

Distributive laws, pseudodistributive laws and decagons

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$$\begin{array}{ccccccc}
 TPTPT & \xrightarrow{TP\lambda T} & TP^2T^2 & \xrightarrow{TP^2m} & TP^2T & \xrightarrow{T\mu T} & TPT \\
 \lambda TPT \downarrow & & & & & & \downarrow \lambda T \\
 PT^2PT & & & & & & PT^2 \\
 PmPT \downarrow & & & & & & \downarrow Pm \\
 PTPT & \xrightarrow{P\lambda T} & P^2T^2 & \xrightarrow{P^2m} & P^2T & \xrightarrow{\mu T} & PT
 \end{array}$$

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1 Background knowledge

- Monads in monoidal and extensive form
- Distributive laws of monads and Beck's theorem
- Pseudomonads in monoidal and extensive form
- Pseudodistributive laws of pseudomonads (until now)

2 Distributive laws of monads in extensive form

- Extending to the Kleisli category extensively
- The five coherence axioms of a pseudodistributive law
- The three redundant coherence axioms

3 Minimal definitions

- Minimal definition of distributive laws
- Minimal definition of pseudodistributive laws
- Future work

Monads in monoidal form

Definition

A monad (in monoidal form) on a category \mathcal{C} consists of a functor $T: \mathcal{C} \rightarrow \mathcal{C}$ and a unit $u: 1 \rightarrow T$ and a multiplication $m: TT \rightarrow T$ rendering commutative

$$\begin{array}{ccccc}
 T & \xrightarrow{uT} & TT & \xleftarrow{Tu} & T \\
 & \searrow \text{id} & \downarrow m & \swarrow \text{id} & \\
 & & T & &
 \end{array}$$

$$\begin{array}{ccc}
 TTT & \xrightarrow{mT} & TT \\
 \downarrow Tm & & \downarrow m \\
 TT & \xrightarrow{m} & T
 \end{array}$$

Monads in extensive form

Definition (Walters, Manes)

A *monad (in extensive form)* on a category \mathcal{C} consists of

- an assignment on objects $X \mapsto TX$;
- for each $X \in \mathcal{C}$, a $u_X: X \rightarrow TX$;
- for each $f: X \rightarrow TY$, a map $f^T: TX \rightarrow TY$;

such that:

- for each $f: X \rightarrow TY$, we have $f^T \cdot u_X = f$ as below

$$\begin{array}{ccc} TX & \xrightarrow{f^T} & TY \\ u_X \uparrow & \nearrow f & \\ X & & \end{array}$$

Monads in extensive form

Definition (Walters, Manes)

- for each $X \in \mathcal{C}$, we have $u_X^T = \text{id}$ as below

$$\begin{array}{ccc} TX & \xrightarrow{u_X^T} & TX \\ u_X \uparrow & \nearrow u_X & \\ X & & \end{array}$$

$$\begin{array}{ccc} TX & \xrightarrow{\text{id}} & TX \\ u_X \uparrow & \nearrow u_X & \\ X & & \end{array}$$

- for each $f: X \rightarrow TY$ and $g: Y \rightarrow TZ$, we have $(g^T \cdot f)^T = g^T \cdot f^T$

$$\begin{array}{ccccc} TX & \xrightarrow{f^T} & TY & \xrightarrow{g^T} & TZ \\ u_X \uparrow & \nearrow f & & & \\ X & & & & \end{array}$$

$$\begin{array}{ccccc} TX & \xrightarrow{(g^T \cdot f)^T} & & \xrightarrow{g^T} & TZ \\ u_X \uparrow & \nearrow f & TY & & \\ X & & & & \end{array}$$

Monads in extensive form

For a $f: X \rightarrow TY$ the induced $f^T: TX \rightarrow TY$ is defined as

$$TX \xrightarrow{Tf} T^2Y \xrightarrow{mY} TY$$

Conversely, $m: T^2 \rightarrow T$ can be recovered by “extending” the identity

$$\begin{array}{ccc} T^2X & \xrightarrow{m^X} & TX \\ \uparrow u_{TX} & \nearrow \text{id} & \\ TX & & \end{array}$$

Theorem (Walters, Manes)

This defines a bijection between monads in monoidal and extensive form.

Distributive laws in monoidal form

Definition (Beck)

A *distributive law* (in monoidal form) between monads (T, u, m) and (P, η, μ) is a natural transformation $\lambda: TP \rightarrow PT$ rendering commutative the four diagrams below

$$\begin{array}{ccc}
 TP & \xrightarrow{\lambda} & PT \\
 \swarrow uP & & \searrow Pu \\
 & P &
 \end{array}
 \qquad
 \begin{array}{ccc}
 TP & \xrightarrow{\lambda} & PT \\
 \swarrow T\eta & & \searrow \eta T \\
 & T &
 \end{array}$$

$$\begin{array}{ccc}
 T^2P & \xrightarrow{T\lambda} & TPT & \xrightarrow{\lambda T} & PTT \\
 mP \downarrow & & & & \downarrow Pm \\
 TP & \xrightarrow{\lambda} & PT & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 TP^2 & \xrightarrow{\lambda P} & PTP & \xrightarrow{P\lambda} & P^2T \\
 T\mu \downarrow & & & & \downarrow \mu T \\
 TP & \xrightarrow{\lambda} & PT & &
 \end{array}$$

Distributive laws in monoidal form

For example, let T be the monad for monoids and P the monad for abelian groups. The usual distributive law $\lambda: TP \rightarrow PT$ takes products of sums to sums of products, e.g.

$$(a + b)(c + d) \mapsto ac + ad + bc + bd$$

Theorem (Beck)

Given monads (T, u, m) and (P, η, μ) , the following are in bijection:

- 1 *distributive laws $\lambda: TP \rightarrow PT$*
- 2 *liftings of the monad P to the category of T -algebras, $\mathbf{Alg}(T)$;*
- 3 *extensions of the monad T to the Kleisli category of P , $\mathbf{Kl}(P)$;*
- 4 *monad structures on PT which are suitably compatible with the monads T and P .*

In this talk we are only interested in extensions to the Kleisli category.

Kleisli category

Definition

Given a monad (P, η, μ) on a category \mathcal{C} , the Kleisli category of P , denoted $\mathbf{Kl}(P)$, is the category as follows

- objects are those of \mathcal{C} ;
- a “Kleisli morphism” $X \rightsquigarrow Y$ is a morphism $X \rightarrow PY$ in \mathcal{C} ;
- the composite of a morphism $f: X \rightarrow PY$ and $g: Y \rightarrow PZ$ is the composite

$$X \xrightarrow{f} PY \xrightarrow{Pg} P^2Z \xrightarrow{\mu Z} PZ$$

Extensions to the Kleisli category

Definition

Given monads (T, u, m) and (P, η, μ) on a category \mathcal{C} . An extension of a monad (T, u, m) to the Kleisli category $\mathbf{Kl}(P)$ is a monad $(\tilde{T}, \tilde{u}, \tilde{m})$ on $\mathbf{Kl}(P)$ such that

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{T} & \mathcal{C} \\
 F \downarrow & & \downarrow F \\
 \mathbf{Kl}(P) & \xrightarrow{\tilde{T}} & \mathbf{Kl}(P)
 \end{array}$$

commutes (where F is the free functor) written $\tilde{T} \cdot F = F \cdot T$; and moreover $\tilde{u} \cdot F = F \cdot u$ and $\tilde{m} \cdot F = F \cdot m$.

Extensions to the Kleisli category

Given a λ , we can define \tilde{T} by sending an $f: X \rightarrow PY$ to

$$TX \xrightarrow{f} TPY \xrightarrow{\lambda Y} PTY$$

If one works out what is required to ensure $\lambda: TP \rightarrow PT$ gives an extension to $\mathbf{Kl}(P)$ with $(\tilde{T}, \tilde{u}, \tilde{m})$ in monoidal form, they naturally arrive at Beck's four coherence axioms.

If one works out what is required to ensure $\lambda: TP \rightarrow PT$ gives an extension to $\mathbf{Kl}(P)$ with $(\tilde{T}, \tilde{u}, \tilde{m})$ in *extensive form*, they arrive at a different set of axioms!

(We will talk about this later on)

Pseudomonads in monoidal form

Definition

A *pseudomonad (in monoidal form)* on a 2-category \mathcal{C} consists of a pseudofunctor equipped with pseudonatural transformations as below

$$T: \mathcal{C} \rightarrow \mathcal{C}, \quad u: 1_{\mathcal{C}} \rightarrow T, \quad m: T^2 \rightarrow T$$

along with three invertible modifications

$$\begin{array}{ccc}
 T & \xrightarrow{uT} & T^2 & \xleftarrow{Tu} & T \\
 & \searrow & \downarrow m & \swarrow & \\
 & & T & & \\
 & \swarrow \alpha & & \nwarrow \beta & \\
 & & T & &
 \end{array}
 \quad
 \begin{array}{ccc}
 T^3 & \xrightarrow{Tm} & T^2 \\
 mT \downarrow & & \downarrow m \\
 T^2 & \xrightarrow{m} & T \\
 & \swarrow \gamma & \\
 & & T
 \end{array}$$

subject to the two coherence axioms.

Pseudomonads in monoidal form

Definition

$$\begin{array}{c}
 T^2 \xrightarrow{TuT} T^3 \begin{array}{l} \nearrow Tm \rightarrow T^2 \\ \searrow mT \rightarrow T^2 \end{array} \begin{array}{l} \nearrow Tm \rightarrow T \\ \searrow m \rightarrow T \end{array} \\
 \Downarrow \gamma \\
 T^2 \begin{array}{l} \nearrow TuT \rightarrow T^3 \\ \searrow TuT \rightarrow T^3 \end{array} \begin{array}{l} \nearrow Tm \rightarrow T^3 \\ \searrow mT \rightarrow T^3 \end{array} \begin{array}{l} \nearrow Tm \rightarrow T^2 \\ \searrow mT \rightarrow T^2 \end{array} \begin{array}{l} \nearrow Tm \rightarrow T \\ \searrow m \rightarrow T \end{array} \\
 \Downarrow T\alpha \\
 \text{id} \\
 \Downarrow \beta T
 \end{array}
 =
 \begin{array}{c}
 T^2 \begin{array}{l} \nearrow TuT \rightarrow T^3 \\ \searrow TuT \rightarrow T^3 \end{array} \begin{array}{l} \nearrow Tm \rightarrow T^3 \\ \searrow mT \rightarrow T^3 \end{array} \begin{array}{l} \nearrow Tm \rightarrow T^2 \\ \searrow mT \rightarrow T^2 \end{array} \begin{array}{l} \nearrow Tm \rightarrow T \\ \searrow m \rightarrow T \end{array} \\
 \Downarrow T\alpha \\
 \text{id} \\
 \Downarrow \beta T
 \end{array}$$

$$\begin{array}{c}
 T^4 \xrightarrow{T^2 m} T^3 \begin{array}{l} \nearrow TmT \\ \searrow mT^2 \end{array} \begin{array}{l} \nearrow Tm \\ \searrow T\gamma \end{array} \begin{array}{l} \nearrow Tm \\ \searrow Tm \end{array} \begin{array}{l} \nearrow Tm \\ \searrow Tm \end{array} \\
 \Downarrow mT^2 \\
 T^3 \begin{array}{l} \nearrow TmT \\ \searrow mT^2 \end{array} \begin{array}{l} \nearrow Tm \\ \searrow T\gamma \end{array} \begin{array}{l} \nearrow Tm \\ \searrow Tm \end{array} \begin{array}{l} \nearrow Tm \\ \searrow Tm \end{array} \\
 \Downarrow mT \\
 T^2 \begin{array}{l} \nearrow Tm \\ \searrow Tm \end{array} \begin{array}{l} \nearrow Tm \\ \searrow Tm \end{array} \begin{array}{l} \nearrow Tm \\ \searrow Tm \end{array} \begin{array}{l} \nearrow Tm \\ \searrow Tm \end{array} \\
 \Downarrow m \\
 T \begin{array}{l} \nearrow Tm \\ \searrow Tm \end{array} \begin{array}{l} \nearrow Tm \\ \searrow Tm \end{array} \begin{array}{l} \nearrow Tm \\ \searrow Tm \end{array} \begin{array}{l} \nearrow Tm \\ \searrow Tm \end{array} \\
 \Downarrow m \\
 T
 \end{array}
 =
 \begin{array}{c}
 T^4 \xrightarrow{T^2 m} T^3 \begin{array}{l} \nearrow Tm \\ \searrow mT^2 \end{array} \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \\
 \Downarrow mT^2 \\
 T^3 \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \\
 \Downarrow mT \\
 T^2 \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \\
 \Downarrow m \\
 T \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \begin{array}{l} \nearrow Tm \\ \searrow mT \end{array} \\
 \Downarrow m \\
 T
 \end{array}$$

Pseudomonads in monoidal form

By results of Kelly, there are three redundant axioms:

$$\begin{array}{ccc}
 T^2 & \xrightarrow{uT^2} & T^3 & \xrightarrow{Tm} & T^2 \\
 \searrow \text{id} & & \swarrow \alpha T & \downarrow & \downarrow m \\
 & & T^2 & \xrightarrow{m} & T \\
 & & \swarrow mT & \leftarrow \gamma & \\
 & & & &
 \end{array}
 =
 \begin{array}{ccccc}
 & & T^3 & \xrightarrow{Tm} & T^2 & \xrightarrow{m} & T \\
 & & \uparrow uT^2 & \downarrow u_m & \uparrow uT & \leftarrow \alpha & \\
 & & T^2 & \xrightarrow{m} & T & & \nearrow \text{id}
 \end{array}$$

and

$$\begin{array}{ccc}
 T^2 & \xrightarrow{T^2u} & T^3 & \xrightarrow{mT} & T^2 \\
 \searrow \text{id} & & \swarrow T\beta^{-1} & \downarrow & \downarrow m \\
 & & T^2 & \xrightarrow{m} & T \\
 & & \swarrow Tm & \leftarrow \gamma^{-1} & \\
 & & & &
 \end{array}
 =
 \begin{array}{ccccc}
 & & T^3 & \xrightarrow{mT} & T^2 & \xrightarrow{m} & T \\
 & & \uparrow T^2u & \downarrow m_u^{-1} & \uparrow Tu & \leftarrow \beta^{-1} & \\
 & & T^2 & \xrightarrow{m} & T & & \nearrow \text{id}
 \end{array}$$

Pseudomonads in monoidal form

and finally

$$\begin{array}{c}
 1_{\mathcal{C}} \xrightarrow{u} T \\
 \begin{array}{ccc}
 & \nearrow^{uT} & T^2 \\
 & \downarrow \alpha & \searrow m \\
 & \text{id} & \longrightarrow T \\
 & \downarrow \beta & \\
 & \searrow_{Tu} & T^2 \\
 & & \nearrow m
 \end{array}
 \end{array}
 =
 \begin{array}{c}
 1_{\mathcal{C}} \begin{array}{ccc}
 \nearrow^u & T & \searrow^{uT} \\
 \searrow_u & & \nearrow_{Tu} \\
 & \downarrow u_u^{-1} & \\
 & T & \nearrow_{Tu}
 \end{array} T^2 \xrightarrow{m} T
 \end{array}$$

Pseudomonads in extensive form

Definition (Marmolejo, Fiore-Gambino-Hyland-Winkskel)

A *pseudomonad (in extensive form)* on a bicategory \mathcal{C} consists of

- an assignation on objects $X \mapsto TX$;
- for each $X \in \mathcal{C}$, a $u_X: X \rightarrow TX$;
- for each $X, Y \in \mathcal{C}$ a functor $(-)^T_{X,Y}: \mathcal{C}(X, TY) \rightarrow \mathcal{C}(TX, TY)$;
- for each $f: X \rightarrow TY$, an isomorphism $\phi_f: f \Rightarrow f^T \cdot u_X$ natural in f ;
- for each $X \in \mathcal{C}$, an isomorphism $\theta_X: u_X^T \Rightarrow \text{id}_{TX}$;
- for each pair $f: X \rightarrow TY$ and $g: Y \rightarrow TZ$, an isomorphism $\delta: (g^T \cdot f)^T \Rightarrow g^T \cdot f^T$ natural in f and g ;

satisfying the two coherence conditions:

Pseudomonads in extensive form

for each $f: X \rightarrow TY$ the diagram

$$\begin{array}{ccccc}
 f^T & \xrightarrow{\phi_f^T} & (f^T u_X)^T & \xrightarrow{\delta_{f, u_X}} & f^T u_X^T \\
 & \searrow & & & \downarrow f^T \theta_X \\
 & & & & f^T \cdot \text{id} \\
 & \searrow \text{unit} & & & \\
 & & & &
 \end{array}$$

commutes;

Pseudomonads in extensive form

for each $f: X \rightarrow TY$, $g: Y \rightarrow TZ$, and $h: Z \rightarrow TV$ the diagram

$$\begin{array}{ccc}
 & (\delta_{h,gf})^T \left((h^T g)^T f \right)^T & \\
 & \swarrow \delta_{h^T g, f} & \searrow \\
 \left((h^T g^T) f \right)^T & & (h^T g)^T f^T \\
 \text{assoc.} \downarrow & & \downarrow \delta_{h,gf}^T \\
 \left(h^T (g^T f) \right)^T & & (h^T g^T) f^T \\
 \delta_{h^T, g^T f} \downarrow & & \downarrow \text{assoc.} \\
 h^T (g^T f)^T & \xrightarrow{h^T \delta_{g,f}} & h^T (g^T f^T)
 \end{array}$$

commutes.

Pseudomonads in extensive form

The redundant axioms now become the assertion that for any objects $X, Y, Z \in \mathcal{C}$ and morphisms $f: X \rightarrow TY$ and $g: Y \rightarrow TZ$ the diagrams

$$\begin{array}{ccc}
 u_X \xrightarrow{\phi_{u_X}} u_X^T u_X & (u_Y^T f)^T \xrightarrow{\delta_{u_Y, f}} u_Y^T f^T & g^T f \xrightarrow{\phi_{g^T f}} (g^T f)^T u_X \\
 \downarrow \theta_X u_X & \downarrow \theta_Y f^T & \downarrow \delta_{g, f} u_X \\
 u_X \xrightarrow{\text{id}} u_X & f^T & g^T f^T u_X \\
 & \uparrow (\theta_Y f)^T & \uparrow g^T \phi_f
 \end{array}$$

commute.

History of pseudodistributive laws

Even when dealing with 2-monads, one often needs to use the pseudo version of distributive law.

Example

Take:

T to be the 2-monad for small symmetric monoidal categories on **Cat**

P to be the 2-monad for small categories with finite products on **Cat**

There is no natural 2-distributive law $\lambda: TP \rightarrow PT$ (one of the pentagons doesn't commute), but there is a natural pseudo-distributive law.

History of pseudodistributive laws

Thus one must allow the required diagrams to only commute up to invertible modifications

$$\begin{array}{ccc}
 TP & \xrightarrow{\lambda} & PT \\
 \swarrow uP & \omega_1 \rightleftarrows & \nearrow Pu \\
 & P &
 \end{array}$$

$$\begin{array}{ccc}
 TP & \xrightarrow{\lambda} & PT \\
 \swarrow T\eta & \omega_2 \Downarrow & \nearrow \eta T \\
 & T &
 \end{array}$$

$$\begin{array}{ccccc}
 T^2P & \xrightarrow{T\lambda} & TPT & \xrightarrow{\lambda T} & PTT \\
 mP \downarrow & & \omega_3 \swarrow & & \downarrow Pm \\
 TP & \xrightarrow{\lambda} & & & PT
 \end{array}$$

$$\begin{array}{ccccc}
 TP^2 & \xrightarrow{\lambda P} & PTP & \xrightarrow{P\lambda} & P^2T \\
 T\mu \downarrow & & \omega_4 \nearrow & & \downarrow \mu T \\
 TP & \xrightarrow{\lambda} & & & PT
 \end{array}$$

subject to “some” coherence conditions.

These coherence conditions are the complicated part!

History of pseudodistributive laws

History of the coherence axioms:

- Kelly considered the “mild case” where both triangles and one of the pentagons commuted strictly. Kelly arrived at five coherence axioms with these “semi-strict” distributive laws.
- Marmolejo considered the general case, requiring the four modifications to satisfy nine coherence axioms.
- Tanaka introduced an extra tenth coherence axiom in this thesis.
- Marmolejo and Wood showed that Tanaka’s extra axiom, in addition to one of the original nine, are redundant. Thus showing eight coherence axioms suffice.
- Today we show five coherence axioms suffice!

Pseudodistributive laws

Theorem (Marmolejo, Cheng-Hyland-Power, Tanaka)

Given pseudomonads (T, u, m) and (P, η, μ) (suppressing the modifications comprising the pseudomonads), the following are equivalent:

- 1 pseudodistributive laws $(\lambda, \omega_1, \omega_2, \omega_3, \omega_4) : TP \rightarrow PT$;
- 2 liftings of the pseudomonad P to the 2-category of pseudo T -algebras, $\mathbf{Ps-Alg}(T)$;
- 3 extensions of the pseudomonad T to the Kleisli bicategory of P , $\mathbf{KI}(P)$;
- 4 pseudomonad structures on PT which are suitably compatible with the pseudomonads T and P .

We will only be interested in extensions to the Kleisli bicategory.

Extending extensively

Suppose we are given monads (T, u, m) and (P, η, μ) on a category \mathcal{C} . Let's work out what it means to extend (T, u, m) to the Kleisli category

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{T} & \mathcal{C} \\
 F \downarrow & & \downarrow F \\
 \mathbf{KI}(P) & \xrightarrow{\tilde{T}} & \mathbf{KI}(P)
 \end{array}$$

where the extension \tilde{T} is defined extensively!

Of course we can just ask for a monad \tilde{T} which is T on objects, has unit constraints $\eta_{TX} \cdot u_X: X \rightarrow PTX$ and

$$(-)^{\tilde{T}} F = F(-)^T: \mathcal{C}(X, TY) \rightarrow \mathbf{KI}(P)(TX, TY), \quad \forall X, Y \in \mathcal{C}$$

Here we mean for a given $\lambda: TP \rightarrow PT$, what conditions will we need?

Extending extensively

Given a $\lambda: TP \rightarrow TP$, we need to define our monad \tilde{T} in extensive form. Clearly \tilde{T} is the same as T on objects.

We define the unit $\tilde{u}_X: X \rightsquigarrow TX$ i.e. $X \rightarrow PTX$ as the composite

$$X \xrightarrow{u_X} TX \xrightarrow{\eta^{TX}} PTX$$

For a general $f: X \rightsquigarrow TY$ i.e. $f: X \rightarrow PTY$, we define our $f^{\tilde{T}}: TX \rightsquigarrow TY$ i.e. $f^{\tilde{T}}: TX \rightarrow PTY$ as the composite

$$TX \xrightarrow{Tf} TPTY \xrightarrow{\lambda^{TY}} PT^2Y \xrightarrow{PmY} PTY$$

Extending extensively

We must first check

$$\begin{array}{ccc}
 TX & \xrightarrow{\tilde{f}T} & TY \\
 \tilde{u}_X \uparrow & & \nearrow f \\
 X & &
 \end{array}$$

That is, any $f: X \rightarrow PTY$ is equal to the bottom composite below (ϕ_f in two dimensions)

$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & PTY & & & & PTY \\
 \searrow u_X & & \downarrow u_{PTY} & \xrightarrow{Pu_{TY}} & id & & \nearrow id \\
 TX & \xrightarrow{Tf} & TPTY & \xrightarrow{\lambda_{TY}} & PT^2Y & \xrightarrow{Pm_Y} & PTY \\
 \searrow \eta_{TX} & & \downarrow \eta_{TPTY} & & \downarrow \eta_{PT^2Y} & & \nearrow \eta_{PTY} \\
 & & PTX & \xrightarrow{PTf} & PTPTY & \xrightarrow{P\lambda_{TY}} & P^2T^2Y \\
 & & & & & \nearrow P^2m_Y & \nearrow \mu_{TY} \\
 & & & & & & P^2TY
 \end{array}$$

This is just the unit axiom with Pu and uP .

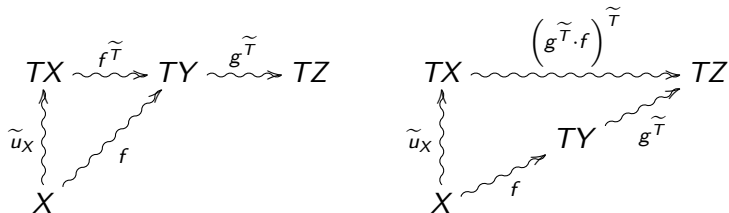
Extending extensively

The axiom that $\tilde{u}_X^T = \text{id}_{TX}$ amounts to the condition (θ_X in two dimensions)

$$\begin{array}{ccccccccc}
 TX & \xrightarrow{TuX} & T^2X & \xrightarrow{T\eta TX} & TPTX & \xrightarrow{\lambda TX} & PT^2X & \xrightarrow{PmX} & PTX \\
 & & & & \downarrow \omega_2 TX & & & & \\
 & & & & \eta T^2X & & & & \\
 & \searrow \text{id} & \xrightarrow{mX} & & TX & \xrightarrow{\eta TX} & & & \\
 & & & & & & & &
 \end{array}$$

This is just the unit axiom with $T\eta$ and ηT .

Extending extensively



Finally, the above axiom amounts to the condition $(\delta_{g,f}$ in two dimensions)

$$\begin{array}{ccccccc}
 TX & \xrightarrow{Tf} & TPTY & \xrightarrow{TPTg} & TPTPTZ & \xrightarrow{TP\lambda TZ} & TP^2 T^2 Z & \xrightarrow{TP^2 mZ} & TP^2 TZ & \xrightarrow{T\mu TZ} & TPTZ \\
 & & \downarrow \lambda TY & & \downarrow \lambda TPTZ & & & & & & \downarrow \lambda TZ \\
 & & PT^2 Y & \xrightarrow{PT^2 g} & PT^2 PTZ & & & \Downarrow \Omega & & & PT^2 Z \\
 & & \downarrow PmY & & \downarrow PmPTZ & & & & & & \downarrow PmZ \\
 & & PT^2 Y & \xrightarrow{PTg} & TPTZ & \xrightarrow{P\lambda TZ} & P^2 T^2 Z & \xrightarrow{P^2 mZ} & P^2 TZ & \xrightarrow{\mu TZ} & PTZ
 \end{array}$$

Distributive laws in extensive form

We thus naturally arrive at the following definition of a distributive law.

Definition

A *distributive law (in extensive form)* between monads (T, u, m) and (P, η, μ) is a natural transformation $\lambda: TP \rightarrow PT$ rendering commutative the three diagrams below

$$\begin{array}{ccc} TP & \xrightarrow{\lambda} & PT \\ & \swarrow uP & \nearrow Pu \\ & P & \end{array}$$

$$\begin{array}{ccc} TP & \xrightarrow{\lambda} & PT \\ & \swarrow T\eta & \nearrow \eta T \\ & T & \end{array}$$

$$\begin{array}{ccccccc} TP^2T & \xrightarrow{TP\lambda T} & TP^2T^2 & \xrightarrow{TP^2m} & TP^2T & \xrightarrow{T\mu T} & TPT \\ \lambda TPT \downarrow & & & & & & \downarrow \lambda T \\ PT^2PT & & & & & & PT^2 \\ PmPT \downarrow & & & & & & \downarrow Pm \\ PTPT & \xrightarrow{P\lambda T} & P^2T^2 & \xrightarrow{P^2m} & P^2T & \xrightarrow{\mu T} & PT \end{array}$$

Mixed distributive laws in extensive form

This also works for mixed distributive laws!

Definition

A *mixed distributive law (in extensive form)* between a comonad (L, ε, δ) and monad (R, η, μ) is a natural transformation $\lambda: LR \rightarrow RL$ rendering commutative the three diagrams below

$$\begin{array}{ccc}
 LR & \xrightarrow{\lambda} & RL \\
 \varepsilon R \searrow & & \swarrow R\varepsilon \\
 & R &
 \end{array}
 \qquad
 \begin{array}{ccc}
 LR & \xrightarrow{\lambda} & RL \\
 L\eta \swarrow & & \nwarrow \eta L \\
 & L &
 \end{array}$$

$$\begin{array}{ccccccc}
 & & & & & & \\
 & & & & & & \\
 & \nearrow \lambda R & & & & & \\
 LR^2 & \xrightarrow{\lambda R} & RLR & \xrightarrow{R\delta R} & RL^2R & \xrightarrow{RL\lambda} & RLRL & \xrightarrow{R\lambda L} & R^2L^2 & \xrightarrow{\mu L^2} & RL^2 \\
 & \searrow \delta R^2 & & & & & & & & & \\
 & & L^2R^2 & \xrightarrow{L\lambda R} & LRLR & \xrightarrow{LR\lambda} & LR^2L & \xrightarrow{L\mu L} & LRL & \xrightarrow{\lambda L} & RL^2
 \end{array}$$

Pseudodistributive laws in monoidal form

Definition

A *pseudodistributive law (in monoidal form)* between pseudomonads (with their modification data suppressed) (T, u, m) and (P, η, μ) is a pseudonatural transformation $\lambda: TP \rightarrow PT$ and four invertible modifications as below

$$\begin{array}{ccc}
 TP & \xrightarrow{\lambda} & PT \\
 \swarrow uP & \nearrow \omega_1 & \nearrow Pu \\
 & P &
 \end{array}$$

$$\begin{array}{ccc}
 TP & \xrightarrow{\lambda} & PT \\
 \swarrow T\eta & \searrow \omega_2 & \nearrow \eta T \\
 & T &
 \end{array}$$

$$\begin{array}{ccccc}
 T^2P & \xrightarrow{T\lambda} & TPT & \xrightarrow{\lambda T} & PTT \\
 mP \downarrow & & \searrow \omega_3 & & \downarrow Pm \\
 TP & \xrightarrow{\lambda} & & \xrightarrow{\lambda} & PT
 \end{array}$$

$$\begin{array}{ccccc}
 TP^2 & \xrightarrow{\lambda P} & PTP & \xrightarrow{P\lambda} & P^2T \\
 T\mu \downarrow & & \nearrow \omega_4 & & \downarrow \mu T \\
 TP & \xrightarrow{\lambda} & & \xrightarrow{\lambda} & PT
 \end{array}$$

satisfying the following five coherence axioms:

Pseudodistributive laws in monoidal form

Definition

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 & & \curvearrowright & & \\
 TP & \xrightarrow{TuP} & T^2P & \xrightarrow{mP} & TP \\
 \text{id} \downarrow & & T\omega_1 \nearrow & & T\lambda \downarrow \\
 TP & \xrightarrow{TPu} & TPT & \xrightarrow{\omega_3} & TP \\
 \lambda \downarrow & & \lambda T \downarrow & & \lambda \downarrow \\
 PT & \xrightarrow{PTu} & PT^2 & \xrightarrow{Pm} & PT \\
 & & \text{id} & & \\
 & & \curvearrowleft & &
 \end{array}
 = \text{id}_\lambda
 \tag{W1}$$

Pseudodistributive laws in monoidal form

Definition

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 & & \curvearrowright & & \\
 TP & \xrightarrow{T\eta P} & TP^2 & \xrightarrow{T\mu} & TP \\
 \text{id} \downarrow & \swarrow \omega_2 P & \lambda P \downarrow & & \downarrow \lambda \\
 TP & \xrightarrow{\eta TP} & PTP & \swarrow \omega_4 & = \text{id}_\lambda \\
 \lambda \downarrow & & P\lambda \downarrow & & \downarrow \lambda \\
 PT & \xrightarrow{\eta PT} & P^2 T & \xrightarrow{\mu T} & PT \\
 & & \text{id} & & \\
 & & \curvearrowleft & &
 \end{array} \quad (W2)$$

Pseudodistributive laws in monoidal form

Definition

$$\begin{array}{ccccc}
 T^3P & \xrightarrow{T^2\lambda} & T^2PT & \xrightarrow{T\lambda T} & TP T^2 & \xrightarrow{\lambda T^2} & PT^3 & & (W3) \\
 \downarrow mTP & & \downarrow TmP & \searrow T\omega_3 & \downarrow TPm & & \downarrow PTm & & \\
 T^2P & & T^2P & \xrightarrow{T\lambda} & TPT & \xrightarrow{\lambda T} & PT^2 & & \\
 \downarrow mP & & \downarrow mP & \searrow \omega_3 & & & \downarrow Pm & & \\
 TP & & TP & \xrightarrow{\lambda} & PT & & PT & &
 \end{array}$$

=

$$\begin{array}{ccccccc}
 T^3P & \xrightarrow{T^2\lambda} & T^2PT & \xrightarrow{T\lambda T} & TP T^2 & \xrightarrow{\lambda T^2} & PT^3 \\
 \downarrow mTP & & \downarrow mPT & \searrow \omega_3 T & \downarrow PmT & & \downarrow PTm \\
 T^2P & \xrightarrow{T\lambda} & TPT & \xrightarrow{\lambda T} & PT^2 & & PT^2 \\
 \downarrow mP & & \downarrow mP & \searrow \omega_3 & \downarrow Pm & & \downarrow Pm \\
 TP & \xrightarrow{\lambda} & TP & \xrightarrow{\lambda} & PT & & PT
 \end{array}$$

Pseudodistributive laws in monoidal form

Definition

$$\begin{array}{ccccc}
 TP^3 & \xrightarrow{\lambda P^2} & PTP^2 & \xrightarrow{P\lambda P} & P^2TP & \xrightarrow{P^2\lambda} & P^3T \\
 \swarrow TP\mu & & \downarrow T\mu P & \nearrow \omega_4 P & \downarrow \mu TP & & \downarrow \mu PT \\
 TP^2 & \xrightarrow{\lambda P} & PTP & \xrightarrow{P\lambda} & P^2T & & \\
 \swarrow T\mu & & \downarrow T\mu & \nearrow \omega_4 & \downarrow \mu T & & \\
 TP & \xrightarrow{\lambda} & PT & & & &
 \end{array}
 \tag{W4}$$

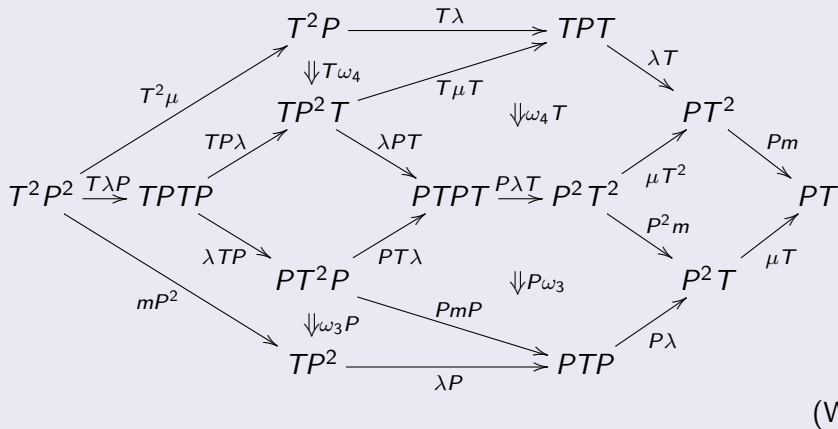
=

$$\begin{array}{ccccc}
 TP^3 & \xrightarrow{\lambda P^2} & PTP^2 & \xrightarrow{P\lambda P} & P^2TP & \xrightarrow{P^2\lambda} & P^3T \\
 \downarrow TP\mu & & \downarrow PT\mu & \nearrow P\omega_4 & \downarrow P\mu T & & \searrow \mu PT \\
 TP^2 & \xrightarrow{\lambda P} & PTP & \xrightarrow{P\lambda} & P^2T & & P^2T \\
 \downarrow T\mu & & \nearrow \omega_4 & & \downarrow \mu T & & \swarrow \mu T \\
 TP & \xrightarrow{\lambda} & PT & & PT & &
 \end{array}$$

Pseudodistributive laws in monoidal form

Definition

The last axiom ensures that the pentagons ω_3 and ω_4 are compatible



is equal to

Pseudodistributive laws in monoidal form

Definition

is equal to

$$\begin{array}{ccccc}
 & & T^2P & \xrightarrow{T\lambda} & TP & \xrightarrow{\lambda T} & PT^2 & & \\
 & & \nearrow^{T^2\mu} & \searrow^{mP} & \Downarrow \omega_3 & \searrow^{Pm} & & & \\
 T^2P^2 & & & & TP & \xrightarrow{\lambda} & PT & & \\
 & & \searrow^{mP^2} & \nearrow^{T\mu} & \Downarrow \omega_4 & \nearrow^{\mu T} & & & \\
 & & TP^2 & \xrightarrow{\lambda P} & PTP & \xrightarrow{P\lambda} & P^2T & &
 \end{array}$$

Pseudodistributive laws in monoidal form

Theorem

A pseudodistributive law $(\lambda, \omega_1, \omega_2, \omega_3, \omega_4) : TP \rightarrow PT$ is equivalently:

- a pseudomonad morphism $(\lambda, \omega_1, \omega_3) : T \rightarrow T$ along P
- a pseudomonad opmorphism $(\lambda, \omega_2, \omega_4) : P \rightarrow P$ along T

such that ω_3 and ω_4 satisfy axiom (W5).

Pseudodistributive laws in extensive form

Definition

A *pseudodistributive law (in extensive form)* between pseudomonads (with their modification data suppressed) (T, u, m) and (P, η, μ) is a pseudonatural transformation $\lambda: TP \rightarrow PT$ and three invertible modifications as below

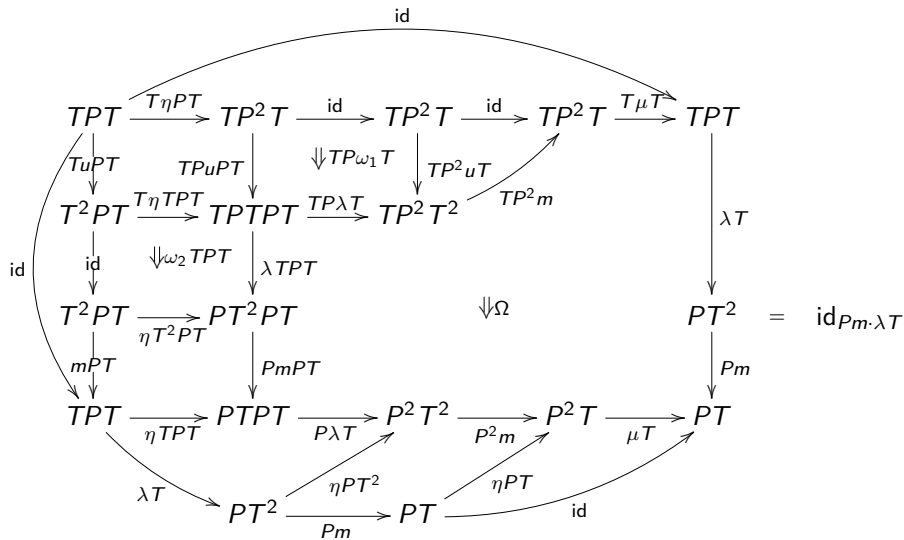
$$\begin{array}{ccc}
 TP & \xrightarrow{\lambda} & PT \\
 \swarrow uP & \omega_1 & \nearrow Pu \\
 & P &
 \end{array}$$

$$\begin{array}{ccc}
 TP & \xrightarrow{\lambda} & PT \\
 \swarrow T\eta & \omega_2 & \nearrow \eta T \\
 & T &
 \end{array}$$

$$\begin{array}{ccccccc}
 TPTPT & \xrightarrow{TP\lambda T} & TP^2T^2 & \xrightarrow{TP^2m} & TP^2T & \xrightarrow{T\mu T} & TPT \\
 \lambda TPT \downarrow & & & & & & \downarrow \lambda T \\
 PT^2PT & & & \Downarrow \Omega & & & PT^2 \\
 PmPT \downarrow & & & & & & \downarrow Pm \\
 PTPT & \xrightarrow{P\lambda T} & P^2T^2 & \xrightarrow{P^2m} & P^2T & \xrightarrow{\mu T} & PT
 \end{array}$$

Pseudodistributive laws in extensive form

satisfying the following two coherence axioms: (D1)



Pseudodistributive laws in extensive form

and

$$\begin{array}{ccccccc}
 & TPTP\lambda T & TPTP^2m & TPT\mu T & & & \\
 TPTPTPT & \longrightarrow & TPTP^2T^2 & \longrightarrow & TPTP^2T & \longrightarrow & TPTPT \\
 \downarrow \lambda TPTPT & & \downarrow \lambda TP^2T^2 & & \downarrow \lambda TP^2T & & \downarrow \lambda TPT \\
 P^2TP\lambda T & & P^2TP^2m & & P^2T\mu T & & TP^2T^2 \\
 PT^2PTPT & \longrightarrow & PT^2P^2T^2 & \longrightarrow & PT^2P^2T & \longrightarrow & PT^2PT \\
 \downarrow PmPTPT & & \downarrow PmP^2T^2 & & \downarrow PmP^2T & & \downarrow PmPT \\
 PTP\lambda T & & PTP^2m & & PT\mu T & & TP^2T^2 \\
 PTPTPT & \longrightarrow & PTP^2T^2 & \longrightarrow & PTP^2T & \longrightarrow & PTPT \\
 \downarrow P\lambda TPT & & \downarrow P\lambda T & & \downarrow P\lambda T & & \downarrow P\lambda T \\
 P^2T^2PT & & P^2T^2 & & P^2T^2 & & TP^2T \\
 \downarrow P^2mPT & & \downarrow P^2m & & \downarrow P^2m & & \downarrow T\mu T \\
 P^2TPT & \longrightarrow & P^3T^2 & \longrightarrow & P^3T & \longrightarrow & P^2T \\
 \downarrow \mu TPT & & \downarrow \mu PT^2 & & \downarrow \mu PT & & \downarrow Pm \\
 PTPT & \longrightarrow & P^2T^2 & \longrightarrow & P^2T & \longrightarrow & PT \\
 & & \downarrow P\lambda T & & \downarrow P^2m & & \downarrow \mu T \\
 & & P^2T & & P^2T & & PT
 \end{array}$$

(D2)

Pseudodistributive laws in extensive form

is equal to

$$\begin{array}{ccccccc}
 & & TPTP\lambda T & & TPTP^2m & & TPT\mu T \\
 & & \longrightarrow & & \longrightarrow & & \longrightarrow \\
 TPTPTPT & \longrightarrow & TPTP^2T^2 & \longrightarrow & TPTP^2T & \longrightarrow & TPTPT \\
 \downarrow \lambda TPTPT & \searrow TP\lambda TPT & & & & & \searrow TP\lambda T \\
 PT^2PTPT & \longrightarrow & TP^2T^2PT & & & & TP^2T^2 \\
 \downarrow PmPTPT & & \searrow TP^2mPT & & \downarrow TP\Omega & & \searrow TP^2m \\
 PTPPTPT & & & & TP^2TPT & \xrightarrow{TP^2\lambda T} & TP^3T^2 & \xrightarrow{TP^3m} & TP^3T & \xrightarrow{TP\mu T} & TP^2T \\
 \downarrow P\lambda TPT & & \downarrow \Omega PT & & T\mu TPT & \downarrow & T\mu PT^2 & \downarrow & T\mu PT & \downarrow & T\mu T \\
 P^2T^2PT & & & & TPTPT & \xrightarrow{TP\lambda T} & TP^2T^2 & \xrightarrow{TP^2m} & TP^2T & \xrightarrow{T\mu T} & TPT \\
 \searrow P^2mPT & & & & \downarrow \lambda TPT & & \downarrow \Omega & & \downarrow \lambda T & & \downarrow \lambda T \\
 & & & & P^2TPT & & & & P^2T & & PT^2 \\
 & & & & \downarrow PmPT & & & & \downarrow Pm & & \downarrow Pm \\
 & & & & PTPT & \xrightarrow{P\lambda T} & P^2T^2 & \xrightarrow{P^2m} & P^2T & \xrightarrow{\mu T} & PT \\
 & & & & \downarrow \mu TPT & & & & & & \\
 & & & & & & & & & &
 \end{array}$$

(D2)

Pseudodistributive laws

In order to justify these two definitions of pseudodistributive laws, we will prove the following theorem.

Theorem

Given two pseudomonads (with their modification data suppressed) (T, u, m) and (P, η, μ) , the following are equivalent:

- 1 *a pseudodistributive law $\lambda: TP \rightarrow PT$ in monoidal form;*
- 2 *a pseudodistributive law $\lambda: TP \rightarrow PT$ in extensive form;*
- 3 *an extension of (T, u, m) to a pseudomonad on the Kleisli bicategory of (P, η, μ) .*

Pseudodistributive laws

We first show that:

Lemma

For a given λ , the data $(\omega_1, \omega_2, \omega_3, \omega_4)$ with axioms $W1$ and $W2$ is in bijection with the data $(\omega_1, \omega_2, \Omega)$ with axiom $D1$.

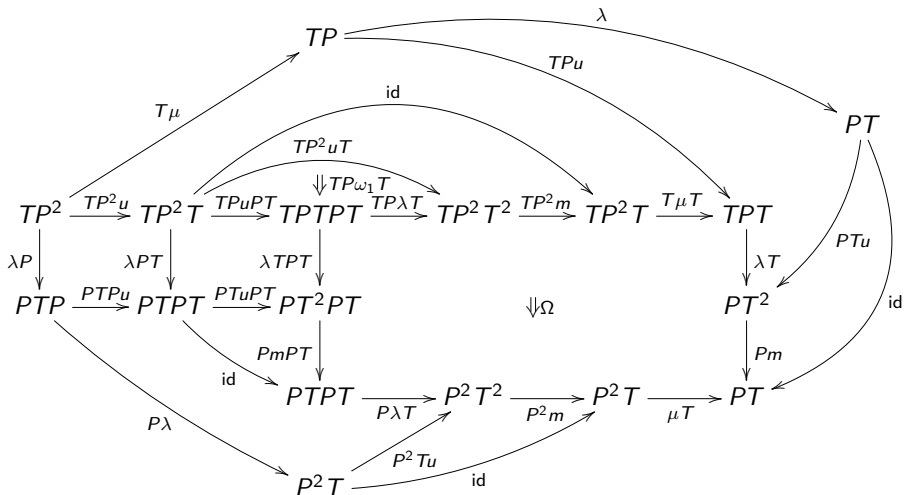
Pseudodistributive laws

From the pentagons, the decagon Ω is constructed as the coherence diagram

$$\begin{array}{ccccccc}
 TPTPT & \xrightarrow{TP\lambda T} & TP^2T^2 & \xrightarrow{TP^2m} & TP^2T & \xrightarrow{T\mu T} & TPT \\
 \downarrow \lambda TPT & & \downarrow \lambda PT^2 & & \downarrow \lambda PT & & \downarrow \lambda T \\
 PT^2PT & \xrightarrow{PT\lambda T} & PTP T^2 & \xrightarrow{PTPm} & PTPT & \Downarrow \omega_4 T & \\
 \downarrow PmPT & & \downarrow P\lambda T^2 & & \downarrow P\lambda T & & \downarrow \lambda T \\
 & & \Downarrow P\omega_3 T & & P^2T^3 & \xrightarrow{P^2Tm} & P^2T^2 & \xrightarrow{\mu T^2} & PT^2 \\
 & & \downarrow P^2mT & & \downarrow P^2m & & \downarrow Pm & & \\
 PTPT & \xrightarrow{P\lambda T} & P^2T^2 & \xrightarrow{P^2m} & P^2T & \xrightarrow{\mu T} & PT
 \end{array}$$

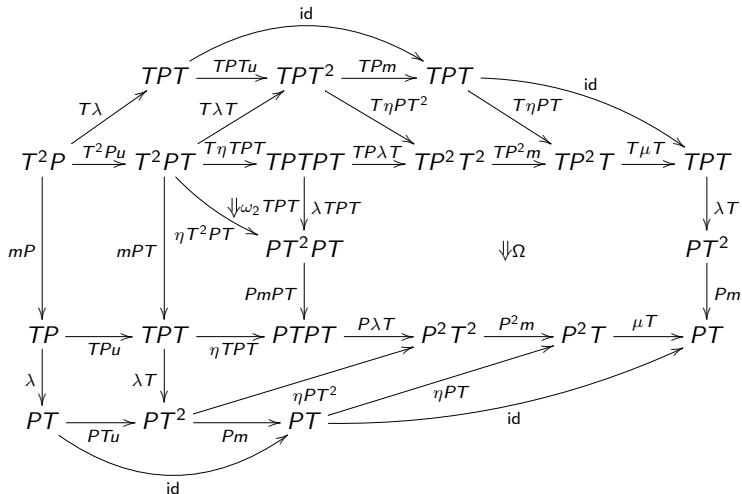
Pseudodistributive laws

Conversely, given the decagon one recovers the pentagon ω_4 as



Pseudodistributive laws

and the pentagon ω_3 as



Pseudodistributive laws

It is not hard to check that this is a bijection & the axioms $W1$, $W2$ correspond to $D1$.

Note that this shows our two definitions of pseudo-distributive laws are equivalent - as Axiom $D2$ involves only decagons (thus only pentagons) - and thus follows from the “pentagon only” axioms $W3$, $W4$, and $W5$.

Conversely, the axioms $W3$, $W4$, and $W5$ follow from restricting $D2$ along the units of the monads.

It just remains to check that we get a pseudomonad on the Kleisli bicategory...

This works since axioms $D1$ and $D2$ are precisely the needed axioms to define a pseudomonad in extensive form!

Redundant axioms

Recall the redundant axioms:

$$\begin{array}{ccc}
 u_X \xrightarrow{\phi_{u_X}} u_X^T u_X & (u_Y^T f)^T \xrightarrow{\delta_{u_Y, f}} u_Y^T f^T & g^T f \xrightarrow{\phi_{g^T f}} (g^T f)^T u_X \\
 \downarrow \text{id} & \downarrow \theta_X u_X & \downarrow \theta_Y f^T \\
 u_X & u_X & u_Y^T f^T \\
 & & \downarrow (\theta_Y f)^T \\
 & & f^T
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & g^T f \xrightarrow{\phi_{g^T f}} (g^T f)^T u_X \\
 & & \downarrow \delta_{g, f} u_X \\
 & & g^T f^T u_X \\
 & & \downarrow g^T \phi_f \\
 & & g^T f^T u_X
 \end{array}$$

The first which says a ϕ (constructed from ω_1) followed by a θ (constructed from ω_2) is the identity, is equivalent to the condition that

$$\begin{array}{ccc}
 1 & \xrightarrow{\eta} & P \\
 & & \searrow uP \\
 & & TP \xrightarrow{\lambda} PT \\
 & & \downarrow \omega_1 \\
 & & TP \\
 & & \downarrow \omega_2 \\
 & & PT \\
 1 & \xrightarrow{u} & T \\
 & & \nearrow T\eta \\
 & & TP \\
 & & \nearrow \eta T \\
 & & PT
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 1 & \xrightarrow{\eta} & P \\
 & & \searrow Pu \\
 & & PT \\
 1 & \xrightarrow{u} & T \\
 & & \nearrow \eta T \\
 & & PT
 \end{array}$$

Redundant axioms

Now to check

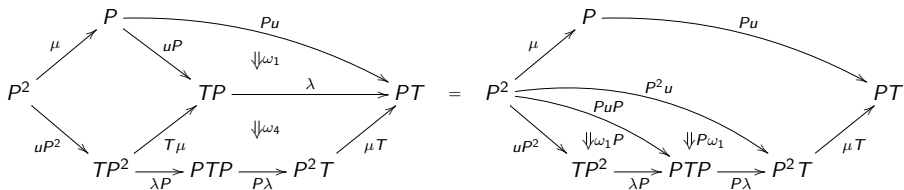
$$\begin{array}{c}
 \begin{array}{ccccc}
 & T^2P & \xrightarrow{T\lambda} & TPT & \xrightarrow{\lambda T} & PT^2 \\
 T^2 \nearrow^{T^2\eta} & & & & & \\
 & & mP & & & Pm \\
 & & \searrow & & \searrow & \\
 & TP & \xrightarrow{\lambda} & PT & & \\
 & \downarrow \omega_3 & & \downarrow \omega_2 & & \\
 & & & & & \\
 T^2 \searrow^m & & T \nearrow^{T\eta} & & & \eta T \\
 & & \searrow & & \searrow & \\
 & & & & & PT
 \end{array}
 & = &
 \begin{array}{ccccc}
 & T^2P & \xrightarrow{T\lambda} & TPT & \xrightarrow{\lambda T} & PT^2 \\
 T^2 \nearrow^{T^2\eta} & & & & & \\
 & & \downarrow T\omega_2 & & \downarrow \omega_2 T & \\
 & & T\eta T & & \eta T^2 & \\
 & & \searrow & & \searrow & \\
 & TP & \xrightarrow{\lambda} & PT & & \\
 & \downarrow \omega_3 & & \downarrow \omega_2 & & \\
 & & & & & \\
 T^2 \searrow^m & & T \nearrow^{T\eta} & & & \eta T \\
 & & \searrow & & \searrow & \\
 & & & & & PT
 \end{array}
 \end{array}$$

Substitute the definition of the pentagon ω_3 in terms of the decagon Ω , then use (noting θ is made from ω_2 and δ from the decagon Ω)

$$\begin{array}{ccc}
 (u_Y^T f)^T & \xrightarrow{\delta_{u_Y, f}} & u_Y^T f^T \\
 & \searrow & \downarrow \theta_Y f^T \\
 & & f^T \\
 (\theta_Y f)^T & \searrow & \\
 & & f^T
 \end{array}$$

Redundant axioms

Dually



follows from the axiom

$$\begin{array}{ccc}
 g^T f & \xrightarrow{\phi_{g^T f}} & (g^T f)^T u_X \\
 & \searrow^{g^T \phi_f} & \downarrow \delta_{g, f} u_X \\
 & & g^T f^T u_X
 \end{array}$$

Minimal definitions

What's the point of distributive laws?

To provide a minimal description of the data and coherence axioms needed to compose monads

So what's the minimal definition??? It depends on your definition of minimal!

- 1 Breaking down into irreducible lowest order components OR
- 2 Total number of sides & coherence axioms.

Minimal definitions

Beck's definition of a distributive law has 16 sides & 4 coherence axioms

The (likely) minimal definition has 12 non-identity sides & 3 coherence axioms!

How do we find it? Combine the earlier decagon conditions with the following result of Marmolejo, Wood and Rosebrugh.

Minimal definitions

Fact (Marmolejo, Wood and Rosebrugh)

Distributive laws $\lambda: TP \rightarrow PT$ of monads (T, u, m) and (P, η, μ) are in bijection with T -algebras $\alpha: TPT \rightarrow PT$ rendering commutative the three diagrams

$$\begin{array}{ccc}
 TPT^2 & \xrightarrow{TPm} & TPT \\
 \alpha T \downarrow & & \downarrow \alpha \\
 PT^2 & \xrightarrow{Pm} & PT
 \end{array}
 \qquad
 \begin{array}{ccc}
 T^2 & \xrightarrow{T\eta T} & TPT \\
 m \downarrow & & \downarrow \alpha \\
 T & \xrightarrow{\eta T} & PT
 \end{array}$$

$$\begin{array}{ccccccc}
 TP^2T & \xrightarrow{TPuPT} & TPTPT & \xrightarrow{\alpha PT} & PTPT & \xrightarrow{P\alpha} & P^2T \\
 T\mu T \downarrow & & & & & & \downarrow \mu T \\
 TPT & \xrightarrow{\hspace{10em}} & & & & & PT \\
 & & \alpha & & & &
 \end{array}$$

Minimal definitions

Theorem

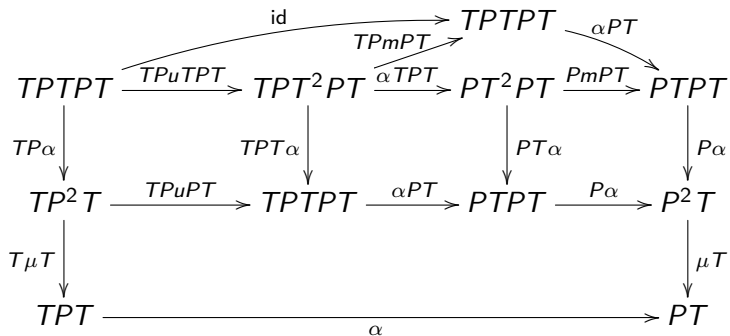
Distributive laws $\lambda: TP \rightarrow PT$ of monads (T, u, m) and (P, η, μ) are in bijection with natural transformations $\alpha: TPT \rightarrow PT$ rendering commutative the three diagrams

$$\begin{array}{ccc}
 TPT & \xrightarrow{\alpha} & PT \\
 \uparrow u_{PT} & \nearrow \text{id} & \\
 PT & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 T^2 & \xrightarrow{T\eta T} & TPT \\
 \downarrow m & & \downarrow \alpha \\
 T & \xrightarrow{\eta T} & PT
 \end{array}$$

$$\begin{array}{ccccc}
 TPTPT & \xrightarrow{TP\alpha} & TP^2T & \xrightarrow{T\mu T} & TPT \\
 \downarrow \alpha_{PT} & & & & \downarrow \alpha \\
 PTPT & \xrightarrow{P\alpha} & P^2T & \xrightarrow{\mu T} & PT
 \end{array}$$

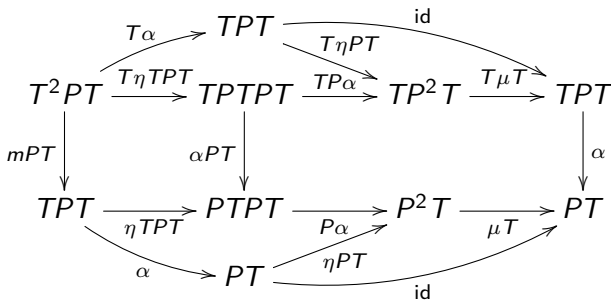
Minimal definitions

Proof is just to show their set of 5 axioms is equivalent to my set of 3 axioms. The (5 axioms) \Rightarrow (3 axioms) direction is just the diagram



Minimal definitions

Conversely, we get the multiplicative algebra axiom as the coherence diagram



just using the square & hexagon axioms.

Minimal definitions

The “op-homomorphism” axiom as

$$\begin{array}{ccccc}
 & & TPm & \rightarrow & TPT & \xrightarrow{\text{id}} & TPT \\
 & & & & \searrow TP\eta T & & \\
 TPT^2 & \xrightarrow{TP\eta T} & TPTPT & \xrightarrow{TP\alpha} & TP^2T & \xrightarrow{T\mu T} & TPT \\
 \downarrow \alpha T & & \downarrow \alpha PT & & & & \downarrow \alpha \\
 PT^2 & \xrightarrow{PT\eta T} & PTPT & \xrightarrow{P\alpha} & P^2T & \xrightarrow{\mu T} & PT \\
 & & Pm & \rightarrow & PT & \xrightarrow{\text{id}} & PT \\
 & & & & \nearrow P\eta T & &
 \end{array}$$

using the square axiom twice & the hexagon

Minimal definitions

and finally the “transformed pentagon” from the triangle axiom

$$\begin{array}{ccccccc}
 & & \text{id} & & & & \\
 & & \curvearrowright & & & & \\
 TP^2T & \xrightarrow{TP\mu PT} & TPTPT & \xrightarrow{TP\alpha} & TP^2T & \xrightarrow{T\mu T} & TPT \\
 & & \downarrow \alpha PT & & & & \downarrow \alpha \\
 & & PTPT & \xrightarrow{P\alpha} & P^2T & \xrightarrow{\mu T} & PT
 \end{array}$$

Note there is also a dual version of this, based on “opalgebras” instead of algebras. Instead of α one would use a morphism

$$PTP \xrightarrow{\text{res}_{Ty}} PT$$

This is the P -embedding structure on Ty .

Minimal definitions in two dimensions

Theorem

Pseudodistributive laws $\lambda: TP \rightarrow PT$ of pseudomonads (T, u, m) and (P, η, μ) are in equivalence with pseudonatural transformations $\alpha: TPT \rightarrow PT$ equipped with three invertible modifications

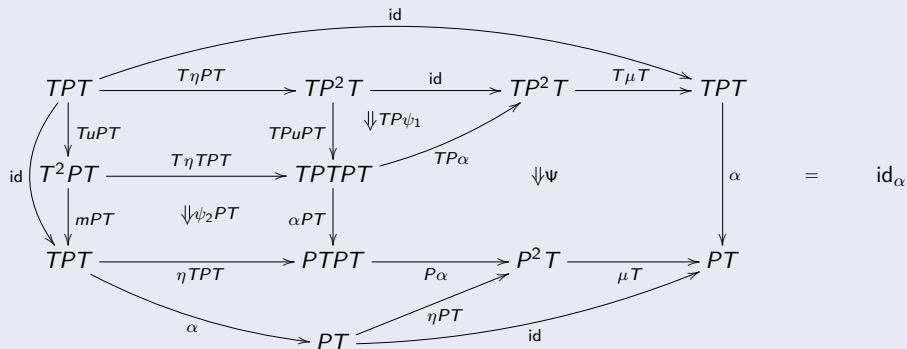
$$\begin{array}{ccc}
 TPT & \xrightarrow{\alpha} & PT \\
 \uparrow u_{PT} & \nearrow \psi_1 & \nearrow \text{id} \\
 PT & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 T^2 & \xrightarrow{T\eta T} & TPT \\
 \downarrow m & \searrow \psi_2 & \downarrow \alpha \\
 T & \xrightarrow{\eta T} & PT
 \end{array}$$

$$\begin{array}{ccc}
 TPTPT & \xrightarrow{TP\alpha} & TP^2T & \xrightarrow{T\mu T} & TPT \\
 \downarrow \alpha_{PT} & & \Downarrow \Psi & & \downarrow \alpha \\
 PTPT & \xrightarrow{P\alpha} & P^2T & \xrightarrow{\mu T} & PT
 \end{array}$$

Minimal definitions in two dimensions

Theorem

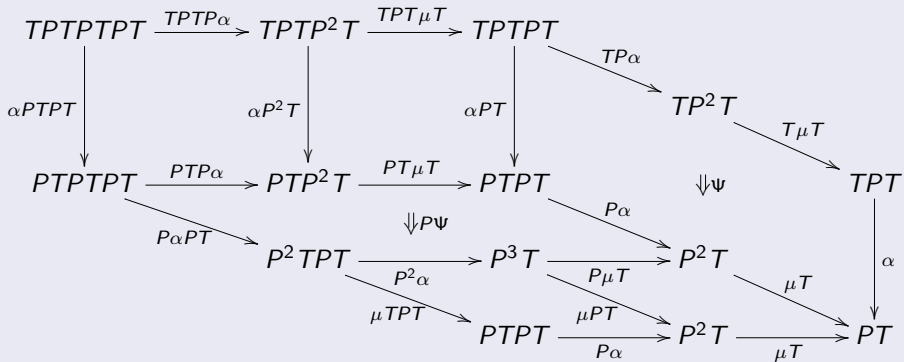
satisfying the two coherence axioms



Minimal definitions in two dimensions

Theorem

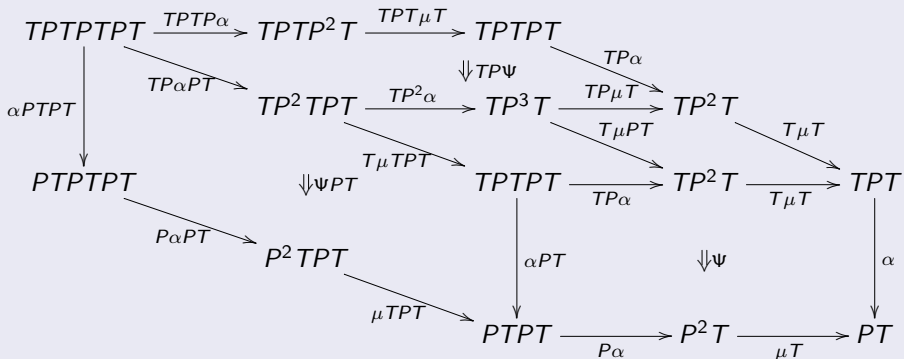
and



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Theorem

is equal to



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We prove that the canonical assignment

$$(\lambda, \omega_1, \omega_2, \Omega) \mapsto (\alpha, \psi_1, \psi_2, \Psi)$$

defines an equivalence. For later proving essential surjectivity we will need

$$\mathcal{H} := \begin{array}{ccccccc} & & TPm & \rightarrow & TPT & \xrightarrow{TP\eta T} & TPT & \xrightarrow{id} & TPT \\ & & & & \downarrow TP\psi_2^{-1} & & & & \\ TPT^2 & \xrightarrow{TPT\eta T} & TPTPT & \xrightarrow{TP\alpha} & TP^2T & \xrightarrow{T\mu T} & TPT & & \\ \alpha T \downarrow & & \alpha PT \downarrow & & & & \downarrow \alpha & & \\ PT^2 & \xrightarrow{PT\eta T} & PTPT & \xrightarrow{P\alpha} & P^2T & \xrightarrow{\mu T} & PT & & \\ & & Pm & \rightarrow & PT & \xrightarrow{P\eta T} & PT & \xrightarrow{id} & PT \end{array}$$

which is in a sense the “op-homomorphism” data for α .

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We first derive the usual three redundant axioms

$$\begin{array}{ccc}
 & PT & \\
 \eta T \nearrow & & \searrow id \\
 T & & TPT \\
 uT \searrow & & \nearrow \alpha \\
 & T^2 & \rightarrow T \\
 & \nearrow T\eta T & \\
 & \xrightarrow{m} &
 \end{array}
 \quad \Downarrow \psi_1
 \quad \Downarrow \psi_2
 \quad =
 \quad
 \begin{array}{ccc}
 & PT & \\
 \eta T \nearrow & & \searrow id \\
 T & & PT \\
 uT \searrow & & \nearrow id \\
 & T^2 & \rightarrow T \\
 & \xrightarrow{m} &
 \end{array}$$

$$\begin{array}{ccc}
 PTPT & \xrightarrow{P\alpha} & P^2T & \xrightarrow{\mu T} & PT \\
 \downarrow uPTPT & & \downarrow uP^2T & & \downarrow uPT \\
 TPTPT & \xrightarrow{TP\alpha} & TP^2T & \xrightarrow{T\mu T} & TPT \xrightarrow{id} PT \\
 \downarrow \alpha PT & & \downarrow \Psi & & \downarrow \alpha \\
 PTPT & \xrightarrow{P\alpha} & P^2T & \xrightarrow{\mu T} & PT
 \end{array}
 \quad \Downarrow \psi_1
 \quad =
 \quad
 \begin{array}{ccc}
 PTPT & \xrightarrow{P\alpha} & P^2T & \xrightarrow{\mu T} & PT \\
 \downarrow uPTPT & & \downarrow uP^2T & & \downarrow uPT \\
 TPTPT & \xrightarrow{\psi_1 P T} & TP^2T & \xrightarrow{id} & TPT \xrightarrow{id} PT \\
 \downarrow \alpha PT & & \downarrow \alpha & & \downarrow \alpha \\
 PTPT & \xrightarrow{P\alpha} & P^2T & \xrightarrow{\mu T} & PT
 \end{array}$$

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and

$$\begin{array}{c}
 \begin{array}{ccccccc}
 TPT^2 & \xrightarrow{TP\eta T} & TPTPT & \xrightarrow{TP\alpha} & TP^2T & \xrightarrow{T\mu T} & TPT \\
 \downarrow \alpha T & & \downarrow \alpha PT & & \Downarrow \psi & & \downarrow \alpha \\
 PT^2 & \xrightarrow{PT\eta T} & PTPT & \xrightarrow{P\alpha} & P^2T & \xrightarrow{\mu T} & PT \\
 & \searrow P_m & \Downarrow P\psi_2 & \nearrow P\eta T & & \nearrow \text{id} & \\
 & & PT & & & &
 \end{array} \\
 = \\
 \begin{array}{ccccccc}
 TPT^2 & \xrightarrow{TP\eta T} & TPTPT & \xrightarrow{TP\alpha} & TP^2T & \xrightarrow{T\mu T} & TPT \\
 \downarrow \alpha T & & \Downarrow TP\psi_2 & & & & \downarrow \alpha \\
 PT^2 & \xrightarrow{PT\eta T} & TPT & \xrightarrow{TP\eta T} & TP^2T & \xrightarrow{\mu T} & PT \\
 & \searrow P_m & \Downarrow \mathcal{H} & \nearrow \text{id} & & \nearrow \text{id} & \\
 & & PT & & & &
 \end{array}
 \end{array}$$

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These five axioms are then used to show (α, \mathcal{H}) is an “op-homomorphism”

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 TPT^3 & \xrightarrow{TPmT} & TPT^2 & & \\
 \alpha T^2 \downarrow & \searrow \mathcal{H}T & \downarrow \alpha T & \searrow TPm & \\
 PT^3 & \xrightarrow{PmT} & PT^2 & \searrow \mathcal{H} & TPT \\
 & \searrow PTm & \searrow Pm & \searrow \alpha & \\
 & & PT^2 & \xrightarrow{Pm} & PT
 \end{array} & = &
 \begin{array}{ccccc}
 TPT^3 & \xrightarrow{TPmT} & TPT^2 & & \\
 \alpha T^2 \downarrow & \searrow TPTm & & \searrow TPm & \\
 PT^3 & & TPT^2 & \xrightarrow{TPm} & TPT \\
 & \searrow PTm & \downarrow \alpha T & \searrow \mathcal{H} & \downarrow \alpha \\
 & & PT^2 & \xrightarrow{Pm} & PT
 \end{array}
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 TPT & & & & \\
 \alpha \downarrow & \searrow TPTu & \searrow id & & \\
 PT & & TPT^2 & \xrightarrow{TPm} & TPT \\
 & \searrow PTu & \downarrow \alpha T & \searrow \mathcal{H} & \downarrow \alpha \\
 & & PT^2 & \xrightarrow{Pm} & PT
 \end{array} & = &
 \begin{array}{ccccc}
 TPT & & & & \\
 \alpha \downarrow & & & \searrow id & \\
 PT & & & & TPT \\
 & \searrow PTu & & \searrow id & \downarrow \alpha \\
 & & PT^2 & \xrightarrow{Pm} & PT
 \end{array}
 \end{array}$$

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and that the modifications are “op-homomorphisms”, meaning they satisfy

$$\begin{array}{ccc}
 PT^2 & \xrightarrow{\text{id}} & PT^2 \xrightarrow{Pm} PT \\
 \searrow uPT^2 & \Downarrow \psi_1 T & \searrow \alpha T \\
 & & TPT^2 \xrightarrow{TPm} TPT \\
 & & \Downarrow \mathcal{H}^{-1} \\
 & & TPT^2 \xrightarrow{TPm} TPT
 \end{array}
 =
 \begin{array}{ccc}
 PT^2 & \xrightarrow{Pm} & PT \xrightarrow{\text{id}} PT \\
 \searrow uPT^2 & \Downarrow \psi_1 & \searrow \alpha \\
 & & TPT^2 \xrightarrow{TPm} TPT
 \end{array}$$

$$\begin{array}{ccc}
 T^3 & \xrightarrow{T\eta T^2} & TPPT^2 \\
 \downarrow mT & \swarrow \psi_2 T & \downarrow \alpha T \\
 T^2 & \xrightarrow{\eta T^2} & PT^2 \\
 \searrow m & & \searrow \mathcal{H} \\
 & & TPT \\
 & & \downarrow \alpha \\
 & & PT
 \end{array}
 =
 \begin{array}{ccc}
 T^3 & \xrightarrow{T\eta T^2} & TPPT^2 \\
 \downarrow mT & \searrow Tm & \downarrow m \\
 T^2 & \xrightarrow{\eta T} & TPT \\
 \searrow m & & \searrow \psi_2 \\
 & & T \\
 & & \downarrow \alpha \\
 & & PT
 \end{array}$$

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and






$$\begin{array}{c}
 TPTPT^2 \xrightarrow{TP\alpha T} TP^2T^2 \xrightarrow{T\mu T^2} TPT^2 \\
 \downarrow \alpha PT^2 \quad \swarrow \psi T \quad \downarrow \alpha T \quad \searrow TPm \\
 PTPT^2 \xrightarrow{P\alpha T} P^2T^2 \xrightarrow{\mu T^2} PT^2 \quad \swarrow \mathcal{H} \quad \searrow Pm \\
 \downarrow PTPm \quad \swarrow P\mathcal{H}^{-1} \quad \downarrow P\alpha \quad \searrow \alpha \\
 PTPT \xrightarrow{P\alpha} P^2T \xrightarrow{\mu T} PT
 \end{array}
 =
 \begin{array}{c}
 TPTPT^2 \xrightarrow{TP\alpha T} TP^2T^2 \xrightarrow{T\mu T^2} TPT^2 \\
 \downarrow \alpha PT^2 \quad \swarrow TPTPm \quad \swarrow TP\mathcal{H}^{-1} \quad \searrow TP^2m \quad \searrow TPm \\
 PTPT^2 \xrightarrow{TP\alpha} TP^2T \xrightarrow{T\mu T} TPT \\
 \downarrow PTPm \quad \downarrow \alpha PT \quad \swarrow \psi \quad \downarrow \alpha \\
 PTPT \xrightarrow{P\alpha} P^2T \xrightarrow{\mu T} PT
 \end{array}$$








These are precisely the axioms needed to prove fully faithfulness.






Future work

Future work includes:

- (1) Marmolejo gave the definition of a distributive law in terms of extension operators $(-)^{\lambda}$. However, finding the definition of a pseudodistributive law in terms of extension operations $(-)^{\lambda}$ has not been practical until now (due to the coherence conditions) - the above “minimal definition” should simplify the calculations significantly.
- (2) Find analogous results for mixed pseudodistributive laws. (Will likely be more complicated).
- (3) To understand how this ties in with KZ case - in which one arrives at a different minimal set of axioms (when they simplify the coherence axioms using the mates correspondence).

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The End

Thank you!