Distributive laws, pseudodistributive laws and decagons

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$$\begin{array}{ccccccc} TPTPT & \xrightarrow{TP\lambda T} & TP^2T^2 & \xrightarrow{TP^2m} & TP^2T & \xrightarrow{T\mu T} & TPT \\ {\scriptstyle \lambda TPT}\downarrow & & & & & & \downarrow{\scriptstyle \lambda T} \\ PT^2PT & & & & & & PT^2 \\ {\scriptstyle PmPT}\downarrow & & & & & & \downarrow{\scriptstyle Pm} \\ PTPT & \xrightarrow[P\lambda T]{} & P^2T^2 & \xrightarrow[P^2m]{} & P^2T & \xrightarrow[\mu T]{} & PT \end{array}$$

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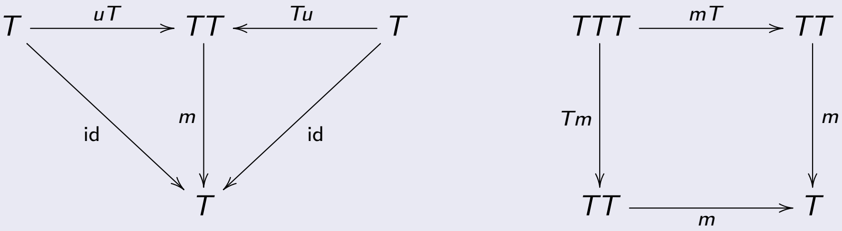
Outline

1. Background knowledge
   - Monads in monoidal and extensive form
   - Distributive laws of monads and Beck's theorem
   - Pseudomonads in monoidal and extensive form
   - Pseudodistributive laws of pseudomonads (until now)
2. Distributive laws of monads in extensive form
   - Extending to the Kleisli category extensively
   - The five coherence axioms of a pseudodistributive law
   - The three redundant coherence axioms
3. Minimal definitions
   - Minimal definition of distributive laws
   - Minimal definition of pseudodistributive laws
   - Future work

Monads in monoidal form

Definition

A monad (in monoidal form) on a category $\mathscr{C}$ consists of a functor $T\colon \mathscr{C} \to \mathscr{C}$ and a unit $u\colon 1 \to T$ and a multiplication $m\colon TT \to T$ rendering commutative



Monads in extensive form

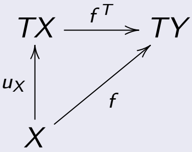
Definition (Walters, Manes)

A *monad (in extensive form)* on a category $\mathscr{C}$ consists of

- an assignation on objects $X \mapsto TX$;
- for each $X \in \mathscr{C}$, a $u_X \colon X \to TX$;
- for each $f \colon X \to TY$, a map $f^T \colon TX \to TY$;

such that:

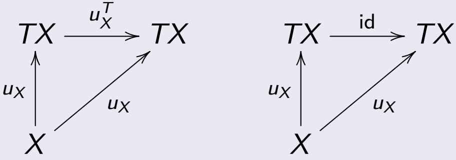
- for each $f \colon X \to TY$, we have $f^T \cdot u_X = f$ as below



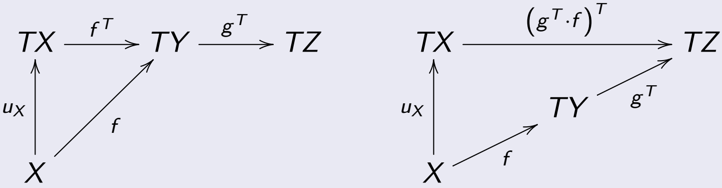
Monads in extensive form

Definition (Walters, Manes)

- for each $X \in \mathscr{C}$, we have $u_X^T = \mathrm{id}$ as below



- for each $f\colon X \to TY$ and $g\colon Y \to TZ$, we have $\left(g^T \cdot f\right)^T = g^T \cdot f^T$



Monads in extensive form

For a $f\colon X \to TY$ the induced $f^T\colon TX \to TY$ is defined as

$$TX \xrightarrow{Tf} T^2Y \xrightarrow{mY} TY$$

Conversely, $m\colon T^2 \to T$ can be recovered by “extending” the identity

$$\begin{array}{ccc} T^2X & \xrightarrow{mX} & TX \\ {\scriptstyle u_{TX}}\uparrow & \nearrow{\scriptstyle \mathrm{id}} & \\ TX & & \end{array}$$

Theorem (Walters, Manes)

This defines a bijection between monads in monoidal and extensive form.

Distributive laws in monoidal form

Definition (Beck)

A *distributive law (in monoidal form)* between monads (T, u, m) and (P, η, μ) is a natural transformation $\lambda \colon TP \to PT$ rendering commutative the four diagrams below

$$\begin{array}{ccc} TP & \xrightarrow{\quad\lambda\quad} & PT \\ & {\scriptstyle uP}\nwarrow \qquad \nearrow{\scriptstyle Pu} & \\ & P & \end{array} \qquad\qquad \begin{array}{ccc} TP & \xrightarrow{\quad\lambda\quad} & PT \\ & {\scriptstyle T\eta}\nwarrow \qquad \nearrow{\scriptstyle \eta T} & \\ & T & \end{array}$$

$$\begin{array}{ccc} T^2P \xrightarrow{T\lambda} TPT & \xrightarrow{\lambda T} & PTT \\ {\scriptstyle mP}\downarrow & & \downarrow{\scriptstyle Pm} \\ TP & \xrightarrow[\lambda]{\qquad\qquad} & PT \end{array} \qquad\qquad \begin{array}{ccc} TP^2 \xrightarrow{\lambda P} PTP & \xrightarrow{P\lambda} & P^2T \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu T} \\ TP & \xrightarrow[\lambda]{\qquad\qquad} & PT \end{array}$$

Distributive laws in monoidal form

For example, let T be the monad for monoids and P the monad for abelian groups. The usual distributive law $\lambda\colon TP \to PT$ takes products of sums to sums of products, e.g.

$$(a+b)(c+d) \mapsto ac+ad+bc+bd$$

Theorem (Beck)

Given monads (T, u, m) and (P, η, μ), the following are in bijection:

1. *distributive laws $\lambda\colon TP \to PT$*
2. *liftings of the monad P to the category of T-algebras, $\mathbf{Alg}(T)$;*
3. *extensions of the monad T to the Kleisli category of P, $\mathbf{Kl}(P)$;*
4. *monad structures on PT which are suitably compatible with the monads T and P.*

In this talk we are only interested in extensions to the Kleisli category.

Kleisli category

Definition

Given a monad (P, η, μ) on a category $\mathscr{C}$, the Kleisli category of P, denoted $\mathbf{Kl}(P)$, is the category as follows

- objects are those of $\mathscr{C}$;
- a "Kleisli morphism" $X \rightsquigarrow Y$ is a morphism $X \to PY$ in $\mathscr{C}$;
- the composite of a morphism $f \colon X \to PY$ and $g \colon Y \to PZ$ is the composite

$$X \xrightarrow{f} PY \xrightarrow{Pg} P^2 Z \xrightarrow{\mu Z} PZ$$

Extensions to the Kleisli category

Definition

Given monads (T, u, m) and (P, η, μ) on a category $\mathscr{C}$. An extension of a monad (T, u, m) to the Kleisli category $\mathbf{Kl}(P)$ is a monad $\left(\widetilde{T}, \widetilde{u}, \widetilde{m}\right)$ on $\mathbf{Kl}(P)$ such that

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{T} & \mathscr{C} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbf{Kl}(P) & \xrightarrow[\widetilde{T}]{} & \mathbf{Kl}(P) \end{array}$$

commutes (where F is the free functor) written $\widetilde{T} \cdot F = F \cdot T$; and moreover $\widetilde{u} \cdot F = F \cdot u$ and $\widetilde{m} \cdot F = F \cdot m$.

Extensions to the Kleisli category

Given a λ, we can define $\widetilde{T}$ by sending an $f\colon X \to PY$ to

$$TX \xrightarrow{f} TPY \xrightarrow{\lambda Y} PTY$$

If one works out what is required to ensure $\lambda\colon TP \to PT$ gives an extension to $\mathbf{Kl}(P)$ with $\left(\widetilde{T}, \widetilde{u}, \widetilde{m}\right)$ in monoidal form, they naturally arrive at Beck's four coherence axioms.

If one works out what is required to ensure $\lambda\colon TP \to PT$ gives an extension to $\mathbf{Kl}(P)$ with $\left(\widetilde{T}, \widetilde{u}, \widetilde{m}\right)$ in *extensive form*, they arrive at a different set of axioms!
(We will talk about this later on)

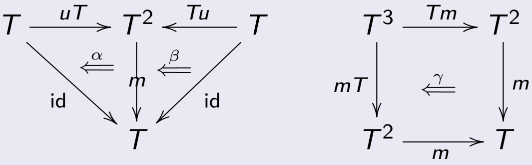
Pseudomonads in monoidal form

Definition

A *pseudomonad (in monoidal form)* on a 2-category $\mathscr{C}$ consists of a pseudofunctor equipped with pseudonatural transformations as below

$$T\colon \mathscr{C} \to \mathscr{C}, \qquad u\colon 1_{\mathscr{C}} \to T, \qquad m\colon T^2 \to T$$

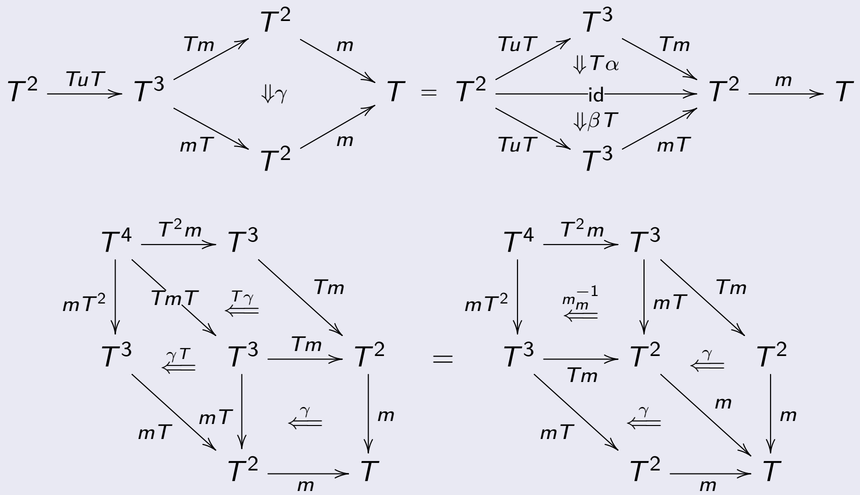
along with three invertible modifications



subject to the two coherence axioms.

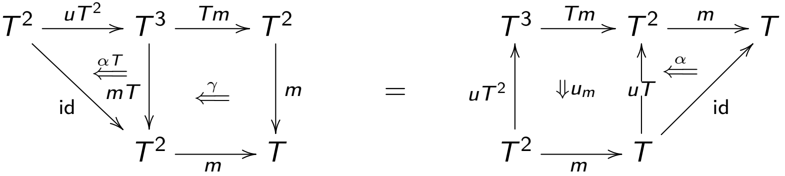
Pseudomonads in monoidal form

Definition

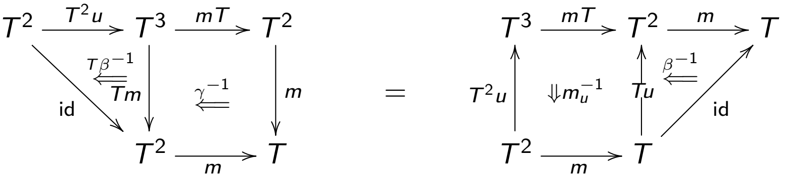


Pseudomonads in monoidal form

By results of Kelly, there are three redundant axioms:

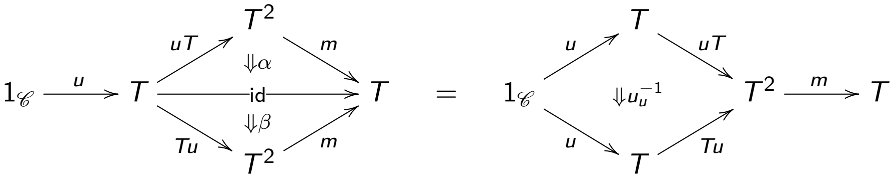


and



Pseudomonads in monoidal form

and finally



Pseudomonads in extensive form

Definition (Marmolejo, Fiore-Gambino-Hyland-Winkskel)

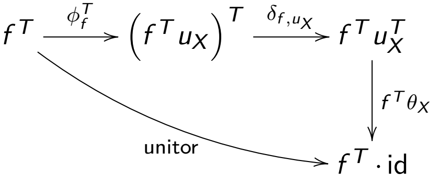
A *pseudomonad (in extensive form)* on a bicategory $\mathscr{C}$ consists of

- an assignation on objects $X \mapsto TX$;
- for each $X \in \mathscr{C}$, a $u_X \colon X \to TX$;
- for each $X, Y \in \mathscr{C}$ a functor $(-)^T_{X,Y} \colon \mathscr{C}(X, TY) \to \mathscr{C}(TX, TY)$;
- for each $f \colon X \to TY$, an isomorphism $\phi_f \colon f \Rightarrow f^T \cdot u_X$ natural in f;
- for each $X \in \mathscr{C}$, an isomorphism $\theta_X \colon u_X^T \Rightarrow \mathrm{id}_{TX}$;
- for each pair $f \colon X \to TY$ and $g \colon Y \to TZ$, an isomorphism $\delta \colon \left(g^T \cdot f\right)^T \Rightarrow g^T \cdot f^T$ natural in f and g;

satisfying the two coherence conditions:

Pseudomonads in extensive form

for each $f\colon X \to TY$ the diagram



commutes;

Pseudomonads in extensive form

for each $f\colon X \to TY$, $g\colon Y \to TZ$, and $h\colon Z \to TV$ the diagram

$$
\begin{array}{ccccc}
 & & \left(\left(h^T g\right)^T f\right)^T & & \\
 & \swarrow{\scriptstyle (\delta_{h,g} f)^T} & & \searrow{\scriptstyle \delta_{h^T g, f}} & \\
\left(\left(h^T g^T\right) f\right)^T & & & & \left(h^T g\right)^T f^T \\
\downarrow{\scriptstyle \text{assoc.}} & & & & \downarrow{\scriptstyle \delta_{h,g} f^T} \\
\left(h^T \left(g^T f\right)\right)^T & & & & \left(h^T g^T\right) f^T \\
\downarrow{\scriptstyle \delta_{h^T, g^T f}} & & & & \downarrow{\scriptstyle \text{assoc.}} \\
h^T \left(g^T f\right)^T & & \xrightarrow[h^T \delta_{g,f}]{} & & h^T \left(g^T f^T\right)
\end{array}
$$

commutes.

Pseudomonads in extensive form

The redundant axioms now become the assertion that for any objects $X, Y, Z \in \mathscr{C}$ and morphisms $f \colon X \to TY$ and $g \colon Y \to TZ$ the diagrams

$$\begin{array}{ccc} u_X & \xrightarrow{\phi_{u_X}} & u_X^T u_X \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \theta_X u_X} \\ & & u_X \end{array} \qquad \begin{array}{ccc} \left(u_Y^T f\right)^T & \xrightarrow{\delta_{u_Y, f}} & u_Y^T f^T \\ & \searrow^{(\theta_Y f)^T} & \downarrow{\scriptstyle \theta_Y f^T} \\ & & f^T \end{array} \qquad \begin{array}{ccc} g^T f & \xrightarrow{\phi_{g^T f}} & \left(g^T f\right)^T u_X \\ & \searrow^{g^T \phi_f} & \downarrow{\scriptstyle \delta_{g,f} u_X} \\ & & g^T f^T u_X \end{array}$$

commute.

History of pseudodistributive laws

Even when dealing with 2-monads, one often needs to use the pseudo version of distributive law.

Example

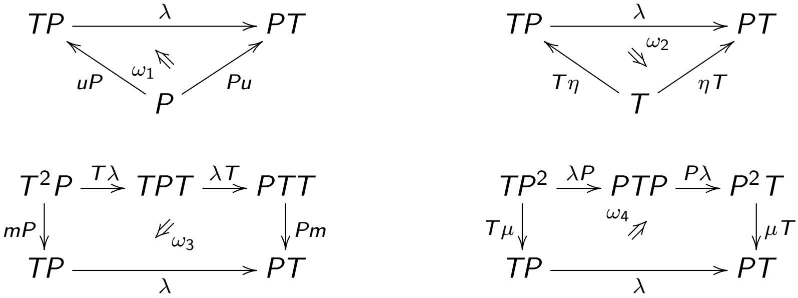
Take:

T to be the 2-monad for small symmetric monoidal categories on **Cat**
P to be the 2-monad for small categories with finite products on **Cat**

There is no natural 2-distributive law $\lambda \colon TP \to PT$ (one of the pentagons doesn't commute), but there is a natural pseudo-distributive law.

History of pseudodistributive laws

Thus one must allow the required diagrams to only commute up to invertible modifications



subject to "some" coherence conditions.

These coherence conditions are the complicated part!

History of pseudodistributive laws

History of the coherence axioms:

- Kelly considered the "mild case" where both triangles and one of the pentagons commuted strictly. Kelly arrived at five coherence axioms with these "semi-strict" distributive laws.
- Marmolejo considered the general case, requiring the four modifications to satisfy nine coherence axioms.
- Tanaka introduced an extra tenth coherence axiom in this thesis.
- Marmolejo and Wood showed that Tanaka's extra axiom, in addition to one of the original nine, are redundant. Thus showing eight coherence axioms suffice.
- Today we show five coherence axioms suffice!

Pseudodistributive laws

Theorem (Marmolejo, Cheng-Hyland-Power, Tanaka)

Given pseudomonads (T, u, m) and (P, η, μ) (suppressing the modifications comprising the pseudomonads), the following are equivalent:

1. *pseudodistributive laws* $(\lambda, \omega_1, \omega_2, \omega_3, \omega_4) : TP \to PT$;
2. *liftings of the pseudomonad P to the 2-category of pseudo T-algebras,* $\mathbf{Ps\text{-}Alg}(T)$;
3. *extensions of the pseudomonad T to the Kleisli bicategory of P,* $\mathbf{Kl}(P)$;
4. *pseudomonad structures on PT which are suitably compatible with the pseudomonads T and P.*

We will only be interested in extensions to the Kleisli bicategory.

Extending extensively

Suppose we are given monads (T, u, m) and (P, η, μ) on a category $\mathscr{C}$. Let's work out what it means to extend (T, u, m) to the Kleisli category

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\;T\;} & \mathscr{C} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbf{Kl}(P) & \xrightarrow[\widetilde{T}]{} & \mathbf{Kl}(P) \end{array}$$

where the extension $\widetilde{T}$ is defined extensively!

Of course we can just ask for a monad $\widetilde{T}$ which is T on objects, has unit constraints $\eta_{TX} \cdot u_X \colon X \to PTX$ and

$$(-)^{\widetilde{T}} F = F(-)^{T} : \mathscr{C}(X, TY) \to \mathbf{Kl}(P)(TX, TY), \qquad \forall X, Y \in \mathscr{C}$$

Here we mean for a given $\lambda \colon TP \to PT$, what conditions will we need?

Extending extensively

Given a $\lambda\colon TP \to TP$, we need to define our monad $\widetilde{T}$ in extensive form. Clearly $\widetilde{T}$ is the same as T on objects.

We define the unit $\widetilde{u_X}\colon X \rightsquigarrow TX$ i.e. $X \to PTX$ as the composite

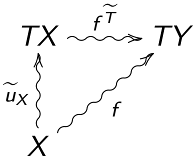
$$X \xrightarrow{uX} TX \xrightarrow{\eta TX} PTX$$

For a general $f\colon X \rightsquigarrow TY$ i.e. $f\colon X \to PTY$, we define our $f^{\widetilde{T}}\colon TX \rightsquigarrow TY$ i.e. $f^{\widetilde{T}}\colon TX \to PTY$ as the composite

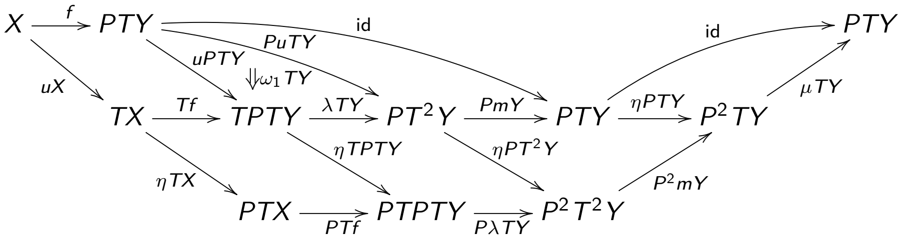
$$TX \xrightarrow{Tf} TPTY \xrightarrow{\lambda TY} PT^2Y \xrightarrow{PmY} PTY$$

Extending extensively

We must first check



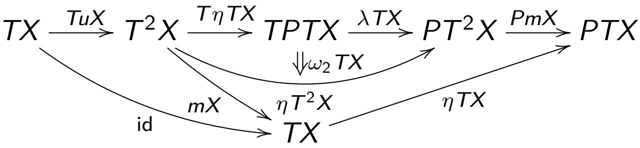
That is, any $f \colon X \to PTY$ is equal to the bottom composite below (ϕ_f in two dimensions)



This is just the unit axiom with Pu and uP.

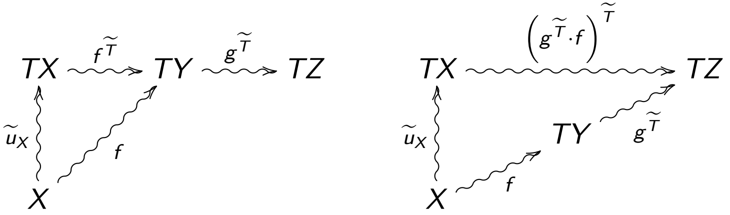
Extending extensively

The axiom that $\widetilde{u}_X^{\widetilde{T}} = \mathrm{id}_{TX}$ amounts to the condition (θ_X in two dimensions)

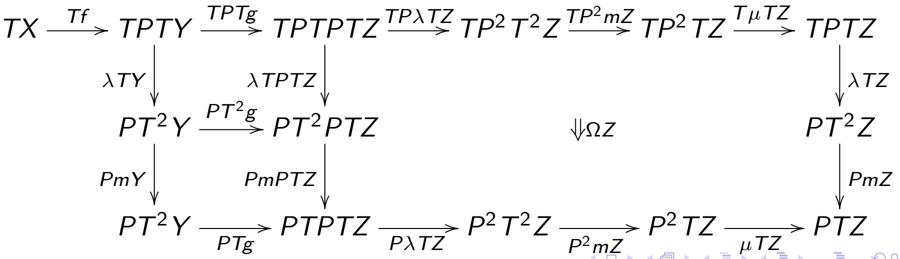


This is just the unit axiom with $T\eta$ and ηT.

Extending extensively



Finally, the above axiom amounts to the condition ($\delta_{g,f}$ in two dimensions)



Distributive laws in extensive form

We thus naturally arrive at the following definition of a distributive law.

Definition

A *distributive law (in extensive form)* between monads (T, u, m) and (P, η, μ) is a natural transformation $\lambda \colon TP \to PT$ rendering commutative the three diagrams below

$$
\begin{array}{ccccc}
TP & & \xrightarrow{\lambda} & & PT \\
& \nwarrow^{uP} & & \nearrow^{Pu} & \\
& & P & &
\end{array}
\qquad
\begin{array}{ccccc}
TP & & \xrightarrow{\lambda} & & PT \\
& \nwarrow^{T\eta} & & \nearrow^{\eta T} & \\
& & T & &
\end{array}
$$

$$
\begin{array}{ccccccc}
TPTPT & \xrightarrow{TP\lambda T} & TP^2T^2 & \xrightarrow{TP^2m} & TP^2T & \xrightarrow{T\mu T} & TPT \\
\downarrow{\scriptstyle \lambda TPT} & & & & & & \downarrow{\scriptstyle \lambda T} \\
PT^2PT & & & & & & PT^2 \\
\downarrow{\scriptstyle PmPT} & & & & & & \downarrow{\scriptstyle Pm} \\
PTPT & \xrightarrow[P\lambda T]{} & P^2T^2 & \xrightarrow[P^2m]{} & P^2T & \xrightarrow[\mu T]{} & PT
\end{array}
$$

Mixed distributive laws in extensive form

This also works for mixed distributive laws!

Definition

A *mixed distributive law (in extensive form)* between a comonad (L, ε, δ) and monad (R, η, μ) is a natural transformation $\lambda \colon LR \to RL$ rendering commutative the three diagrams below

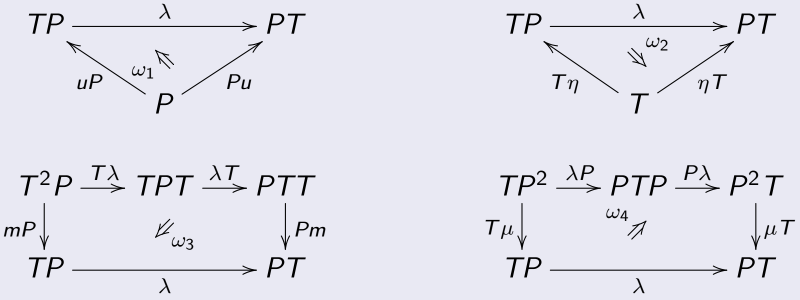
$$\begin{array}{ccc} LR & \xrightarrow{\ \lambda\ } & RL \\ & {\scriptstyle \varepsilon R}\searrow \quad \swarrow{\scriptstyle R\varepsilon} & \\ & R & \end{array} \qquad\qquad \begin{array}{ccc} LR & \xrightarrow{\ \lambda\ } & RL \\ & {\scriptstyle L\eta}\nwarrow \quad \nearrow{\scriptstyle \eta L} & \\ & L & \end{array}$$

$$\begin{array}{ccccccccccc} & & RLR & \xrightarrow{R\delta R} & RL^2R & \xrightarrow{RL\lambda} & RLRL & \xrightarrow{R\lambda L} & R^2L^2 & & \\ & {\scriptstyle \lambda R}\nearrow & & & & & & & & \searrow{\scriptstyle \mu L^2} & \\ LR^2 & & & & & & & & & & RL^2 \\ & {\scriptstyle \delta R^2}\searrow & & & & & & & & \nearrow{\scriptstyle \lambda L} & \\ & & L^2R^2 & \xrightarrow[L\lambda R]{} & LRLR & \xrightarrow[LR\lambda]{} & LR^2L & \xrightarrow[L\mu L]{} & LRL & & \end{array}$$

Pseudodistributive laws in monoidal form

Definition

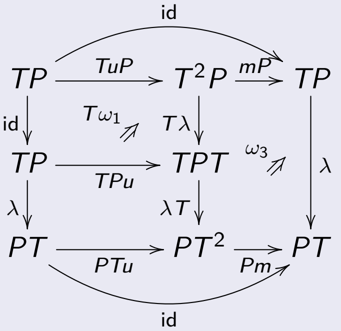
A *pseudodistributive law (in monoidal form)* between pseudomonads (with their modification data suppressed) (T, u, m) and (P, η, μ) is a pseudonatural transformation $\lambda \colon TP \to PT$ and four invertible modifications as below



satisfying the following five coherence axioms:

Pseudodistributive laws in monoidal form

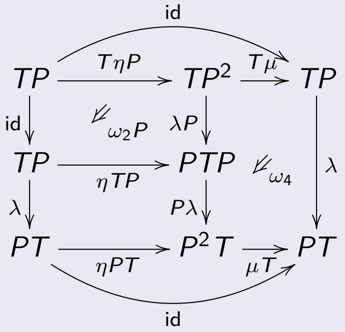
Definition



$$= \quad \mathrm{id}_\lambda \qquad \text{(W1)}$$

Pseudodistributive laws in monoidal form

Definition

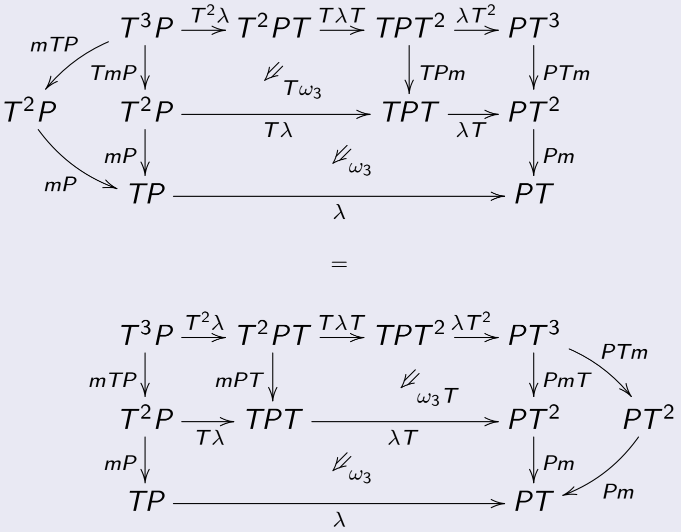


$$= \quad \mathrm{id}_\lambda \qquad \text{(W2)}$$

Pseudodistributive laws in monoidal form

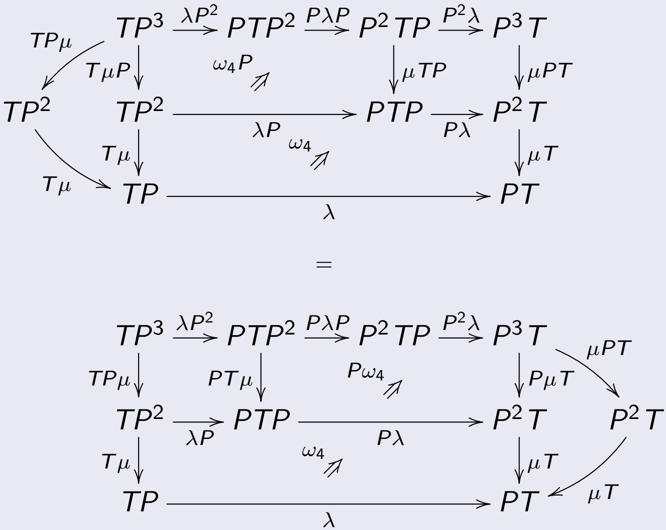
Definition

(W3)



Pseudodistributive laws in monoidal form

Definition

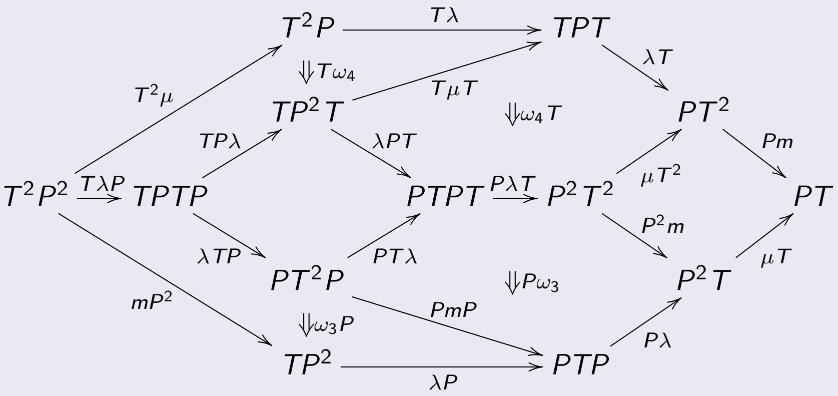


(W4)

Pseudodistributive laws in monoidal form

Definition

The last axiom ensures that the pentagons ω_3 and ω_4 are compatible



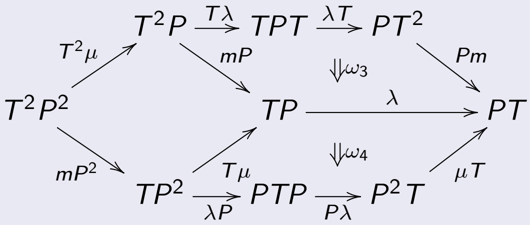
(W5)

is equal to

Pseudodistributive laws in monoidal form

Definition

is equal to



Pseudodistributive laws in monoidal form

Theorem

A pseudodistributive law $(\lambda, \omega_1, \omega_2, \omega_3, \omega_4) : TP \to PT$ *is equivalently:*

- *a pseudomonad morphism* $(\lambda, \omega_1, \omega_3) : T \to T$ *along* P
- *a pseudomonad opmorphism* $(\lambda, \omega_2, \omega_4) : P \to P$ *along* T

such that ω_3 *and* ω_4 *satisfy axiom (W5).*

Pseudodistributive laws in extensive form

Definition

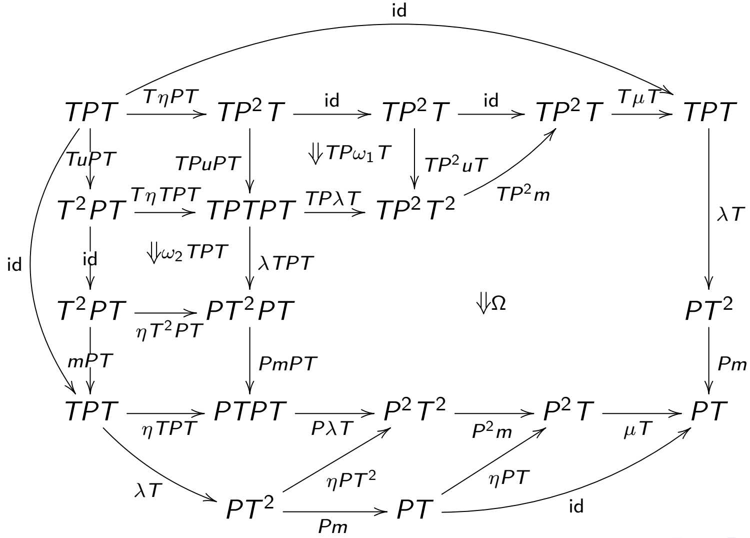
A *pseudodistributive law (in extensive form)* between pseudomonads (with their modification data suppressed) (T, u, m) and (P, η, μ) is a pseudonatural transformation $\lambda \colon TP \to PT$ and three invertible modifications as below

$$\begin{array}{ccccc} TP & & \xrightarrow{\lambda} & & PT \\ & \nwarrow^{uP} & \Leftarrow \omega_1 & {}^{Pu}\nearrow & \\ & & P & & \end{array} \qquad \begin{array}{ccccc} TP & & \xrightarrow{\lambda} & & PT \\ & \nwarrow^{T\eta} & \Downarrow \omega_2 & {}^{\eta T}\nearrow & \\ & & T & & \end{array}$$

$$\begin{array}{ccccccc} TPTPT & \xrightarrow{TP\lambda T} & TP^2T^2 & \xrightarrow{TP^2m} & TP^2T & \xrightarrow{T\mu T} & TPT \\ {\scriptstyle \lambda TPT}\downarrow & & & & & & \downarrow{\scriptstyle \lambda T} \\ PT^2PT & & & \Downarrow\Omega & & & PT^2 \\ {\scriptstyle PmPT}\downarrow & & & & & & \downarrow{\scriptstyle Pm} \\ PTPT & \xrightarrow[P\lambda T]{} & P^2T^2 & \xrightarrow[P^2m]{} & P^2T & \xrightarrow[\mu T]{} & PT \end{array}$$

Pseudodistributive laws in extensive form

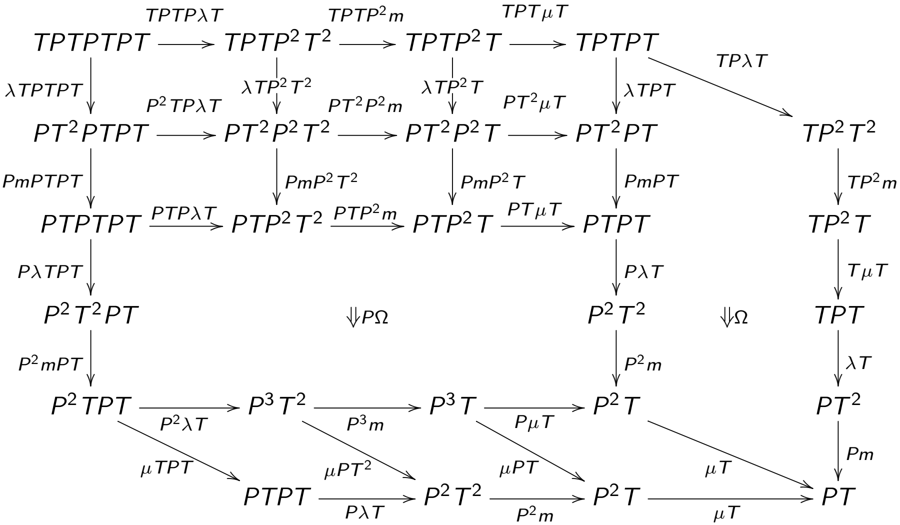
satisfying the following two coherence axioms: (D1)



$$= \quad \mathrm{id}_{Pm\cdot\lambda T} \qquad (1)$$

Pseudodistributive laws in extensive form

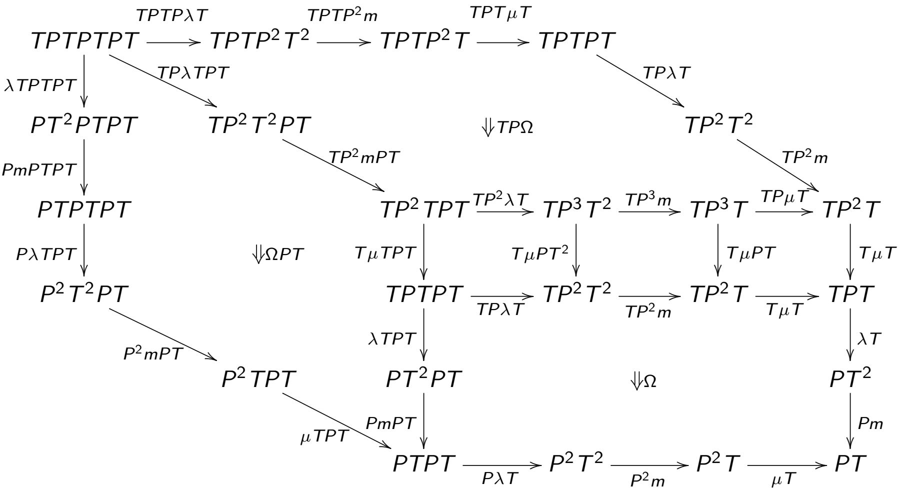
and



(D2)

Pseudodistributive laws in extensive form

is equal to



(D2)

Pseudodistributive laws

In order to justify these two definitions of pseudodistributive laws, we will prove the following theorem.

Theorem

Given two pseudomonads (with their modification data suppressed) (T, u, m) and (P, η, μ), the following are equivalent:

1. *a pseudodistributive law $\lambda \colon TP \to PT$ in monoidal form;*
2. *a pseudodistributive law $\lambda \colon TP \to PT$ in extensive form;*
3. *an extension of (T, u, m) to a pseudomonad on the Kleisli bicategory of (P, η, μ).*

Pseudodistributive laws

We first show that:

Lemma

For a given λ, the data $(\omega_1, \omega_2, \omega_3, \omega_4)$ with axioms W1 and W2 is in bijection with the data $(\omega_1, \omega_2, \Omega)$ with axiom D1.

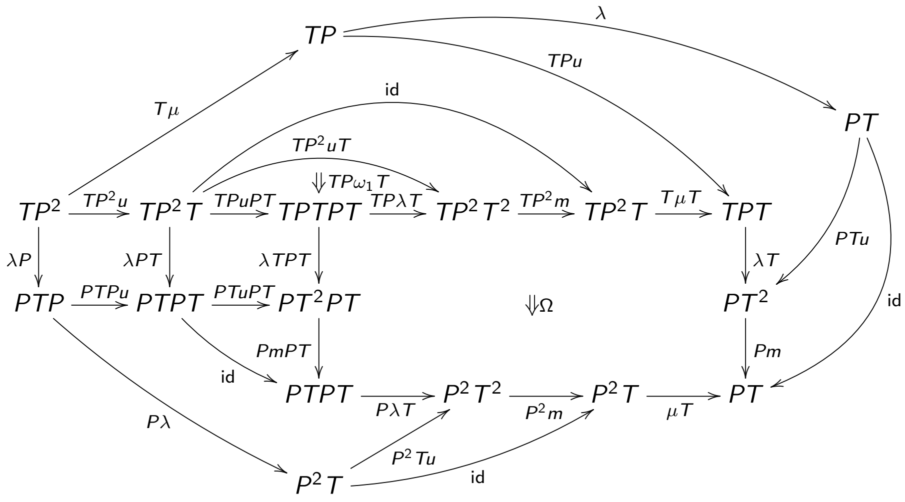
Pseudodistributive laws

From the pentagons, the decagon Ω is constructed as the coherence diagram

$$
\begin{array}{ccccccc}
TPTPT & \xrightarrow{TP\lambda T} & TP^2T^2 & \xrightarrow{TP^2m} & TP^2T & \xrightarrow{T\mu T} & TPT \\
\downarrow{\scriptstyle \lambda TPT} & & \downarrow{\scriptstyle \lambda PT^2} & & \downarrow{\scriptstyle \lambda PT} & & \\
PT^2PT & \xrightarrow[PT\lambda T]{} & PTPT^2 & \xrightarrow{PTPm} & PTPT & \Downarrow \omega_4 T & \downarrow{\scriptstyle \lambda T} \\
 & & \downarrow{\scriptstyle P\lambda T^2} & & \downarrow{\scriptstyle P\lambda T} & & \\
\downarrow{\scriptstyle PmPT} & \Downarrow P\omega_3 T & P^2T^3 & \xrightarrow{P^2Tm} & P^2T^2 & \xrightarrow[\mu T^2]{} & PT^2 \\
 & & \downarrow{\scriptstyle P^2mT} & & \downarrow{\scriptstyle P^2m} & & \downarrow{\scriptstyle Pm} \\
PTPT & \xrightarrow[P\lambda T]{} & P^2T^2 & \xrightarrow[P^2m]{} & P^2T & \xrightarrow[\mu T]{} & PT
\end{array}
$$

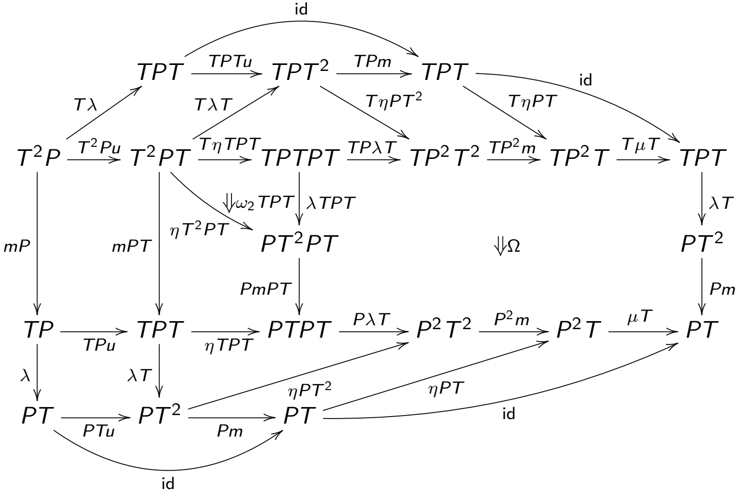
Pseudodistributive laws

Conversely, given the decagon one recovers the pentagon ω_4 as



Pseudodistributive laws

and the pentagon ω_3 as



Pseudodistributive laws

It is not hard to check that this is a bijection & the axioms W1, W2 correspond to D1.

Note that this shows our two definitions of pseudo-distributive laws are equivalent - as Axiom D2 involves only decagons (thus only pentagons) - and thus follows from the "pentagon only" axioms W3, W4, and W5.

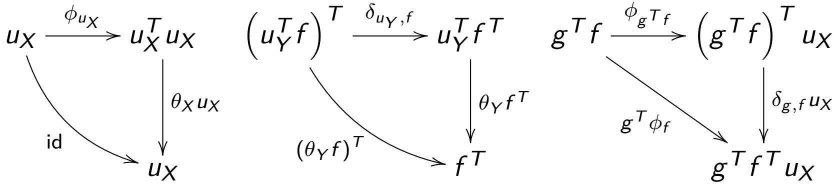
Conversely, the axioms W3, W4, and W5 follow from restricting D2 along the units of the monads.

It just remains to check that we get a pseudomonad on the Kleisli bicategory...

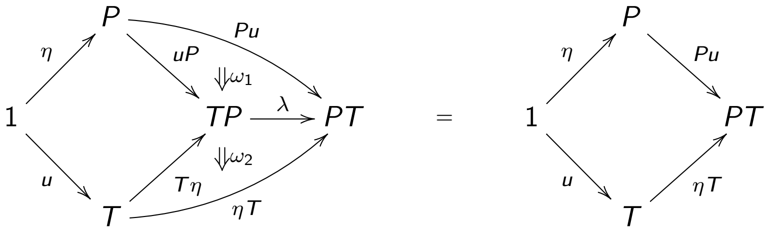
This works since axioms D1 and D2 are precisely the needed axioms to define a pseudomonad in extensive form!

Redundant axioms

Recall the redundant axioms:

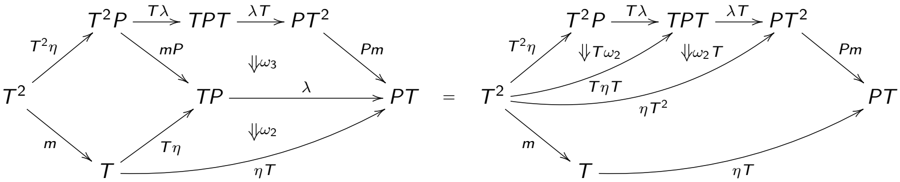


The first which says a ϕ (constructed from ω_1) followed by a θ (constructed from ω_2) is the identity, is equivalent to the condition that

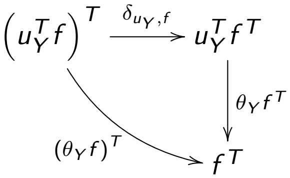


Redundant axioms

Now to check

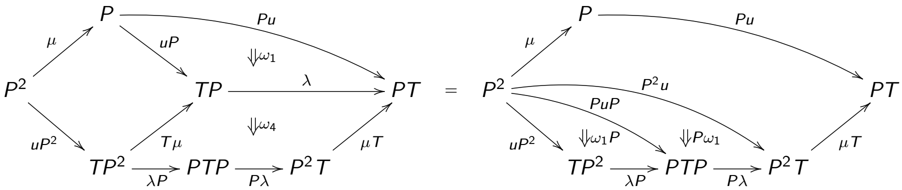


Substitute the definition of the pentagon ω_3 in terms of the decagon Ω, then use (noting θ is made from ω_2 and δ from the decagon Ω)

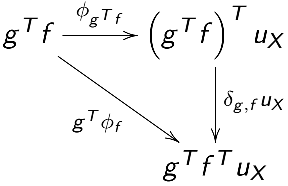


Redundant axioms

Dually



follows from the axiom



Minimal definitions

What's the point of distributive laws?

To provide a minimal description of the data and coherence axioms needed to compose monads

So what's the minimal definition??? It depends on your definition of minimal!

1. Breaking down into irreducible lowest order components OR
2. Total number of sides & coherence axioms.

Minimal definitions

Beck's definition of a distributive law has 16 sides & 4 coherence axioms

The (likely) minimal definition has 12 non-identity sides & 3 coherence axioms!

How do we find it? Combine the earlier decagon conditions with the following result of Marmolejo, Wood and Rosebrugh.

Minimal definitions

Fact (Marmolejo, Wood and Rosebrugh)

Distributive laws $\lambda\colon TP \to PT$ *of monads* (T, u, m) *and* (P, η, μ) *are in bijection with* T*-algebras* $\alpha\colon TPT \to PT$ *rendering commutative the three diagrams*

$$
\begin{array}{ccc}
TPT^2 & \xrightarrow{TPm} & TPT \\
\downarrow{\scriptstyle \alpha T} & & \downarrow{\scriptstyle \alpha} \\
PT^2 & \xrightarrow[Pm]{} & PT
\end{array}
\qquad\qquad
\begin{array}{ccc}
T^2 & \xrightarrow{T\eta T} & TPT \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \alpha} \\
T & \xrightarrow[\eta T]{} & PT
\end{array}
$$

$$
\begin{array}{ccccccc}
TP^2T & \xrightarrow{TPuPT} & TPTPT & \xrightarrow{\alpha PT} & PTPT & \xrightarrow{P\alpha} & P^2T \\
\downarrow{\scriptstyle T\mu T} & & & & & & \downarrow{\scriptstyle \mu T} \\
TPT & & & \xrightarrow[\alpha]{} & & & PT
\end{array}
$$

Minimal definitions

Theorem

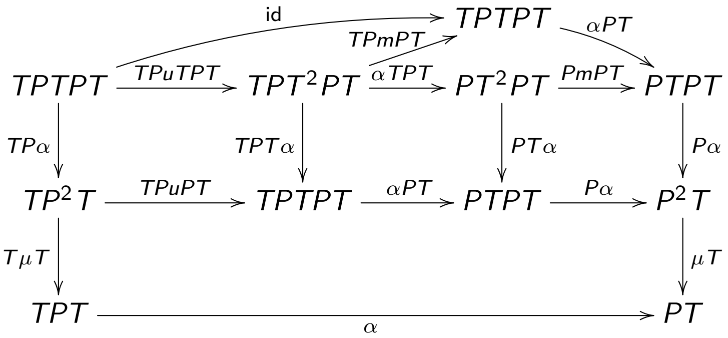
Distributive laws $\lambda\colon TP \to PT$ of monads (T, u, m) and (P, η, μ) are in bijection with natural transformations $\alpha\colon TPT \to PT$ rendering commutative the three diagrams

$$
\begin{array}{ccc}
TPT & \xrightarrow{\alpha} & PT \\
{\scriptstyle uPT}\uparrow & \nearrow{\scriptstyle \mathrm{id}} & \\
PT & &
\end{array}
\qquad
\begin{array}{ccc}
T^2 & \xrightarrow{T\eta T} & TPT \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \alpha} \\
T & \xrightarrow[\eta T]{} & PT
\end{array}
$$

$$
\begin{array}{ccccc}
TPTPT & \xrightarrow{TP\alpha} & TP^2T & \xrightarrow{T\mu T} & TPT \\
{\scriptstyle \alpha PT}\downarrow & & & & \downarrow{\scriptstyle \alpha} \\
PTPT & \xrightarrow[P\alpha]{} & P^2T & \xrightarrow[\mu T]{} & PT
\end{array}
$$

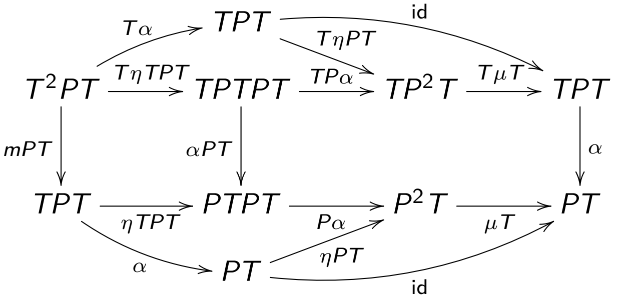
Minimal definitions

Proof is just to show their set of 5 axioms is equivalent to my set of 3 axioms. The (5 axioms) => (3 axioms) direction is just the diagram



Minimal definitions

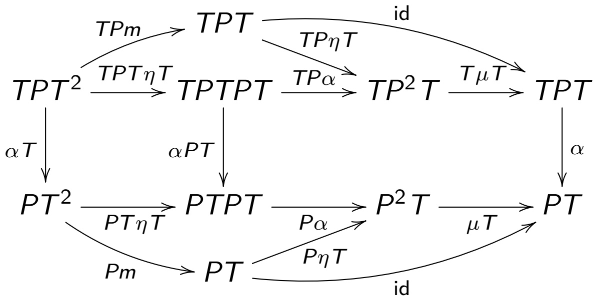
Conversely, we get the multiplicative algebra axiom as the coherence diagram



just using the square & hexagon axioms.

Minimal definitions

The “op-homomorphism” axiom as



using the square axiom twice & the hexagon

Minimal definitions

and finally the "transformed pentagon" from the triangle axiom

$$\begin{array}{ccccccc} TP^2T & \xrightarrow{TPuPT} & TPTPT & \xrightarrow{TP\alpha} & TP^2T & \xrightarrow{T\mu T} & TPT \\ & & \Big\downarrow{\scriptstyle \alpha PT} & & & & \Big\downarrow{\scriptstyle \alpha} \\ & & PTPT & \xrightarrow[P\alpha]{} & P^2T & \xrightarrow[\mu T]{} & PT \end{array}
\qquad \text{with } TP^2T \xrightarrow{\mathrm{id}} TP^2T$$

Note there is a also a dual version of this, based on "opalgebras" instead of algebras. Instead of α one would use a morphism

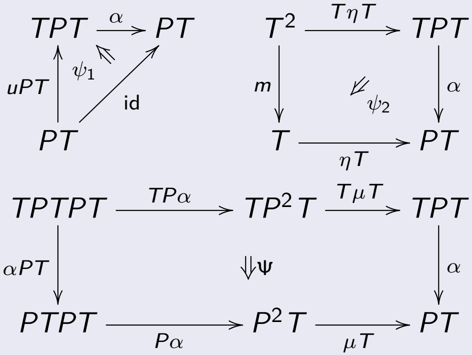
$$PTP \xrightarrow{\mathbf{res}_{Ty}} PT$$

This is the P-embedding structure on Ty.

Minimal definitions in two dimensions

Theorem

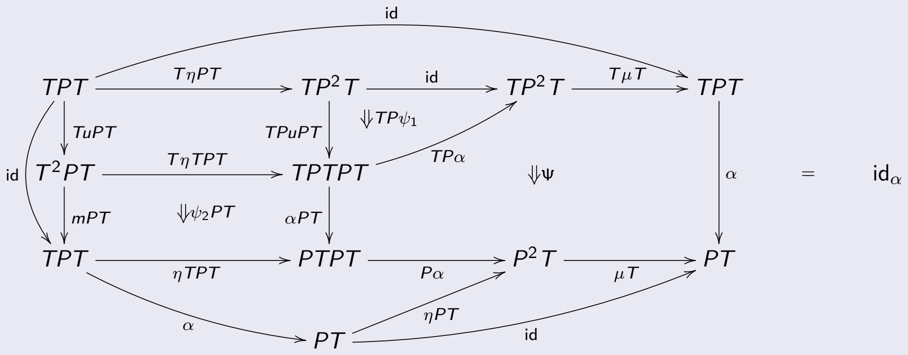
Pseudodistributive laws $\lambda\colon TP \to PT$ *of pseudomonads* (T, u, m) *and* (P, η, μ) *are in equivalence with pseudonatural transformations* $\alpha\colon TPT \to PT$ *equipped with three invertible modifications*



Minimal definitions in two dimensions

Theorem

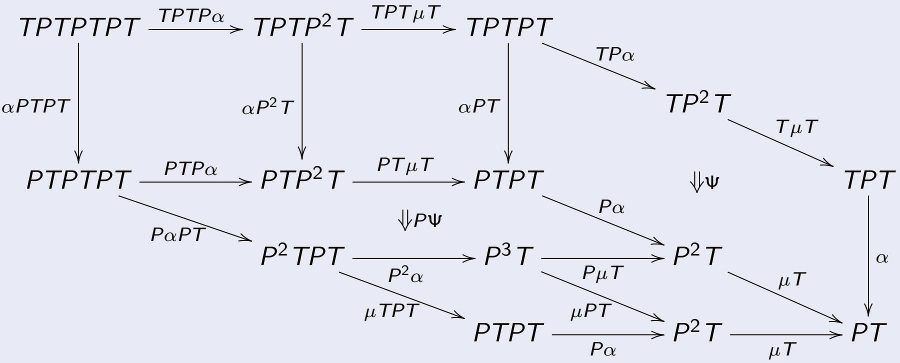
satisfying the two coherence axioms



Minimal definitions in two dimensions

Theorem

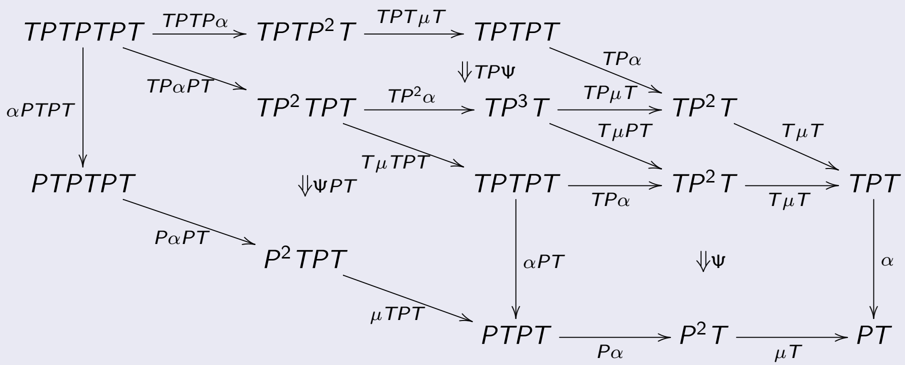
and



Minimal definitions in two dimensions

Theorem

is equal to



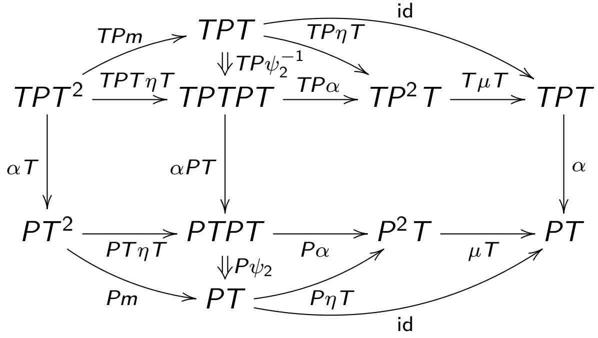
Minimal definitions in two dimensions

We prove that the canonical assignation

$$(\lambda, \omega_1, \omega_2, \Omega) \mapsto (\alpha, \psi_1, \psi_2, \Psi)$$

defines an equivalence. For later proving essential surjectivity we will need

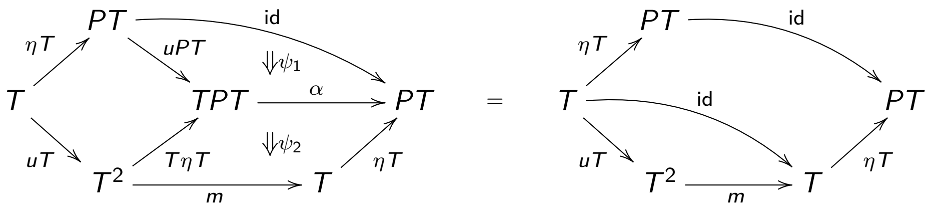
$\mathcal{H} \quad :=$

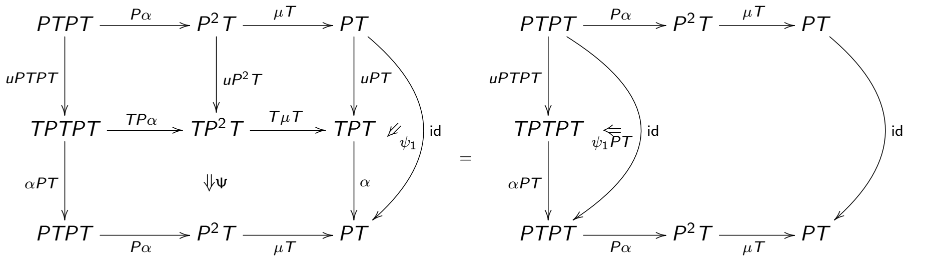


which is in a sense the "op-homomorphism" data for α.

Minimal definitions in two dimensions

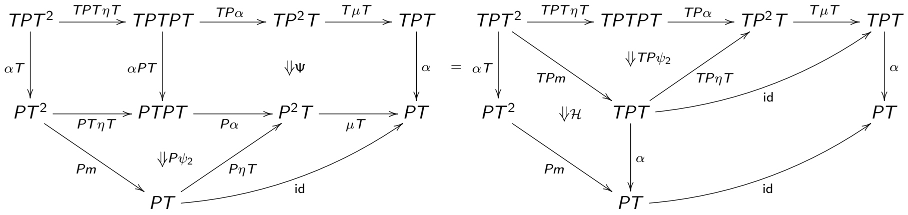
We first derive the usual three redundant axioms





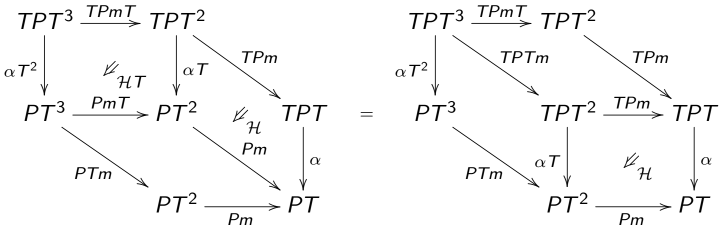
Minimal definitions in two dimensions

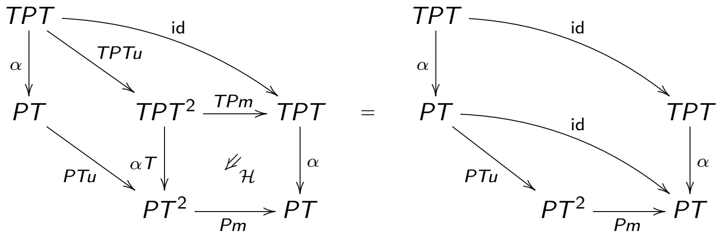
and



Minimal definitions in two dimensions

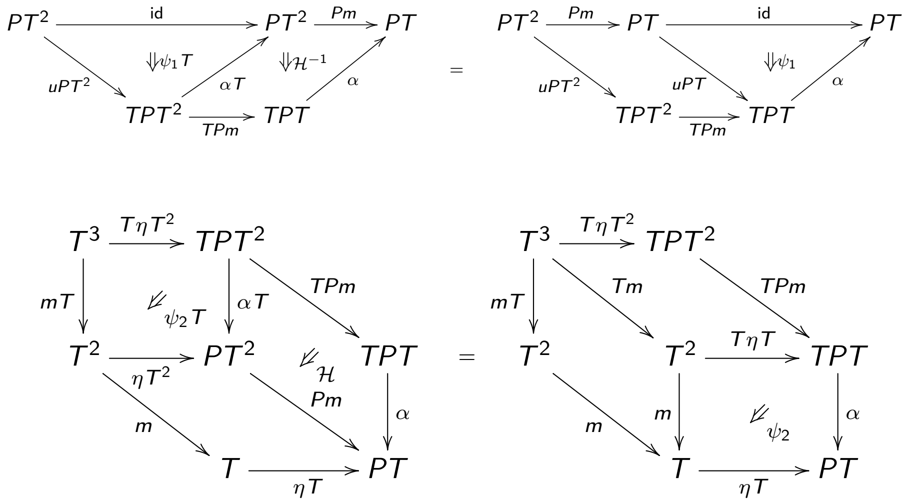
These five axioms are then used to show $(\alpha, \mathcal{H})$ is an "op-homomorphism"





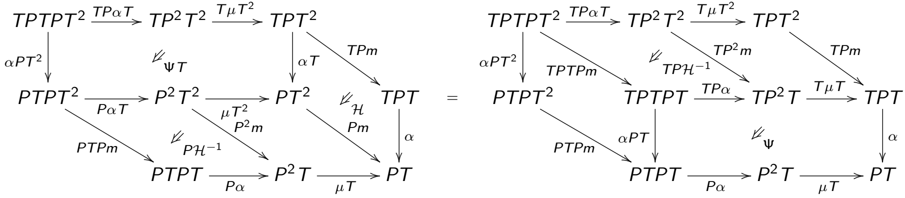
Minimal definitions in two dimensions

and that the modifications are "op-homomorphisms", meaning they satisfy



Minimal definitions in two dimensions

and



These are precisely the axioms needed to prove fully faithfulness.

Future work

Future work includes:

(1) Marmolejo gave the definition of a distributive law in terms of extension operators $(-)^{\lambda}$. However, finding the definition of a pseudodistributive law in terms of extension operations $(-)^{\lambda}$ has not been practical until now (due to the coherence conditions) - the above "minimal definition" should simplify the calculations significantly.

(2) Find analogous results for mixed pseudodistributive laws. (Will likely be more complicated).

(3) To understand how this ties in with KZ case - in which one arrives at a different minimal set of axioms (when they simplify the coherence axioms using the mates correspondence).

J. Beck, *Distributive laws*, in Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), Springer, Berlin, 1969, pp. 119–140.

E. Cheng, M. Hyland, and J. Power, *Pseudo-distributive laws*, Electronic Notes in Theoretical Computer Science, 83 (2003).

M. Fiore, N. Gambino, M. Hyland, and G. Winskel, *Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures*, Selecta Math. (N.S.), 24 (2018), pp. 2791–2830.

N. Gambino and G. Lobbia, *On the formal theory of pseudomonads and pseudodistributive laws*, Theory Appl. Categ., 37 (2021), pp. No. 2, 14–56.

G. M. Kelly, *On MacLane's conditions for coherence of natural associativities, commutativities, etc*, J. Algebra, 1 (1964), pp. 397–402.

- ——, *Coherence theorems for lax algebras and for distributive laws*, (1974), pp. 281–375. Lecture Notes in Math., Vol. 420.
- E. G. Manes, *Algebraic theories*, Springer-Verlag, New York-Heidelberg, 1976.
- F. Marmolejo, *Doctrines whose structure forms a fully faithful adjoint string*, Theory Appl. Categ., 3 (1997), pp. No. 2, 24–44.
- ——, *Distributive laws for pseudomonads*, Theory Appl. Categ., 5 (1999), pp. No. 5, 91–147.
- F. Marmolejo, R. D. Rosebrugh, and R. J. Wood, *A basic distributive law*, J. Pure Appl. Algebra, 168 (2002), pp. 209–226.
- F. Marmolejo and R. J. Wood, *Coherence for pseudodistributive laws revisited*, Theory Appl. Categ., 20 (2008), pp. No. 5, 74–84.
- ——, *Monads as extension systems—no iteration is necessary*, Theory Appl. Categ., 24 (2010), pp. No. 4, 84–113.

- ——, *No-iteration pseudomonads*, Theory Appl. Categ., 28 (2013), pp. No. 14, 371–402.
- R. Street, *The formal theory of monads*, J. Pure Appl. Algebra, 2 (1972), pp. 149–168.
- M. Tanaka, *Pseudo-Distributive Laws and a Unified Framework for Variable Binding*, PhD thesis, University of Edinburgh, 2004.
- C. Walker, *Distributive laws via admissibility*, Appl. Categ. Structures, 27 (2019), pp. 567–617.
- R. F. C. Walters, *A categorical approach to universal algebra*, PhD thesis, Australian National University, 1970.

The End

Thank you!