A topos-theoretic proof of Shelah’s eventual categoricity conjecture

Christian Espíndola

Brno, MUNI

April 30th, 2020
Abstract elementary classes

Definition

An abstract elementary class (AEC) $K$ is a category equivalent to an accessible category with directed colimits whose morphisms are monomorphisms, that admits an embedding $F: K \rightarrow A$ into a finitely accessible category preserving directed colimits and monomorphisms which is, in addition:

1. Full on isomorphisms: for every isomorphism $h: F(A) \rightarrow F(B)$ there is an isomorphism $m: A \rightarrow B$ with $F(m) = h$.

2. Nearly full: for every commutative triangle: $F(A) \rightarrow F(B) \rightarrow F(C)$ with $F(f) \circ F(g)$ there is $m: A \rightarrow C$ with $F(m) = h$. 

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Abstract elementary classes

FACT (Beke-Rosicky 2012): An AEC automatically admits an iso-full, nearly full embedding $E: K \rightarrow \text{Emb}(\Sigma)$ (for some signature $\Sigma$) preserving directed colimits. It also has eventually a Löwenheim-Skolem number $\lambda$: for every substructure $i: A \rightarrow F(B)$ there is $E(f): F(C) \rightarrow F(B)$ such that $i$ factors through $E(f)$ and $|E(C)| \leq |A| + \lambda$.

FACT (Lieberman-Rosicky-Vasey 2019): For each object $A$ of an AEC, $|E(A)|$ coincides with its internal size $|A|$ defined as follows. If $r(A)$ is the least regular cardinal $\lambda$ such that $A$ is $\lambda$-presentable, then:

$$|A| = \begin{cases} \kappa & \text{if } r(A) = \kappa \\ r(A) & \text{if } r(A) \text{ is limit} \end{cases}$$
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$$|A| = \begin{cases} \kappa & \text{if } r(A) = \kappa^+ \\ r(A) & \text{if } r(A) \text{ is limit} \end{cases}$$
Shelah’s conjecture

Conjecture (ZFC) For every AEC there is a cardinal \( \kappa \) such that if the AEC is categorical in some \( \lambda > \kappa \) then it is categorical in every \( \lambda' > \kappa \).

There is a proof of the conjecture (Shelah-Vasey 2019) assuming GCH and large cardinals. In fact, the use of large cardinals is to guarantee that the AEC will eventually satisfy the amalgamation property:

\[ N \xrightarrow{A} M \xrightarrow{B} N' \]

We will give a short topos-theoretic proof of the conjecture assuming GCH and that the AEC satisfies amalgamation.
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The Scott adjunction

Given an accessible category $A$ with $\kappa$-directed colimits, its $\kappa$-Scott topos $S_\kappa(A)$ is the full subcategory of the presheaf $\mathbf{Set}^A$ given by those functors $F: A \to \mathbf{Set}$ preserving $\kappa$-directed colimits.

Given a topos $E$, its category of $\kappa$-points $\text{pt}_\kappa(E)$ (i.e., geometric morphisms to $\mathbf{Set}$ whose inverse images preserve all $\kappa$-small limits) is an accessible category with $\kappa$-directed colimits.

Theorem (Henry-Di Liberti) There is a $(2,1)$-adjunction: $S_\kappa: \text{Acc}^\kappa \to \text{Top}_\kappa: \text{pt}_\kappa$ between the category of accessible categories with $\kappa$-directed colimits and the category of $\kappa$-exact localization of presheaf toposes given by the Scott functor $S_\kappa$ and the functor $\text{pt}_\kappa$. 

Christian Espíndola (Brno, MUNI)
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Theorem

*(Henry-Di Liberti)* There is a (2-)adjunction:

$$S : \text{Acc}_\kappa \rightleftharpoons \text{Top}_\kappa : pt_\kappa$$

between the category of accessible categories with $\kappa$-directed colimits and the category of $\kappa$-exact localization of presheaf toposes given by the Scott functor $S_\kappa$ and the functor $pt_\kappa$. 
Proof idea

Let $K$ be an AEC with amalgamation that is categorical in some successor $\lambda > \kappa$. A model $M$ is $\mu^+\text{-saturated}$ if for every morphism $N \to N'$ between models of size $\mu$, every morphism $N \to M$ can be extended.

Consider the following diagram of toposes and inverse images of geometric morphisms given by restriction:

$$
\begin{array}{ccc}
S \kappa^+ (K \geq \kappa^+) & \sim & S \lambda^+ (K \geq \lambda^+) \\
S \kappa^+ (Sat \kappa^+ (K)) & & S \lambda^+ (Sat \lambda^+ (K)) \\
\end{array}
$$
Proof idea

Let \( \mathcal{K} \) be an AEC with amalgamation that is categorical in some successor \( \lambda > \kappa \).

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A model $M$ is $\mu^+$-saturated if for every morphism $N \to N'$ between models of size $\mu$, every morphism $N \to M$ can be extended:

\[
\begin{array}{ccc}
N' & \xrightarrow{\text{dashed}} & M \\
\uparrow & & \downarrow \\
N & \xrightarrow{} & M
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\begin{array}{ccc}
S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) & \longrightarrow & S_\lambda(\mathcal{K}_{\geq \lambda}) \\
\downarrow & & \downarrow I_R \\
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FACT (Rosicky 1997): $\mathcal{K}_{\geq \lambda}$ coincides with $Sat_\lambda(\mathcal{K})$. Therefore the right morphism is an isomorphism.
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The AEC $\mathcal{K}$, as any accessible category, can be axiomatized up to equivalence by basic sentences through a theory $\mathbb{T}$ in $\mathcal{L}_{\mu^+\mu}$, and can have models not just in $\text{Set}$ but in any $\mu$-coherent category (i.e., in a category with enough $\text{Set}$-like properties).
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The syntactic category $C_\mathbb{T}$ is defined through the following universal property:
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The syntactic category $C_\mathbb{T}$ is defined through the following universal property:

Then $\mathcal{K}_{\geq \kappa^+}$ is axiomatized by basic sentences through $\mathbb{T}_{\kappa^+}$.
Proof idea

The $\kappa^+$-classifying topos of $T\kappa^+$ (Espíndola 2017), $\text{Set}^{[T\kappa^+]}\kappa^+$ is defined through the following universal property: $C_{\text{Set}^{[T\kappa^+]}\kappa^+}$ $E\kappa^+ - \text{small limit preserving}$
Proof idea

The $\kappa^+$-classifying topos of $T_{\kappa^+}$ (Espindola 2017), $\text{Set}[T_{\kappa^+}]_{\kappa^+}$ is defined through the following universal property:

$$
\mathcal{C}_T \xrightarrow{\kappa^+-\text{small limit preserving}} \text{Set}[T_{\kappa^+}]_{\kappa^+}
$$
Proof idea

Suppose \( \text{Sat}^\kappa + (K) \) is axiomatizable by \( T^{\text{sat}}_\kappa + \). Then the morphism \( C^{T^\kappa +} \rightarrow C^{T^{\text{sat}}_\kappa +} \) induces a morphism between the corresponding \( \kappa + \)-classifying toposes \( f^*: \text{Set}[[T^\kappa +]]_{\kappa +} \rightarrow \text{Set}[[T^{\text{sat}}_\kappa +]]_{\kappa +} \).

Then:

\[ \text{Set}[[T^\kappa +]]_{\kappa +} \cong \text{Set}[[T^{\text{sat}}_\kappa +]]_{\kappa +} \]

We now want to deduce from the fact that the right morphism is an isomorphism that \( f^* \) is an equivalence, i.e., every model of size \( \kappa + \) is \( \kappa + \)-saturated. Then \( K \) is \( \kappa + \)-categorical since there is a unique such model (Rosicky 1997).
Proof idea

Suppose $Sat_{\kappa^+}(\mathcal{K})$ is axiomatizable by $T_{\kappa^+}^{sat}$. Then the morphism $C_{T_{\kappa^+}} \rightarrow C_{T_{\kappa^+}^{sat}}$ induces a morphism between the corresponding $\kappa^+$-classifying toposes $f^* : Set[T_{\kappa^+}]_{\kappa^+} \rightarrow Set[T_{\kappa^+}^{sat}]_{\kappa^+}$.  

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$$
\begin{align*}
\text{Set}[T_{\kappa^+}]_{\kappa^+} & \xrightarrow{\eta^*_{\text{Set}[T_{\kappa^+}]_{\kappa^+}}} S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) & S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) & \xrightarrow{\text{R}} S_{\lambda}(\mathcal{K}_{\geq \lambda}) \\
\text{Set}[T_{\kappa^+}^{sat}]_{\kappa^+} & \xrightarrow{\eta^*_{\text{Set}[T_{\kappa^+}^{sat}]_{\kappa^+}}} S_{\kappa^+}(Sat_{\kappa^+}(\mathcal{K})) & S_{\kappa^+}(Sat_{\kappa^+}(\mathcal{K})) & \xrightarrow{\text{R}} S_{\lambda}(Sat_{\lambda}(\mathcal{K})) \\
\end{align*}
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We now want to deduce from the fact that the right morphism is an isomorphism that $f^*$ is an equivalence, i.e., every model of size $\kappa^+$ is $\kappa^+$-saturated. Then $\mathcal{K}$ is $\kappa^+$-categorical since there is a unique such model (Rosicky 1997).
Proof idea

Suppose $Sat_{\kappa^+}(\mathcal{K})$ is axiomatizable by $T_{\kappa^+}^{sat}$. Then the morphism $C_{T_{\kappa^+}} \rightarrow C_{T_{\kappa^+}^{sat}}$ induces a morphism between the corresponding $\kappa^+$-classifying toposes $f^*: \text{Set}[T_{\kappa^+}]_{\kappa^+} \rightarrow \text{Set}[T_{\kappa^+}^{sat}]_{\kappa^+}$. Then:

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\text{Set}[T_{\kappa^+}]_{\kappa^+} \xrightarrow{\eta^*_{\text{Set}[T_{\kappa^+}]}_{\kappa^+}} \text{Set}[T_{\kappa^+}^{sat}]_{\kappa^+} \xrightarrow{\eta^*_{\text{Set}[T_{\kappa^+}^{sat}]}_{\kappa^+}} \text{Set}[T_{\kappa^+}^{sat}]_{\kappa^+} \\
\downarrow f^* \quad \downarrow \quad \downarrow

\text{Set}[T_{\kappa^+}]_{\kappa^+} \xrightarrow{\eta^*_{\text{Set}[T_{\kappa^+}]}_{\kappa^+}} \text{Set}[T_{\kappa^+}^{sat}]_{\kappa^+} \xrightarrow{\eta^*_{\text{Set}[T_{\kappa^+}^{sat}]}_{\kappa^+}} \text{Set}[T_{\kappa^+}^{sat}]_{\kappa^+} \\
\downarrow \quad \downarrow \quad \downarrow

S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) \quad \text{Set}(\mathcal{K})_{\geq \kappa^+} \quad S_{\lambda}(\mathcal{K}_{\geq \lambda})
$$

We now want to deduce from the fact that the right morphism is an isomorphism that $f^*$ is an equivalence, i.e., every model of size $\kappa^+$ is $\kappa^+$-saturated. Then $\mathcal{K}$ is $\kappa^+$-categorical since there is a unique such model (Rosicky 1997)
Proof idea

We will prove that in fact $\text{Sat}^κ + (K)$ can be axiomatized and, moreover, if $τ_D$ is the dense (alternatively, atomic) Grothendieck topology on $K$:

$$\text{Set}[\text{Sat}^κ +] = \text{Sh}(K^{op}, τ_D)$$

This is based in the following:

**Theorem**

Let $T^κ$ axiomatize $K ≥ κ$. Then the $κ^+$-classifying topos of $T^κ$ is equivalent to the presheaf topos $\text{Set}K^κ$. Moreover, the canonical embedding of the syntactic category is given by (note that $K^κ$ ∈ $C^{T^κ} → \text{Set}$):

$$C^{T^κ} \ni M : C^{T^κ} → \text{Set} \quad M ↦ M(X)$$

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$$\text{Set}[\mathcal{T}_{\kappa^+}^{\text{sat}}]_{\kappa^+} \cong \text{Sh}(\mathcal{K}_{\kappa}^{\text{op}}, \tau_D)$$
Proof idea

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$$\text{Set}[\mathbb{T}_{\kappa^+}^{\text{sat}}]_{\kappa^+} \cong \mathcal{S}h(\mathcal{K}_{\kappa}^{op}, \tau_D)$$

This is based in the following:

**Theorem**

Let $\mathbb{T}_{\kappa}$ axiomatize $\mathcal{K}_{\geq \kappa}$. Then the $\kappa^+$-classifying topos of $\mathbb{T}_{\kappa}$ is equivalent to the presheaf topos $\text{Set}^{\mathcal{K}_{\kappa}}$. Moreover, the canonical embedding of the syntactic category is given by (note that $\mathcal{K}_{\kappa} \ni M : \mathcal{C}_{\mathbb{T}_{\kappa}} \rightarrow \text{Set}$):

$$\mathcal{C}_{\mathbb{T}_{\kappa}} \xrightarrow{\text{ev}} \text{Set}^{\mathcal{K}_{\kappa}}$$

$$X \xrightarrow{\quad \quad M \mapsto M(X) \quad \quad} M$$
Proof idea

Every model of $T$ is a $\kappa^+$-filtered colimit of models in $K$. We have:

$$C^T_{\kappa} \cong \text{ev} \lim_{\to i} M_i = \text{ev} \lim_{\to i} M_i$$

This proves the universal property when $E = \text{Set}$. 

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Every model of $\mathcal{T}$ is a $\kappa^+$-filtered colimit of models in $\mathcal{K}_\kappa$. We have:

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\text{Set} \\
\text{Set}^{\mathcal{K}_\kappa} } \xleftarrow{\lim_{i} \text{ev}_{M_i}}
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\text{Set} & & \\
\end{array}
\]

This proves the universal property when $\mathcal{E} = \text{Set}$. 
Proof idea

Let now $E$ be the $\kappa^{+}$-classifying topos of $T_\kappa$. Then it has enough $\kappa^{+}$-points by the infinitary Deligne completeness theorems (Espíndola 2017). We have:

$$C_{T_\kappa}S \cong \text{ev} \circ N \circ G_{\kappa} \circ \text{ev}.$$ Now every object $F$ in $S_{\kappa}$ can be written as $F \sim \lim_{\rightarrow i} \left[ M_i, - \right] \text{Mod}_\lambda(T)$, i.e.:

$$\lim_{\rightarrow i} \lim_{\leftarrow j} \left[ \phi_{ij}, - \right] C_{T_\kappa}, - \sim \lim_{\rightarrow i} \lim_{\leftarrow j} \text{ev}(\phi_{ij})$$
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Let now \( \mathcal{E} \) be the \( \kappa^+ \)-classifying topos of \( T_\kappa \). Then it has enough \( \kappa^+ \)-points by the infinitary Deligne completeness theorems (Espindola 2017).
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\begin{align*}
\mathcal{C}_{\mathbb{T}_\kappa} & \xrightarrow{\text{ev}} \mathcal{S}et^{\mathcal{C}_\kappa} \\
N & \downarrow \quad \mathcal{E} \\
\mathcal{E} & \xrightarrow{G} \mathcal{S}et^{\mathcal{C}_\kappa} \\
\mathcal{S}et^I & \xrightarrow{\text{ev}_i} \mathcal{S}et
\end{align*}
$$
Proof idea

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Now every object $F$ in $\text{Set}^{\mathcal{K}_\kappa}$ can be written as $F \cong \lim_{i} [M_i, -]_{\text{Mod}_\lambda(\mathbb{T})}$, i.e.:

$$\lim_{i} \lim_{j} [\phi_{ij}, -]_{\mathcal{C}_{\mathbb{T}}} \cong \lim_{i} \lim_{j} [[\phi_{ij}, -]_{\mathcal{C}_{\mathbb{T}}}, -]_{\text{Mod}_\lambda(\mathbb{T})} \cong \lim_{i} \lim_{j} \text{ev}(\phi_{ij})$$
Proof idea

Since $G$ preserves colimits and $\kappa^+$-small limits, we have:

$$G(F) \cong \lim_{\to} \lim_{\leftarrow} G \circ \text{ev}(\phi_{ij})$$

and hence $G$ is completely determined by its values on $\text{ev}(C_T\kappa)$, which land in $E$. Since $E$ also preserves colimits and $\kappa^+$-small limits, the whole $G$ lands in $E$.

This proves the universal property when $E$ is $\kappa^+$-classifying topos of $T\kappa$.

Since this later satisfies the same universal property, we must have $E \cong \text{Set}_{K\kappa}$. This finishes the proof.
Since $G$ preserves colimits and $\kappa^+$-small limits, we have:

$$G(F) \cong \lim_{i} \lim_{j} G \circ \text{ev}(\phi_{ij})$$

and hence $G$ is completely determined by its values on $\text{ev}(\mathcal{C}_{T,\kappa})$, which land in $\mathcal{E}$. Since $E$ also preserves colimits and $\kappa^+$-small limits, the whole $G$ lands in $\mathcal{E}$. 

This proves the universal property when $E$ is $\kappa^+$-classifying topos of $T_{\kappa}$. Since this later satisfies the same universal property, we must have $E \cong \text{Set}$. This finishes the proof.
Proof idea

Since $G$ preserves colimits and $\kappa^+$-small limits, we have:

$$G(F) \cong \lim_{i} \lim_{j} G \circ \text{ev}(\phi_{ij})$$

and hence $G$ is completely determined by its values on $\text{ev}(\mathcal{C}_{T_\kappa})$, which land in $\mathcal{E}$. Since $E$ also preserves colimits and $\kappa^+$-small limits, the whole $G$ lands in $\mathcal{E}$.

This proves the universal property when $\mathcal{E}$ is $\kappa^+$-classifying topos of $T_\kappa$.

Since this later satisfies the same universal property, we must have $\mathcal{E} \cong \text{Set}^{\mathcal{K}_\kappa}$. This finishes the proof.
Proof idea

Coming back to $\mathsf{Sat}^{\kappa}$, we have the following situation:

$$\mathsf{C} \mathsf{T}^{\kappa} \mathsf{S} \mathsf{et} \kappa = \mathsf{C} \mathsf{T}^{\kappa} \mathsf{S} \mathsf{et} \left[ \mathsf{T}^{\kappa} \right] \kappa = \mathsf{C} \mathsf{T} \mathsf{sat}^{\kappa} \mathsf{S} \mathsf{h} \left( \mathsf{K} \mathsf{op}^{\kappa}, \tau \mathsf{D} \right) \mathsf{ev} \mathsf{f} \mathsf{∗} \mathsf{M} \sim \lim_{\to} \mathsf{Ni} \mathsf{f} \mathsf{∗}$$

FACT (e.g. Johnstone's Elephant): The embedding $\mathsf{S} \mathsf{h} \left( \mathsf{K} \mathsf{op}^{\kappa}, \tau \mathsf{D} \right) \to \mathsf{S} \mathsf{et} \kappa$ factors through $\mathsf{f} \mathsf{∗}$ if and only if $\mathsf{f} \mathsf{∗}$ is dense ($\mathsf{f} \mathsf{∗}(0) = 0$, or alternatively $\mathsf{f} \mathsf{∗}(C) \neq 0$ for $C \neq 0$).
Coming back to $Sat_{\kappa^+}(\mathcal{K})$, we have the following situation:
Coming back to $Sat_{\kappa^+}(\mathcal{K})$, we have the following situation:

\[
\begin{array}{ccc}
C_{\mathbb{T}_\kappa} & \xrightarrow{\text{ev}} & Set^{\mathcal{K}_{\kappa}} \\
\downarrow & & \\
C_{\mathbb{T}_{\kappa^+}} & \rightarrow & Set[\mathbb{T}_{\kappa^+}]_{\kappa^+} \\
\downarrow & & \\
C_{\mathbb{T}_{\kappa^+}^{\text{sat}}} & \rightarrow & Sh(K_{\kappa}^{\text{op}}, \tau_D) \\
\end{array}
\]
Coming back to $Sat_{\kappa^+}(\mathcal{K})$, we have the following situation:

\[
\begin{array}{ccc}
C_{T_{\kappa}} & \xrightarrow{ev} & Set^{\mathcal{K}_{\kappa}} \\
\downarrow & & \downarrow f^* \quad f_* \\
C_{T_{\kappa}^+} & \rightarrow & Set[\mathcal{T}_{\kappa^+}]_{\kappa^+} \\
\uparrow & & \downarrow \\
C_{T_{\kappa}^{sat}} & \rightarrow & Sh(K_{\kappa}^{op}, \tau_D)
\end{array}
\]

FACT (e.g. Johnstone’s Elephant): The embedding $Sh(K_{\kappa}^{op}, \tau_D) \hookrightarrow Set^{\mathcal{K}_{\kappa}}$ factors through $f_*$ if and only if $f_*$ is dense ($f_*(0) = 0$, or alternatively $f^*(C) \neq 0$ for $C \neq 0$).
Proof idea

If $M$ is a saturated model of size $\kappa^+$, for every $p : N \to N'$ in $K_{\kappa}$, each $N \to M$ extends to some $p' : N' \to M$. This is the same as saying that $M$ factors through $\text{Set} K_{\kappa} \to \text{Set}$ maps $p^* : [N', -] \to [N, -]$ to an epimorphism, since:

$$\lim_{\to} \text{ev} N_i([N, -]) = \lim_{\to} [N, N_i] \sim [N, \lim_{\to} N_i] \sim [N, M]$$

It follows that $M$ factors through $\text{Sh}(K_{\kappa}^{\text{op}}, \tau_D)$ and that $\text{Set} K_{\kappa}^{\text{op}}$ is the $\kappa^+$-classifying topos of $T_{\text{sat}}^{\kappa^+}$.

FACT (Rosicky 1997): $\kappa^+$-saturated models exist. This can also be seen topos-theoretically by noticing that $\text{Sh}(K_{\kappa}^{\text{op}}, \tau_D)$ has enough $\kappa^+$-points (Espindola 2017). The uniqueness of $\kappa^+$-saturated models of size $\kappa^+$ can be seen as well by noticing that $\text{Sh}(K_{\kappa}^{\text{op}}, \tau_D)$ is two-valued and Boolean (Barr-Makkai 1987, Espindola 2017).
Proof idea

If $M$ is a saturated model of size $\kappa^{++}$, for every $p: N \to N'$ in $K_{\kappa}$, each $N \to M$ extends to some $p': N' \to M$. This is the same as saying that $M: \text{Set} \to \text{Set}$ maps $p^*: [N', -] \to [N, -]$ to an epimorphism, since:

$\lim_{\to} - \to \text{eval}_N \circ i \circ (N, -) = \lim_{\to} - \to i \circ N \circ \text{eval}_N \sim = [N, \lim_{\to} - \to \text{eval}_N \circ N \circ i] \sim = [N, M]$

It follows that $M$ factors through $\text{sh}(K_{\kappa^{op}}, \tau_D)$ and that $\text{sh}(K_{\kappa^{op}}, \tau_D)$ is the $\kappa^{++}$-classifying topos of $T_{\text{sat}}\kappa^{++}$.

**FACT (Rosicky 1997):** $\kappa^{++}$-saturated models exist. This can also be seen topos-theoretically by noticing that $\text{sh}(K_{\kappa^{op}}, \tau_D)$ has enough $\kappa^{++}$-points (Espindola 2017). The uniqueness of $\kappa^{++}$-saturated models of size $\kappa^{++}$ can be seen as well by noticing that $\text{sh}(K_{\kappa^{op}}, \tau_D)$ is two-valued and Boolean (Barr-Makkai 1987+ Espindola 2017).
Proof idea

If $M$ is a saturated model of size $\kappa^+$, for every $p : N \to N'$ in $\mathcal{K}_\kappa$, each $N \to M$ extends to some $p' : N' \to M$. This is the same as saying that $M : \text{Set}^{\mathcal{K}_\kappa} \to \text{Set}$ maps $p^* : [N', -] \to [N, -]$ to an epimorphism, since:

$$\lim_i \text{ev}_{N_i}([N, -]) = \lim_i [N, N_i] \cong [N, \lim_i N_i] \cong [N, M]$$
Proof idea

If \( M \) is a saturated model of size \( \kappa^+ \), for every \( p : N \to N' \) in \( \mathcal{K}_\kappa \), each \( N \to M \) extends to some \( p' : N' \to M \). This is the same as saying that \( M : \text{Set}^{\mathcal{K}_\kappa} \to \text{Set} \) maps \( p^* : [N', -] \to [N, -] \) to an epimorphism, since:

\[
\lim_i \text{ev}_{N_i}([N, -]) = \lim_i [N, N_i] \cong [N, \lim_i N_i] \cong [N, M]
\]

It follows that \( M \) factors through \( a : \text{Set}^{\mathcal{K}_\kappa} \to \text{Sh}(K_\kappa^{op}, \tau_D) \) and that \( \text{Sh}(K_\kappa^{op}, \tau_D) \) is the \( \kappa^+ \)-classifying topos of \( \mathbb{T}_{\kappa^+}^{\text{sat}} \).
If $M$ is a saturated model of size $\kappa^+$, for every $p : N \to N'$ in $\mathcal{K}_\kappa$, each $N \to M$ extends to some $p' : N' \to M$. This is the same as saying that $M : \text{Set}^{\mathcal{K}_\kappa} \to \text{Set}$ maps $p^* : [N', -] \to [N, -]$ to an epimorphism, since:

$$\lim_{i} \text{ev}_{N_i}([N, -]) \cong \lim_{i} [N, N_i] \cong [N, \lim_{i} N_i] \cong [N, M]$$

It follows that $M$ factors through $a : \text{Set}^{\mathcal{K}_\kappa} \to Sh(K^{\text{op}}_\kappa, \tau_D)$ and that $Sh(K^{\text{op}}_\kappa, \tau_D)$ is the $\kappa^+$-classifying topos of $\mathbb{T}^{\text{sat}}_{\kappa^+}$.

FACT (Rosicky 1997): $\kappa^+$-saturated models exist. This can also be seen topos-theoretically by noticing that $Sh(K^{\text{op}}_\kappa, \tau_D)$ has enough $\kappa^+$-points (Espindola 2017). The uniqueness of $\kappa^+$-saturated models of size $\kappa^+$ can be seen as well by noticing that $Sh(K^{\text{op}}_\kappa, \tau_D)$ is two-valued and Boolean (Barr-Makkai 1987+ Espindola 2017)

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\[ \text{Set}[^{\mathbb{T}_\kappa^+}]_{\kappa^+} \xrightarrow{\eta_{\text{Set}[\mathbb{T}_\kappa^+],\kappa^+}} S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) \xrightarrow{\eta_{\text{Sh}(\mathbb{K}_\kappa^\text{op},\tau_D)}} S_{\lambda}(\mathcal{K}_{\geq \lambda}) \]

\[ \text{Sh}(\mathbb{K}_\kappa^\text{op},\tau_D) \xrightarrow{f^*} S_{\kappa^+}(\text{Sat}_{\kappa^+}(\mathcal{K})) \xrightarrow{\eta_{\text{Sh}(\mathbb{K}_\kappa^\text{op},\tau_D)}} S_{\lambda}(\text{Sat}_\lambda(\mathcal{K})) \]

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Wrapping up

To prove that $\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \cong \text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^{\text{op}}, \tau_D)$ we show that the embedding $\text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^{\text{op}}, \tau_D) \hookrightarrow \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ is an isomorphism, for which we in turn show that any basic sequent valid in $\text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^{\text{op}}, \tau_D)$ will also be valid in $\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}$.
Wrapping up

\[
\begin{array}{c}
\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \xrightarrow{\eta^*_{\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}}} S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) \xrightarrow{\text{f}^*} S_{\lambda}(\mathcal{K}_{\geq \lambda}) \\
\downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
\text{Sh}(\mathcal{K}_{\kappa}^{\text{op}}, \tau_D) \xrightarrow{\eta^*_{\text{Sh}(\mathcal{K}_{\kappa}^{\text{op}}, \tau_D)}} S_{\kappa^+}(\text{Sat}_{\kappa^+}(\mathcal{K})) \xrightarrow{\text{f}^*} S_{\lambda}(\text{Sat}_{\lambda}(\mathcal{K})) \\
\end{array}
\]

To prove that \(\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \cong \text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^{\text{op}}, \tau_D)\) we show that the embedding \(\text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^{\text{op}}, \tau_D) \hookrightarrow \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}\) is an isomorphism, for which we in turn show that any basic sequent valid in \(\text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^{\text{op}}, \tau_D)\) will also be valid in \(\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}\).

Inspection of the diagram shows that any such basic sequent \(\forall x(\phi \rightarrow \psi)\) valid in \(\text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^{\text{op}}, \tau_D)\) is also valid in \(S_{\lambda}(\mathcal{K}_{\geq \lambda})\), and hence in the unique model of size \(\lambda\).
Wrapping up

Consider the presheaf category \( \text{Set}^{K_{\geq \kappa^{+} \leq \lambda}} \). The interpretation of the sentence \( \forall x (\phi \rightarrow \psi) \) corresponds to a subobject \( S \hookrightarrow 1 \).

**FACT (Kripke-Joyal semantics):**

\( S = 0 \) if and only if for every morphism \( M \rightarrow N \) in \( K_{\geq \kappa^{+} \leq \lambda} \) there is a morphism \( N \rightarrow N' \) with \( N' \not\models \forall x (\phi \rightarrow \psi) \).

We conclude that \( S \neq 0 \).

Assume now that \( K \) is \( \kappa \)-categorical. Then \( \text{Set}^{K_{\kappa^{+}^{+}}} \), and hence \( \text{Set}^{[T_{\kappa^{+}^{+}}]}_{\kappa^{+}^{+}} \), is two-valued. Thus the interpretation of \( \forall x (\phi \rightarrow \psi) \) in \( \text{Set}^{[T_{\kappa^{+}^{+}}]}_{\kappa^{+}^{+}} \) corresponds to a subobject \( T \) that is either 0 or 1. It is hence enough to prove it is not 0.

This is the last missing piece, which is proven through an infinitary generalization of a completeness theorem of Joyal:
Consider the presheaf category $\text{Set}^{K_{\geq \kappa^+, \leq \lambda}}$. The interpretation of the sentence $\forall x (\phi \rightarrow \psi)$ corresponds to a subobject $S \rightarrow 1$. 

**FACT (Kripke-Joyal semantics):** $S = 0$ if and only if for every morphism $M \rightarrow N$ in $K_{\geq \kappa^+, \leq \lambda}$ there is a morphism $N \rightarrow N'$ with $N' \not\equiv \forall x (\phi \rightarrow \psi)$. We conclude that $S \neq 0$. Assume now that $K$ is $\kappa$-categorical. Then $\text{Set}^K_{\kappa}$, and hence $\text{Set}^{\left[ T_{\kappa^+} \right]_{\kappa^+, \leq \lambda}}$, is two-valued. Thus the interpretation of $\forall x (\phi \rightarrow \psi)$ in $\text{Set}^{\left[ T_{\kappa^+} \right]_{\kappa^+, \leq \lambda}}$ corresponds to a subobject $T$ that is either 0 or 1. It is hence enough to prove it is not 0. This is the last missing piece, which is proven through an infinitary generalization of a completeness theorem of Joyal:
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Assume now that \( \mathcal{K} \) is \( \kappa \)-categorical. Then \( \text{Set}^{\mathcal{K}_{\kappa^+}, \leq \lambda} \) is two-valued. Thus the interpretation of \( \forall x(\phi \rightarrow \psi) \) in \( \text{Set}^{\mathcal{T}_{\kappa^+}, \leq \lambda} \) corresponds to a subobject \( T \) that is either 0 or 1. It is hence enough to prove it is not 0.

This is the last missing piece, which is proven through an infinitary generalization of a completeness theorem of Joyal.
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We conclude that $S \neq 0$. 

We assume now that $\mathcal{K}$ is $\kappa$-categorical. Then $\text{Set}^{\mathcal{K}_{\kappa^+, \kappa^+}}$ and hence $\text{Set}^{[\mathcal{T}_{\kappa^+}]_{\kappa^+}}$ is two-valued. Thus the interpretation of $\forall x(\phi \rightarrow \psi)$ in $\text{Set}^{[\mathcal{T}_{\kappa^+}]_{\kappa^+}}$ corresponds to a subobject $T$ that is either 0 or 1. It is hence enough to prove it is not 0. This is the last missing piece, which is proven through an infinitary generalization of a completeness theorem of Joyal.
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Assume now that $\mathcal{K}$ is $\kappa$-categorical. Then $\text{Set}^{\mathcal{K}_{\kappa}}$, and hence $\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}$, is two-valued. Thus the interpretation of $\forall x (\phi \rightarrow \psi)$ in $\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ corresponds to a subobject $T$ that is either 0 or 1. It is hence enough to prove it is not 0.
Consider the presheaf category \( \text{Set}^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}} \). The interpretation of the sentence \( \forall x (\phi \to \psi) \) corresponds to a subobject \( S \to 1 \).

**FACT (Kripke-Joyal semantics):** \( S = 0 \) if and only if for every morphism \( M \to N \) in \( \mathcal{K}_{\geq \kappa^+, \leq \lambda} \) there is a morphism \( N \to N' \) with \( N' \not\models \forall x (\phi \to \psi) \). We conclude that \( S \neq 0 \).

Assume now that \( \mathcal{K} \) is \( \kappa \)-categorical. Then \( \text{Set}^ {\mathcal{K}_{\kappa}} \), and hence \( \text{Set}^ {\mathcal{T}_{\kappa^+}_{\kappa^+}} \), is two-valued. Thus the interpretation of \( \forall x (\phi \to \psi) \) in \( \text{Set}^ {\mathcal{T}_{\kappa^+}_{\kappa^+}} \) corresponds to a subobject \( T \) that is either 0 or 1. It is hence enough to prove it is not 0. This is the last missing piece, which is proven through an infinitary generalization of a completeness theorem of Joyal:
Wrapping up

Theorem

The evaluation functor: $\text{ev}: S_{\mathbb{K}}[T_{\kappa}^+] \rightarrow S_{\mathbb{K} \geq \kappa, \leq \lambda}$ preserves the interpretation of the sentence $\forall x (\phi \rightarrow \psi)$.

Proof.

$C(T_{\kappa}^+)^{\mathbb{K}} \subseteq S_{\mathbb{K}}[T_{\kappa}^+] \subseteq S_{\mathbb{K}^+}$.

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The evaluation functor:

\[ \text{ev} : \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \to \text{Set}^{\mathcal{K}_{\geq \kappa^+}, \leq \lambda} \]

preserves the interpretation of the sentence \( \forall x(\phi \to \psi) \).
Wrapping up

**Theorem**

The evaluation functor:

\[ \text{ev} : \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \rightarrow \text{Set}^{\mathcal{K}_{\geq \kappa^+}, \leq \lambda} \]

preserves the interpretation of the sentence \( \forall x (\phi \rightarrow \psi) \).

**Proof.**

\[
\begin{array}{cccc}
C_{(\mathbb{T}_{\kappa^+})_{\lambda}} & \xrightarrow{Y'} & \text{Set}[\mathbb{T}_{\kappa^+}]_{\lambda^+} \\
g & \uparrow & & \uparrow g^* \\
C_{\mathbb{T}_{\kappa^+}} & \xrightarrow{Y} & \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}
\end{array}
\]
Wrapping up

It is enough to prove that the interpretation of $\forall x (\phi \to \psi)$ is preserved by the canonical morphism $g^*: \text{Set}^{\text{T}\kappa^+} \rightarrow \text{Set}^{\text{T}\kappa^+\lambda^+}$, since this latter is the $\lambda^+$-classifying topos of $\text{T}\kappa^+$. An entirely analogous proof to a previous theorem shows that this must be the presheaf topos $\text{Set}_K^{\geq \kappa^+, \leq \lambda^+}$. This follows immediately since $g$ preserves the interpretation of $\forall x (\phi \to \psi)$ (by the syntactic construction of the syntactic categories), and a theorem of Butz and Johnstone (1998) proves that the interpretation of $\forall x (\phi \to \psi)$ is preserved by $Y$ and $Y'$. This completes the proof.
Wrapping up

It is enough to prove that the interpretation of $\forall x (\phi \to \psi)$ is preserved by the canonical morphism $g^*: \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \to \text{Set}[\mathbb{T}_{\kappa^+}]_{\lambda^+}$, since this latter is the $\lambda^+$-classifying topos of $\mathbb{T}_{\kappa^+}$, and an entirely analogous proof to a previous theorem shows that this must be the presheaf topos $\text{Set}^{\mathbb{K}_{\geq \kappa^+, \leq \lambda}}$.
Wrapping up

It is enough to prove that the interpretation of $\forall x (\phi \rightarrow \psi)$ is preserved by the canonical morphism $g^* : \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \rightarrow \text{Set}[\mathbb{T}_{\kappa^+}]_{\lambda^+}$, since this latter is the $\lambda^+$-classifying topos of $\mathbb{T}_{\kappa^+}$, and an entirely analogous proof to a previous theorem shows that this must be the presheaf topos $\text{Set}^{\mathcal{K}_{\geq \kappa^+}, \leq \lambda}$. This follows immediately since $g$ preserves the interpretation of $\forall x (\phi \rightarrow \psi)$ (by the syntactic construction of the syntactic categories), and a theorem of Butz and Johnstone (1998) proves that the interpretation of $\forall x (\phi \rightarrow \psi)$ is preserved by $Y$ and $Y'$. This completes the proof.
Wrapping up

We conclude that categoricity in $\kappa$ and $\lambda$ implies categoricity in $\kappa +$. Repeating the argument we conclude categoricity in $\kappa^{++}$, and so on. For a limit $\mu$, we simply consider the diagram:

$\text{Set}[\text{T}_{\kappa +}]_{\kappa +}$

$\text{Set}[\text{T}_{\kappa^{++}}]_{\kappa^{++}}$

$\vdots$

$\text{Sh}(\text{Kop}_{\kappa}, \tau_{D})$

$\text{Sh}(\text{Kop}_{\kappa +}, \tau_{D})$

$\vdots$

This also serves for the case in which $\lambda$ is limit.
We conclude that categoricity in $\kappa$ and $\lambda$ implies categoricity in $\kappa^+$. Repeating the argument we conclude categoricity in $\kappa^{++}$, and so on. For a limit $\mu$, we simply consider the diagram:

$$
\begin{align*}
Set[\mathbb{T}_{\kappa^+}]_{\kappa^+} & \rightarrow Set[\mathbb{T}_{\kappa^{++}}]_{\kappa^{++}} & \rightarrow & \cdots & \rightarrow Set[\mathbb{T}_{\mu}]_{\mu} \\
\downarrow & & & & \downarrow \\
Sh(K_{\kappa}^{op}, \tau_D) & \rightarrow Sh(K_{\kappa^+}^{op}, \tau_D) & \rightarrow & \cdots & \rightarrow E
\end{align*}
$$

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We conclude that categoricity in $\kappa$ and $\lambda$ implies categoricity in $\kappa^+$. Repeating the argument we conclude categoricity in $\kappa^{++}$, and so on. For a limit $\mu$, we simply consider the diagram:

\[
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\downarrow & \downarrow & \vdots \\
\mathcal{S}h(K^{\text{op}}_\kappa, \tau_D) & \longrightarrow \mathcal{S}h(K^{\text{op}}_{\kappa^+}, \tau_D) \longrightarrow \cdots \longrightarrow \mathcal{E}
\end{align*}
\]

This also serves for the case in which $\lambda$ is limit.
Wrapping up

Proof of Shelah's eventual categoricity conjecture.

FACT (Hanf numbers): For every AEC $K$ there is a cardinal $\kappa$ such that if $K$ is categorical in some $\lambda > \kappa$, it is categorical in unboundedly many cardinals. Since categoricity in a pair of cardinals implies categoricity in all cardinals in between, we conclude that there is a tail of cardinals where $K$ is categorical. QED

Remark: Assume GCH. Let $K$ be an accessible category with all morphisms monomorphisms, directed bounds and amalgamation. Then the same proof outlined also proves that there is a cardinal $\kappa$ such that if $K$ is $\lambda$-categorical for some $\lambda \supseteq \kappa$ (i.e., it has only one object of internal size $\lambda$ up to isomorphism) then it is $\lambda'$-categorical for every $\lambda' \supseteq \kappa$. The hypotheses on $K$ can be spared assuming instead a proper class of strongly compact cardinals.
Wrapping up

*Proof of Shelah’s eventual categoricity conjecture.*
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Remark

Assume $GCH$. Let $\mathcal{K}$ be an accessible category with all morphisms monomorphisms, directed bounds and amalgamation. Then the same proof outlined also proves that there is a cardinal $\kappa$ such that if $\mathcal{K}$ is $\lambda$-categorical for some $\lambda \geq \kappa$ (i.e., it has only one object of internal size $\lambda$ up to isomorphism) then it is $\lambda'$-categorical for every $\lambda' \geq \kappa$. The hypotheses on $\mathcal{K}$ can be spared assuming instead a proper class of strongly compact cardinals.
Thank you!