A topos-theoretic proof of Shelah's eventual categoricity conjecture

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Abstract elementary classes

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An abstract elementary class (AEC) $\mathcal{K}$ is a category equivalent to an accessible category with directed colimits whose morphisms are monomorphisms, that admits an embedding $F : \mathcal{K} \to \mathcal{A}$ into a finitely accessible category preserving directed colimits and monomorphisms which is, in addition:

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F(A) \arrow[rr, "F(f)"] \arrow[dr, "h"'] & & F(B) \\
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FACT (Lieberman-Rosicky-Vasey 2019): For each object A of an AEC, $|E(A)|$ coincides with its internal size $|A|$ defined as follows. If $r(A)$ is the least regular cardinal λ such that A is λ-presentable, then:

$$|A| = \begin{cases} \kappa & \text{if } r(A) = \kappa^+ \\ r(A) & \text{if } r(A) \text{ is limit} \end{cases}$$

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There is a proof of the conjecture (Shelah-Vasey 2019) assuming *GCH* and large cardinals. In fact, the use of large cardinals is to guarantee that the AEC will eventually satisfy the amalgamation property:

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N \arrow[r, dashed] & A \\
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We will give a short topos-theoretic proof of the conjecture assuming *GCH* and that the AEC satisfies amalgamation.

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Given an accessible category A with κ-directed colimits, its κ-Scott topos $S_\kappa(A)$ is the full subcategory of the presheaf $\mathcal{S}et^A$ given by those functors $F : A \to \mathcal{S}et$ preserving κ-directed colimits.

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Given a topos $\mathcal{E}$, its category of κ-points $pt_{\kappa}(\mathcal{E})$(i.e., geometric morphisms to $\mathcal{S}et$ whose inverse images preserve all κ-small limits) is an accessible category with κ-directed colimits.

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Theorem

(Henry-Di Liberti) There is a (2-)adjunction:

$$\begin{tikzcd}
S : Acc_\kappa \arrow[rr, bend left] & & Top_\kappa : pt_\kappa \arrow[ll, bend left]
\end{tikzcd}$$

between the category of accessible categories with κ-directed colimits and the category of κ-exact localization of presheaf toposes given by the Scott functor S_κ and the functor pt_κ.

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A model M is μ^+-saturated if for every morphism $N \to N'$ between models of size μ, every morphism $N \to M$ can be extended:

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S_{\kappa^+}(\mathcal{K}_{\geq\kappa^+}) \arrow[r] \arrow[d] & S_\lambda(\mathcal{K}_{\geq\lambda}) \arrow[d, "\cong"] \\
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The AEC $\mathcal{K}$, as any accessible category, can be axiomatized up to equivalence by basic sentences through a theory $\mathbb{T}$ in $\mathcal{L}_{\mu^+,\mu}$, and can have models not just in $\mathcal{S}et$ but in any μ-coherent category (i.e., in a category with enough $\mathcal{S}et$-like properties).

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$$\begin{tikzcd}
\mathcal{C}_\mathbb{T} \arrow[r, dashed, "\mu-coherent"] & \mathcal{D} \\
M_0 \arrow[u, mapsto] \arrow[r, dashed, mapsto] & M \arrow[u, mapsto]
\end{tikzcd}$$

Then $\mathcal{K}_{\geq\kappa^+}$ is axiomatized by basic sentences through $\mathbb{T}_{\kappa^+}$.

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The κ^+-classifying topos of $\mathbb{T}_{\kappa^+}$ (Espindola 2017), $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ is defined through the following universal property:

$$\begin{tikzcd}
\mathcal{C}_{\mathbb{T}} \arrow[rr] \arrow[dr] & & \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \arrow[dl, dashed, "\kappa^+-\text{small limit preserving}"] \\
& \mathcal{E} &
\end{tikzcd}$$

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Suppose $Sat_{\kappa^+}(\mathcal{K})$ is axiomatizable by $\mathbb{T}^{sat}_{\kappa^+}$. Then the morphism $\mathcal{C}_{\mathbb{T}_{\kappa^+}} \to \mathcal{C}_{\mathbb{T}^{sat}_{\kappa^+}}$ induces a morphism between the corresponding κ^+-classifying toposes $f^* : \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \to \mathcal{S}et[\mathbb{T}^{sat}_{\kappa^+}]_{\kappa^+}$.

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$$\begin{tikzcd}
\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \arrow[r, "\eta^*_{\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}}"] \arrow[d, "f^*"'] & S_{\kappa^+}(\mathcal{K}_{\geq\kappa^+}) \arrow[r] \arrow[d] & S_\lambda(\mathcal{K}_{\geq\lambda}) \arrow[d, "\cong"] \\
\mathcal{S}et[\mathbb{T}^{sat}_{\kappa^+}]_{\kappa^+} \arrow[r, "\eta^*_{\mathcal{S}et[\mathbb{T}^{sat}_{\kappa^+}]_{\kappa^+}}"] & S_{\kappa^+}(Sat_{\kappa^+}(\mathcal{K})) \arrow[r] & S_\lambda(Sat_\lambda(\mathcal{K}))
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\end{tikzcd}$$

We now want to deduce from the fact that the right morphism is an isomorphism that f^* is an equivalence, i.e., every model of size κ^+ is κ^+-saturated. Then $\mathcal{K}$ is κ^+-categorical since there is a unique such model (Rosicky 1997)

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We will prove that in fact $Sat_{\kappa^+}(\mathcal{K})$ can be axiomatized and, moreover, if τ_D is the dense (alternatively, atomic) Grothendieck topology on K_κ^{op}:

$$\mathcal{S}et[\mathbb{T}_{\kappa^+}^{sat}]_{\kappa^+} \cong \mathcal{S}h(K_\kappa^{op}, \tau_D)$$

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This is based in the following:

Theorem

Let $\mathbb{T}_\kappa$ axiomatize $\mathcal{K}_{\geq\kappa}$. Then the κ^+-classifying topos of $\mathbb{T}_\kappa$ is equivalent to the presheaf topos $\mathcal{S}et^{\mathcal{K}_\kappa}$. Moreover, the canonical embedding of the syntactic category is given by (note that $\mathcal{K}_\kappa \ni M : \mathcal{C}_{\mathbb{T}_\kappa} \to \mathcal{S}et$):

$$\begin{tikzcd}
\mathcal{C}_{\mathbb{T}_\kappa} \arrow[r, "ev"] & \mathcal{S}et^{\mathcal{K}_\kappa} \\
X \arrow[r, mapsto] & M \mapsto M(X)
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This proves the universal property when $\mathcal{E} = \mathcal{S}et$.

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Now every object F in $\mathcal{S}et^{\mathcal{K}_\kappa}$ can be written as $F \cong \varinjlim_i [M_i, -]_{Mod_\lambda(\mathbb{T})}$, i.e.:

$$\varinjlim_i [\varinjlim_j [\phi_{ij}, -]_{\mathcal{C}_\mathbb{T}}, -]_{Mod_\lambda(\mathbb{T})} \cong \varinjlim_i \varprojlim_j [[\phi_{ij}, -]_{\mathcal{C}_\mathbb{T}}, -]_{Mod_\lambda(\mathbb{T})} \cong \varinjlim_i \varprojlim_j ev(\phi_{ij})$$

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Since G preserves colimits and κ^+-small limits, we have:

$$G(F) \cong \varinjlim_i \varprojlim_j G \circ ev(\phi_{ij})$$

and hence G is completely determined by its values on $ev(\mathcal{C}_{\mathbb{T}_\kappa})$, which land in $\mathcal{E}$. Since E also preserves colimits and κ^+-small limits, the whole G lands in $\mathcal{E}$.

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This proves the universal property when $\mathcal{E}$ is κ^+-classifying topos of $\mathbb{T}_\kappa$. Since this later satisfies the same universal property, we must have $\mathcal{E} \cong \mathcal{S}et^{\mathcal{K}_\kappa}$. This finishes the proof.

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\mathcal{C}_{\mathbb{T}_{\kappa^+}} \arrow[r] \arrow[d] & \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \arrow[u, bend right, "f_*"'] \arrow[d, bend right] & \mathcal{S}et \\
\mathcal{C}_{\mathbb{T}^{sat}_{\kappa^+}} \arrow[r] & \mathcal{S}h(K_\kappa^{op}, \tau_D) \arrow[u, bend right] \arrow[ur, dashed] &
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FACT (e.g. Johnstone's Elephant): The embedding $\mathcal{S}h(K_\kappa^{op}, \tau_D) \hookrightarrow \mathcal{S}et^{\mathcal{K}_\kappa}$ factors through f_* if and only if f_* is dense ($f_*(0) = 0$, or alternatively $f^*(C) \neq 0$ for $C \neq 0$).

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If M is a saturated model of size κ^+, for every $p : N \to N'$ in $\mathcal{K}_\kappa$, each $N \to M$ extends to some $p' : N' \to M$. This is the same as saying that $M : \mathcal{S}et^{\mathcal{K}_\kappa} \to \mathcal{S}et$ maps $p^* : [N', -] \to [N, -]$ to an epimorphism, since:

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It follows that M factors through $a : \mathcal{S}et^{\mathcal{K}_\kappa} \to \mathcal{S}h(K_\kappa^{op}, \tau_D)$ and that $\mathcal{S}h(K_\kappa^{op}, \tau_D)$ is the κ^+-classifying topos of $\mathbb{T}^{sat}_{\kappa^+}$.

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FACT (Rosicky 1997): κ^+-saturated models exist. This can also be seen topos-theoretically by noticing that $\mathcal{S}h(K_\kappa^{op}, \tau_D)$ has enough κ^+-points (Espindola 2017). The uniqueness of κ^+-saturated models of size κ^+ can be seen as well by noticing that $\mathcal{S}h(K_\kappa^{op}, \tau_D)$ is two-valued and Boolean (Barr-Makkai 1987+ Espindola 2017)

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\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \arrow[r, "\eta^*_{\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}}"] \arrow[d, "f^*"'] & S_{\kappa^+}(\mathcal{K}_{\geq\kappa^+}) \arrow[r] \arrow[d] & S_{\lambda}(\mathcal{K}_{\geq\lambda}) \arrow[d, "\cong"] \\
\mathcal{S}h(K_{\kappa}^{op}, \tau_D) \arrow[r, "\eta^*_{\mathcal{S}h(K_{\kappa}^{op}, \tau_D)}"] & S_{\kappa^+}(Sat_{\kappa^+}(\mathcal{K})) \arrow[r] & S_{\lambda}(Sat_{\lambda}(\mathcal{K}))
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To prove that $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \cong \mathcal{S}h(Mod_\kappa(\mathbb{T})^{op}, \tau_D)$ we show that the embedding $\mathcal{S}h(Mod_\kappa(\mathbb{T})^{op}, \tau_D) \hookrightarrow \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ is an isomorphism, for which we in turn show that any basic sequent valid in $\mathcal{S}h(Mod_\kappa(\mathbb{T})^{op}, \tau_D)$ will also be valid in $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$.

Wrapping up

$$\begin{tikzcd}
\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \arrow[r, "\eta^*_{\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}}"] \arrow[d, "f^*"'] & S_{\kappa^+}(\mathcal{K}_{\geq\kappa^+}) \arrow[r] \arrow[d] & S_\lambda(\mathcal{K}_{\geq\lambda}) \arrow[d, "\cong"] \\
\mathcal{S}h(K_\kappa^{op}, \tau_D) \arrow[r, "\eta^*_{\mathcal{S}h(K_\kappa^{op}, \tau_D)}"] & S_{\kappa^+}(Sat_{\kappa^+}(\mathcal{K})) \arrow[r] & S_\lambda(Sat_\lambda(\mathcal{K}))
\end{tikzcd}$$

To prove that $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \cong \mathcal{S}h(Mod_\kappa(\mathbb{T})^{op}, \tau_D)$ we show that the embedding $\mathcal{S}h(Mod_\kappa(\mathbb{T})^{op}, \tau_D) \hookrightarrow \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ is an isomorphism, for which we in turn show that any basic sequent valid in $\mathcal{S}h(Mod_\kappa(\mathbb{T})^{op}, \tau_D)$ will also be valid in $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$.

Inspection of the diagram shows that any such basic sequent $\forall \mathbf{x}(\phi \to \psi)$ valid in $\mathcal{S}h(Mod_\kappa(\mathbb{T})^{op}, \tau_D)$ is also valid in $S_\lambda(\mathcal{K}_{\geq\lambda})$, and hence in the unique model of size λ.

Wrapping up

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Consider the presheaf category $\mathcal{S}et^{\mathcal{K}_{\geq\kappa^+,\leq\lambda}}$. The interpretation of the sentence $\forall \mathbf{x}(\phi \to \psi)$ corresponds to a subobject $S \rightarrowtail 1$.

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FACT (Kripke-Joyal semantics): $S = 0$ if and only if for every morphism $M \to N$ in $\mathcal{K}_{\geq\kappa^+,\leq\lambda}$ there is a morphism $N \to N'$ with $N' \not\models \forall\mathbf{x}(\phi \to \psi)$.

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We conclude that $S\neq 0$.

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We conclude that $S \neq 0$.
Assume now that $\mathcal{K}$ is κ-categorical. Then $\mathcal{S}et^{\mathcal{K}_\kappa}$, and hence $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$, is two-valued. Thus the interpretation of $\forall\mathbf{x}(\phi \to \psi)$ in $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ corresponds to a subobject T that is either 0 or 1. It is hence enough to prove it is not 0.

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Wrapping up

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Theorem

The evaluation functor:

$$ev : \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \to \mathcal{S}et^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}}$$

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Proof.

$$\begin{tikzcd}
\mathcal{C}_{(\mathbb{T}_{\kappa^+})_\lambda} \arrow[r, "Y'"] & \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\lambda^+} \\
\mathcal{C}_{\mathbb{T}_{\kappa^+}} \arrow[u, "g"] \arrow[r, "Y"'] & \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \arrow[u, "g^*"']
\end{tikzcd}$$

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It is enough to prove that the interpretation of $\forall \mathbf{x}(\phi \to \psi)$ is preserved by the canonical morphism $g^* : \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \to \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\lambda^+}$, since this latter is the λ^+-classifying topos of $\mathbb{T}_{\kappa^+}$, and an entirely analogous proof to a previous theorem shows that this must be the presheaf topos $\mathcal{S}et^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}}$.

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Wrapping up

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We conclude that categoricity in κ and λ implies categoricity in κ^+. Repeating the argument we conclude categoricity in κ^{++}, and so on. For a limit μ, we simply consider the diagram:

$$\begin{tikzcd}
\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \arrow[r] \arrow[d] & \mathcal{S}et[\mathbb{T}_{\kappa^{++}}]_{\kappa^{++}} \arrow[r] \arrow[d] & \dots \arrow[r] & \mathcal{S}et[\mathbb{T}_{\mu}]_{\mu} \arrow[d, dashed] \\
\mathcal{S}h(K_{\kappa}^{op}, \tau_D) \arrow[r] & \mathcal{S}h(K_{\kappa^+}^{op}, \tau_D) \arrow[r] & \dots \arrow[r] & \mathcal{E}
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\mathcal{S}h(K_{\kappa}^{op}, \tau_D) \arrow[r] & \mathcal{S}h(K_{\kappa^+}^{op}, \tau_D) \arrow[r] & \ldots \arrow[r] & \mathcal{E}
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This also serves for the case in which λ is limit.

Wrapping up

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Proof of Shelah's eventual categoricity conjecture.

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Remark

Assume *GCH*. Let $\mathcal{K}$ be an accessible category with all morphisms monomorphisms, directed bounds and amalgamation. Then the same proof outlined also proves that there is a cardinal κ such that if $\mathcal{K}$ is λ-categorical for some $\lambda \trianglerighteq \kappa$ (i.e., it has only one object of internal size λ up to isomorphism) then it is λ'-categorical for every $\lambda' \trianglerighteq \kappa$. The hypotheses on $\mathcal{K}$ can be spared assuming instead a proper class of strongly compact cardinals.

Thank you!