcal{L} \to \mathcal{K}$ is *accessible*—that involves preservation of colimits.

Theorem

Let $\mathcal{K}$ be an accessible category with all morphisms monomorphisms, and let $\mathcal{L}$ be an accessibly-embedded, cofinal subcategory of $\mathcal{K}$. If:

- *$\mathcal{K}$ has an $\aleph_0$-continuous weakly stable independence notion.*
- *$\mathcal{L}$ has a stable independence notion.*

Then $\mathcal{K}$ has a stable independence notion.

Proof: (Sketch) Let $\downarrow$ be the $\aleph_0$-continuous, weakly stable notion on $\mathcal{K}$. We must show that $\mathcal{K}_\downarrow$ is accessible.

By $\aleph_0$-continuity, $\mathcal{K}$ has directed colimits, meaning $\mathcal{L}$ has directed bounds. Since the restriction of $\downarrow$ to $\mathcal{L}$ will be weakly stable (check!), the canonicity theorem ensures that it is, in fact, stable.

This implies that the restriction of $\downarrow$ to $\mathcal{L}$ is accessible, meaning $\mathcal{L}$ is accessible. Take μ with $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{L}_\downarrow$ all μ-accessible.

Let $\mathcal{K}_\mu$ be the subcategory on the μ-presentable objects of $\mathcal{K}$. Let $\mathcal{K}^*$ be the set of all μ-directed colimits of $\mathcal{K}_\mu$-morphisms in $\mathcal{K}_\downarrow$. It suffices to show that $\mathcal{K}^* = \mathcal{K}^2$, the full arrow category. Notice that $\mathcal{L}^2 \subseteq \mathcal{K}^*$, in any case.

To achieve this, we show:

(i) $\mathcal{K}^*$ is closed under composition.

(ii) $\mathcal{K}^*$ is left cancellable.

(iii) If $M \in \mathcal{K}$, there is a morphism $M \to N$ in $\mathcal{K}^*$ with $N \in \mathcal{L}$.

We show sufficiency, but give only brief sketches of the proofs of (i)-(iii).

(i) $\mathcal{K}^*$ is closed under composition.

(ii) $\mathcal{K}^*$ is left cancellable.

(iii) If $M \in \mathcal{K}$, there is a morphism $M \to N$ in $\mathcal{K}^*$ with $N \in \mathcal{L}$.

Sufficient: for any $f : M \to N$ in $\mathcal{K}^2$, (iii) implies there is $g : M \to M'$ in $\mathcal{K}^*$ with $M' \in \mathcal{L}$. There is $h : M' \to M^*$ with M λ-saturated, $\lambda > \mu$. Then there is $t : N \to M^*$ with $tf = hg$.

By cofinality, there is $p : M^* \to L$ with $L \in \mathcal{L}$, and $ph \in \mathcal{L}^2 \subseteq \mathcal{K}^*$. By (i), $phg \in \mathcal{K}^*$. Since $ptf = phg$, (ii) implies that $f \in \mathcal{K}^*$, and we are done.

Proof of (i): Given composable f and g in $\mathcal{K}^*$, we can decompose them as μ-directed colimits and, with a little fiddling, ensure that (enough of) the arrows in these colimits are composable—this gives $gf \in \mathcal{K}^*$.

(i) $\mathcal{K}^*$ is closed under composition.

(ii) $\mathcal{K}^*$ is left cancellable.

(iii) If $M \in \mathcal{K}$, there is a morphism $M \to N$ in $\mathcal{K}^*$ with $N \in \mathcal{L}$.

Sufficient: for any $f : M \to N$ in $\mathcal{K}^2$, (iii) implies there is $g : M \to M'$ in $\mathcal{K}^*$ with $M' \in \mathcal{L}$. There is $h : M' \to M^*$ with M λ-saturated, $\lambda > \mu$. Then there is $t : N \to M^*$ with $tf = hg$.

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Proof of (ii): Quite finicky. Skip...

(i) $\mathcal{K}^*$ is closed under composition.

(ii) $\mathcal{K}^*$ is left cancellable.

(iii) If $M \in \mathcal{K}$, there is a morphism $M \to N$ in $\mathcal{K}^*$ with $N \in \mathcal{L}$.

Sufficient: for any $f : M \to N$ in $\mathcal{K}^2$, (iii) implies there is $g : M \to M'$ in $\mathcal{K}^*$ with $M' \in \mathcal{L}$. There is $h : M' \to M^*$ with M λ-saturated, $\lambda > \mu$. Then there is $t : N \to M^*$ with $tf = hg$.

By cofinality, there is $p : M^* \to L$ with $L \in \mathcal{L}$, and $ph \in \mathcal{L}^2 \subseteq \mathcal{K}^*$. By (i), $phg \in \mathcal{K}^*$. Since $ptf = phg$, (ii) implies that $f \in \mathcal{K}^*$, and we are done.

Proof of (iii): By contradiction. Uses well μ-filtrability of $\mathcal{K}$ ([LRV]); that is, objects are colimits of smooth chains.

We turn now to deriving stable independence in certain algebraic categories known to be Galois-stable.

We will, of course, use the theorem just discussed. But of critical importance, too, is:

Lemma

Let $\mathcal{K}$ be an AEC. If

1. *$\mathcal{K}$ has the amalgamation property,*
2. *$\mathcal{K}$ is Galois-stable, and*
3. *types in $\mathcal{K}$ are $< \aleph_0$-short over models,*

then there is a stable independence notion on a full, cofinal subcategory of $\mathcal{K}$—consisting of sufficiently saturated models.

Proof: (Sketch) Essentially the same as the construction of stable independence over saturated models of a stable first-order theory: $\aleph_0$-shortness stands in for compactness (cf. [V]).

Thus Galois-stability, in an AEC with amalgamation and $\aleph_0$-shortness, buys us much of what we need to derive existence of stable independence on the AEC itself.

We still require a weakly stable, $\aleph_0$-continuous independence notion. But we already have an excellent candidate:

Lemma

Let $\mathcal{K}$ be a category, and $\mathcal{M}$ an almost nice family of morphisms.

- *$\mathcal{K}_{\mathcal{M}}$ has a weakly stable independence notion consisting of the $\mathcal{M}$-effective squares.*
- *If $\mathcal{M}$ is closed under directed colimits, this independence notion is $\aleph_0$-continuous [LRV2].*

So, if our AEC is formed by taking an almost nice, direct-colimit-closed set of morphisms, we are in business. In all of the algebraic examples to follow, happily, this is the case.

We now stand on the shoulders of giants—Kucera and Mazari-Armida—who have done all of the model-theoretic heavy lifting. First, a template:

Proposition

For any ring with unit R, the category of left R-modules, R-$\mathbf{Mod}_{pure}$, has a stable independence notion.

Proof: By [KMA], R-$\mathbf{Mod}_{pure}$ forms an AEC, has amalgamation, is stable, and types are $< \aleph_0$-short over models. So, by the lemma, there is stable independence on a cofinal, full subcategory.

By inspection, or the fact that pure monomorphisms are the left half of a coherent WFS, pure monomorphisms are almost nice, closed under directed colimits. So $\mathcal{M}$-effective squares are weakly stable, $\aleph_0$-continuous.

The main theorem then yields stable independence on R-$\mathbf{Mod}_{pure}$.

We note that this is a special case of the result of [LPRV]: the latter was obtained by other means, but discovered by these.

The same template can be applied to, e.g. the following categories of modules. Here R is an integral domain.

1. Torsion R-modules with pure monomorphisms.
2. R-divisible modules (for any nonzero m and nonzero $r \in R$, there is n with $m = rn$) with pure monomorphisms.

Similarly, we obtain stable independence on many categories of groups, including:

1. Abelian groups and (pure) monos.
2. (Reduced) Torsion-free abelian groups with pure monos.
3. Abelian p-groups with (pure) monos.

We turn now to something which is still in search of applications: higher-dimensional stable independence on an abstract category.

Definition

Let $\mathcal{K}$ be a category. For $n \geq 1$, we define an *n-dimensional stable independence relation* on $\mathcal{K}$, Γ, and its induced category $\mathcal{K}_\Gamma$ by induction on n:

- We say Γ is a 1-dimensional stable independence notion on $\mathcal{K}$ just in case it is $\mathrm{Mor}(\mathcal{K})$. In this case, define $\mathcal{K}^\Gamma = \mathcal{K}$.
- An $(n+1)$-dimensional stable independence relation on $\mathcal{K}$ consists of a pair (Γ_n, Γ), where
  1. Γ_n is an n-dimensional stable independence relation on $\mathcal{K}$.
  2. Γ is a stable independence notion on $\mathcal{K}^{\Gamma_n}$
- Given $(n+1)$-dimensional $\Gamma_{n+1} = (\Gamma_n, \Gamma)$ on $\mathcal{K}$, define $(\mathcal{K}^{\Gamma_{n+1}} = \mathcal{K}^{\Gamma_n})^\Gamma$, whose objects are morphisms of $\mathcal{K}^{\Gamma_n}$ and whose morphisms are Γ-independent squares (that is, $\mathcal{K}_\downarrow$ with $\downarrow = \Gamma$).

That is too much to internalize in one sitting, of course.

(As an exercise, one might check that 2-dimensional stable independence notions correspond to stable independence notions as already defined.)

The best-case—and presumably rare—scenario is the following:

Definition

We say that a category $\mathcal{K}$ is *excellent* if for all $n \geq 1$, $\mathcal{K}$ has an n-dimensional stable independence relation Γ_n such that $\mathcal{K}^{\Gamma_n}$ has directed colimits.

We return to our favorite special case: $\mathcal{K}$ locally presentable.

Theorem

Let $\mathcal{K}$ be a locally presentable category, and let $\mathcal{M}$ be a nice, accessible, $\aleph_0$-continuous class of morphisms in $\mathcal{K}$. If $\mathcal{K}_{\mathcal{M}}$ has a stable independence relation, it is excellent.

Theorem

Let $\mathcal{K}$ be a locally presentable category, and let $\mathcal{M}$ be a nice, accessible, $\aleph_0$-continuous class of morphisms in $\mathcal{K}$. If $\mathcal{K}_{\mathcal{M}}$ has a stable independence relation, it is excellent.

Proof: (Sketch) We wish to proceed by induction on dimension.

Recall that, under these hypotheses, $\mathcal{K}_{\mathcal{M}}$ has stable independence just in case $\mathcal{M}$ is cofibrantly generated.

So really, the inductive step involves showing that, given the above assumptions, the class of $\mathcal{M}$-effective morphisms in $\mathcal{K}^2$—call it $\mathcal{M}!$—is well-behaved in exactly the same ways:

- $\mathcal{M}!$ is cofibrantly generated in $\mathcal{K}^2$.
- $\mathcal{M}!$ is nice, $\aleph_0$-continuous, and accessible.

Nearly everything is just bookkeeping, aside from showing $\mathcal{M}!$ is a coclan and that it is cofibrantly generated (easier via stability!).

There are plenty of open questions, some of which have already occurred naturally in this discussion:

- What happens if we weaken different conditions in the definition of stable independence, particularly uniqueness?
- Do superstability, simplicity, etc. admit clean category-theoretic characterizations, e.g. via properties of WFSs?
- How much use can we get out of stable independence and related constructions, e.g. independent sequences, in an abstract category?
- In what other contexts can we get stability via the induced route taken here?
- What is excellence good for in an abstract category? Does it correspond to any existing notions?

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