

A topos-theoretic proof of Shelah's eventual categoricity conjecture

Christian Espíndola

Brno, MUNI

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Abstract elementary classes

Definition

An abstract elementary class (AEC) \mathcal{K} is a category equivalent to an accessible category with directed colimits whose morphisms are monomorphisms, that admits an embedding $F : \mathcal{K} \rightarrow \mathcal{A}$ into a finitely accessible category preserving directed colimits and monomorphisms which is, in addition:

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- 2 Nearly full: for every commutative triangle:

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FACT (Lieberman-Rosicky-Vasey 2019): For each object A of an AEC, $|E(A)|$ coincides with its internal size $|A|$ defined as follows. If $r(A)$ is the least regular cardinal λ such that A is λ -presentable, then:

$$|A| = \begin{cases} \kappa & \text{if } r(A) = \kappa^+ \\ r(A) & \text{if } r(A) \text{ is limit} \end{cases}$$

Shelah's conjecture

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Conjecture

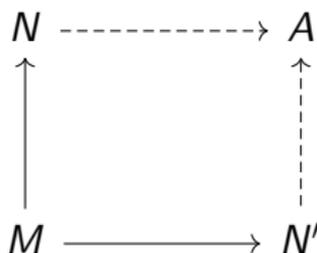
(ZFC) For every AEC there is a cardinal κ such that if the AEC is categorical in some $\lambda > \kappa$ then it is categorical in every $\lambda' > \kappa$.

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There is a proof of the conjecture (Shelah-Vasey 2019) assuming *GCH* and large cardinals. In fact, the use of large cardinals is to guarantee that the AEC will eventually satisfy the amalgamation property:

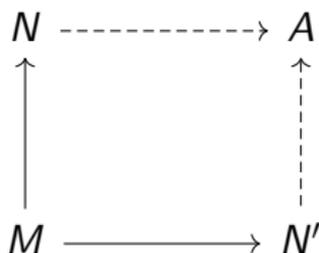


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We will give a short topos-theoretic proof of the conjecture assuming *GCH* and that the AEC satisfies amalgamation.

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Given a topos \mathcal{E} , its category of κ -points $pt_\kappa(\mathcal{E})$ (i.e., geometric morphisms to $\mathcal{S}et$ whose inverse images preserve all κ -small limits) is an accessible category with κ -directed colimits.

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Theorem

(Henry-Di Liberti) *There is a (2-)adjunction:*

$$S : \text{Acc}_\kappa \rightleftarrows \text{Top}_\kappa : pt_\kappa$$

between the category of accessible categories with κ -directed colimits and the category of κ -exact localization of presheaf toposes given by the Scott functor S_κ and the functor pt_κ .

Proof idea

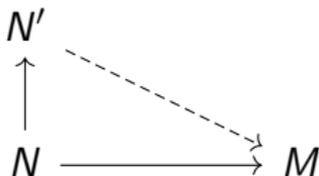
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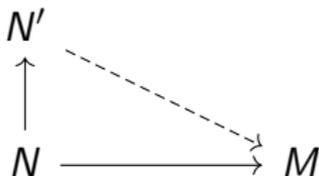
A model M is μ^+ -saturated if for every morphism $N \rightarrow N'$ between models of size μ , every morphism $N \rightarrow M$ can be extended:



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Consider the following diagram of toposes and inverse images of geometric morphisms given by restriction:

$$\begin{array}{ccc} S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) & \longrightarrow & S_{\lambda}(\mathcal{K}_{\geq \lambda}) \\ \downarrow & & \downarrow \cong \\ S_{\kappa^+}(\text{Sat}_{\kappa^+}(\mathcal{K})) & \longrightarrow & S_{\lambda}(\text{Sat}_{\lambda}(\mathcal{K})) \end{array}$$

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The AEC \mathcal{K} , as any accessible category, can be axiomatized up to equivalence by basic sentences through a theory \mathbb{T} in $\mathcal{L}_{\mu^+, \mu}$, and can have models not just in Set but in any μ -coherent category (i.e., in a category with enough Set -like properties).

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$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}} & \overset{\mu\text{-coherent}}{\dashrightarrow} & \mathcal{D} \\ \uparrow & & \uparrow \\ M_0 & \dashrightarrow & M \end{array}$$

Then $\mathcal{K}_{\geq \kappa^+}$ is axiomatized by basic sentences through \mathbb{T}_{κ^+} .

Proof idea

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The κ^+ -classifying topos of \mathbb{T}_{κ^+} (Espindola 2017), $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ is defined through the following universal property:

$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}} & \xrightarrow{\quad} & \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \\ & \searrow & \swarrow \\ & \mathcal{E} & \end{array}$$

κ^+ -small limit preserving

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Suppose $\text{Sat}_{\kappa^+}(\mathcal{K})$ is axiomatizable by $\mathbb{T}_{\kappa^+}^{\text{sat}}$. Then the morphism $\mathcal{C}_{\mathbb{T}_{\kappa^+}} \rightarrow \mathcal{C}_{\mathbb{T}_{\kappa^+}^{\text{sat}}}$ induces a morphism between the corresponding κ^+ -classifying toposes $f^* : \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \rightarrow \text{Set}[\mathbb{T}_{\kappa^+}^{\text{sat}}]_{\kappa^+}$.

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$$\begin{array}{ccccc} Set[\mathbb{T}_{\kappa^+}]_{\kappa^+} & \xrightarrow{\eta_{Set[\mathbb{T}_{\kappa^+}]_{\kappa^+}}^*} & S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) & \xrightarrow{\quad} & S_{\lambda}(\mathcal{K}_{\geq \lambda}) \\ \downarrow f^* & & \downarrow & & \downarrow \cong \\ Set[\mathbb{T}_{\kappa^+}^{sat}]_{\kappa^+} & \xrightarrow{\eta_{Set[\mathbb{T}_{\kappa^+}^{sat}]_{\kappa^+}}^*} & S_{\kappa^+}(Sat_{\kappa^+}(\mathcal{K})) & \xrightarrow{\quad} & S_{\lambda}(Sat_{\lambda}(\mathcal{K})) \end{array}$$

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We now want to deduce from the fact that the right morphism is an isomorphism that f^* is an equivalence, i.e., every model of size κ^+ is κ^+ -saturated. Then \mathcal{K} is κ^+ -categorical since there is a unique such model (Rosicky 1997)

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We will prove that in fact $Sat_{\kappa^+}(\mathcal{K})$ can be axiomatized and, moreover, if τ_D is the dense (alternatively, atomic) Grothendieck topology on K_κ^{op} :

$$Set[\mathbb{T}_{\kappa^+}^{sat}]_{\kappa^+} \cong Sh(K_\kappa^{op}, \tau_D)$$

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This is based in the following:

Theorem

Let \mathbb{T}_{κ} axiomatize $\mathcal{K}_{\geq \kappa}$. Then the κ^+ -classifying topos of \mathbb{T}_{κ} is equivalent to the presheaf topos $Set^{\mathcal{K}_{\kappa}}$. Moreover, the canonical embedding of the syntactic category is given by (note that $\mathcal{K}_{\kappa} \ni M : \mathcal{C}_{\mathbb{T}_{\kappa}} \rightarrow Set$):

$$\mathcal{C}_{\mathbb{T}_{\kappa}} \xrightarrow{ev} Set^{\mathcal{K}_{\kappa}}$$

$$X \longmapsto M \mapsto M(X)$$

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This proves the universal property when $\mathcal{E} = \mathit{Set}$.

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$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}_\kappa} & \xrightarrow{ev} & \mathcal{S}et^{\mathcal{K}_\kappa} \\ & \searrow N & \swarrow \text{dotted} \\ & \mathcal{E} & \\ & \downarrow E & \swarrow G \\ \mathcal{S}et^I & \xrightarrow{ev_i} & \mathcal{S}et \end{array}$$

The diagram shows a commutative square with a diagonal arrow. The top-left node is $\mathcal{C}_{\mathbb{T}_\kappa}$, the top-right node is $\mathcal{S}et^{\mathcal{K}_\kappa}$, the bottom-left node is $\mathcal{S}et^I$, and the bottom-right node is $\mathcal{S}et$. The top horizontal arrow is labeled ev . The left vertical arrow is labeled E . The bottom horizontal arrow is labeled ev_i . The right vertical arrow is labeled G . A diagonal arrow labeled N points from $\mathcal{C}_{\mathbb{T}_\kappa}$ to \mathcal{E} . A dotted arrow points from $\mathcal{S}et^{\mathcal{K}_\kappa}$ to \mathcal{E} . A dashed arrow labeled G points from $\mathcal{S}et^{\mathcal{K}_\kappa}$ to $\mathcal{S}et^I$.

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 & \downarrow E & \swarrow G \\
 & \text{Set}^I & \\
 & \xrightarrow{\text{ev}_i} & \text{Set}
 \end{array}$$

\downarrow (from $\text{Set}^{\mathcal{K}_\kappa}$ to Set)

Now every object F in $\text{Set}^{\mathcal{K}_\kappa}$ can be written as $F \cong \varinjlim_i [M_i, -]_{\text{Mod}_\lambda(\mathbb{T})}$, i.e.:

$$\varinjlim_i [\varinjlim_j [\phi_{ij}, -]_{\mathcal{C}_\mathbb{T}}, -]_{\text{Mod}_\lambda(\mathbb{T})} \cong \varinjlim_i \varprojlim_j [[\phi_{ij}, -]_{\mathcal{C}_\mathbb{T}}, -]_{\text{Mod}_\lambda(\mathbb{T})} \cong \varinjlim_i \varprojlim_j \text{ev}(\phi_{ij})$$

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Since G preserves colimits and κ^+ -small limits, we have:

$$G(F) \cong \lim_{i \rightarrow} \lim_{\leftarrow j} G \circ \text{ev}(\phi_{ij})$$

and hence G is completely determined by its values on $\text{ev}(\mathcal{C}_{\mathbb{T}_\kappa})$, which land in \mathcal{E} . Since E also preserves colimits and κ^+ -small limits, the whole G lands in \mathcal{E} .

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This proves the universal property when \mathcal{E} is κ^+ -classifying topos of \mathbb{T}_κ . Since this later satisfies the same universal property, we must have $\mathcal{E} \cong \text{Set}^{\mathcal{K}_\kappa}$. This finishes the proof.

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 \downarrow & & \begin{array}{c} f^* \left(\uparrow \right) \\ \downarrow \end{array} f_* \\
 \mathcal{C}_{\mathbb{T}_{\kappa^+}} & \longrightarrow & \mathbf{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \\
 \downarrow & & \left(\uparrow \right) \\
 \mathcal{C}_{\mathbb{T}_{\kappa^+}^{sat}} & \longrightarrow & \mathbf{Sh}(K_{\kappa}^{op}, \tau_D)
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 \xrightarrow{M \cong \lim_{\rightarrow i} ev_{N_i}} \\
 \xrightarrow{\quad \quad \quad} \mathbf{Set}
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FACT (e.g. Johnstone's Elephant): The embedding $Sh(K_{\kappa}^{op}, \tau_D) \hookrightarrow Set^{\mathcal{K}_{\kappa}}$ factors through f_* if and only if f_* is dense ($f_*(0) = 0$, or alternatively $f^*(C) \neq 0$ for $C \neq 0$).

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If M is a saturated model of size κ^+ , for every $p : N \rightarrow N'$ in \mathcal{K}_κ , each $N \rightarrow M$ extends to some $p' : N' \rightarrow M$. This is the same as saying that $M : \mathcal{S}et^{\mathcal{K}_\kappa} \rightarrow \mathcal{S}et$ maps $p^* : [N', -] \rightarrow [N, -]$ to an epimorphism, since:

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It follows that M factors through $a : \mathcal{S}et^{\mathcal{K}_\kappa} \rightarrow \mathcal{S}h(K_\kappa^{op}, \tau_D)$ and that $\mathcal{S}h(K_\kappa^{op}, \tau_D)$ is the κ^+ -classifying topos of $\mathbb{T}_{\kappa^+}^{sat}$.

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FACT (Rosicky 1997): κ^+ -saturated models exist. This can also be seen topos-theoretically by noticing that $\mathcal{S}h(K_\kappa^{op}, \tau_D)$ has enough κ^+ -points (Espindola 2017). The uniqueness of κ^+ -saturated models of size κ^+ can be seen as well by noticing that $\mathcal{S}h(K_\kappa^{op}, \tau_D)$ is two-valued and Boolean (Barr-Makkai 1987+ Espindola 2017)

Wrapping up

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$$\begin{array}{ccccc} \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} & \xrightarrow{\eta_{\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}}^*} & \mathcal{S}_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) & \longrightarrow & \mathcal{S}_\lambda(\mathcal{K}_{\geq \lambda}) \\ \downarrow f^* & & \downarrow & & \downarrow \cong \\ \mathcal{S}h(\mathcal{K}_\kappa^{op}, \mathcal{T}_D) & \xrightarrow{\eta_{\mathcal{S}h(\mathcal{K}_\kappa^{op}, \mathcal{T}_D)}^*} & \mathcal{S}_{\kappa^+}(\mathcal{S}at_{\kappa^+}(\mathcal{K})) & \longrightarrow & \mathcal{S}_\lambda(\mathcal{S}at_\lambda(\mathcal{K})) \end{array}$$

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 \downarrow f^* & & \downarrow & & \downarrow \cong \\
 \mathcal{S}h(\mathcal{K}_\kappa^{op}, \mathcal{T}_D) & \xrightarrow{\eta_{\mathcal{S}h(\mathcal{K}_\kappa^{op}, \mathcal{T}_D)}^*} & \mathcal{S}_{\kappa^+}(\mathcal{S}at_{\kappa^+}(\mathcal{K})) & \xrightarrow{\quad} & \mathcal{S}_\lambda(\mathcal{S}at_\lambda(\mathcal{K}))
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To prove that $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \cong \mathcal{S}h(\mathcal{M}od_\kappa(\mathbb{T})^{op}, \mathcal{T}_D)$ we show that the embedding $\mathcal{S}h(\mathcal{M}od_\kappa(\mathbb{T})^{op}, \mathcal{T}_D) \hookrightarrow \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ is an isomorphism, for which we in turn show that any basic sequent valid in $\mathcal{S}h(\mathcal{M}od_\kappa(\mathbb{T})^{op}, \mathcal{T}_D)$ will also be valid in $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$.

Wrapping up

$$\begin{array}{ccccc}
 \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} & \xrightarrow{\eta_{\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}}^*} & \mathcal{S}_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) & \xrightarrow{\quad} & \mathcal{S}_{\lambda}(\mathcal{K}_{\geq \lambda}) \\
 \downarrow f^* & & \downarrow & & \downarrow \cong \\
 \text{Sh}(\mathcal{K}_{\kappa}^{\text{op}}, \tau_D) & \xrightarrow{\eta_{\text{Sh}(\mathcal{K}_{\kappa}^{\text{op}}, \tau_D)}^*} & \mathcal{S}_{\kappa^+}(\text{Sat}_{\kappa^+}(\mathcal{K})) & \xrightarrow{\quad} & \mathcal{S}_{\lambda}(\text{Sat}_{\lambda}(\mathcal{K}))
 \end{array}$$

To prove that $\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \cong \text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^{\text{op}}, \tau_D)$ we show that the embedding $\text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^{\text{op}}, \tau_D) \hookrightarrow \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ is an isomorphism, for which we in turn show that any basic sequent valid in $\text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^{\text{op}}, \tau_D)$ will also be valid in $\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}$.

Inspection of the diagram shows that any such basic sequent $\forall \mathbf{x}(\phi \rightarrow \psi)$ valid in $\text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^{\text{op}}, \tau_D)$ is also valid in $\mathcal{S}_{\lambda}(\mathcal{K}_{\geq \lambda})$, and hence in the unique model of size λ .

Wrapping up

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Consider the presheaf category $\mathcal{S}et^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}}$. The interpretation of the sentence $\forall \mathbf{x}(\phi \rightarrow \psi)$ corresponds to a subobject $S \twoheadrightarrow 1$.

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FACT (Kripke-Joyal semantics): $S = 0$ if and only if for every morphism $M \rightarrow N$ in $\mathcal{K}_{\geq \kappa^+, \leq \lambda}$ there is a morphism $N \rightarrow N'$ with $N' \not\models \forall \mathbf{x}(\phi \rightarrow \psi)$.

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Assume now that \mathcal{K} is κ -categorical. Then $\mathcal{S}et^{\mathcal{K}_\kappa}$, and hence $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$, is two-valued. Thus the interpretation of $\forall \mathbf{x}(\phi \rightarrow \psi)$ in $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ corresponds to a subobject T that is either 0 or 1. It is hence enough to prove it is not 0.

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Wrapping up

Theorem

The evaluation functor:

$$ev : \mathit{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \rightarrow \mathit{Set}^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}}$$

preserves the interpretation of the sentence $\forall \mathbf{x}(\phi \rightarrow \psi)$.

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Proof.

$$\begin{array}{ccc} \mathcal{C}_{(\mathbb{T}_{\kappa^+})_{\lambda}} & \xrightarrow{\gamma'} & \text{Set}[\mathbb{T}_{\kappa^+}]_{\lambda^+} \\ \uparrow g & & \uparrow g^* \\ \mathcal{C}_{\mathbb{T}_{\kappa^+}} & \xrightarrow{\gamma} & \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \end{array}$$

Wrapping up

It is enough to prove that the interpretation of $\forall \mathbf{x}(\phi \rightarrow \psi)$ is preserved by the canonical morphism $g^* : \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \rightarrow \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\lambda^+}$, since this latter is the λ^+ -classifying topos of \mathbb{T}_{κ^+} , and an entirely analogous proof to a previous theorem shows that this must be the presheaf topos $\mathcal{S}et^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}}$.

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Wrapping up

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We conclude that categoricity in κ and λ implies categoricity in κ^+ . Repeating the argument we conclude categoricity in κ^{++} , and so on. For a limit μ , we simply consider the diagram:

$$\begin{array}{ccccccc} \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} & \longrightarrow & \text{Set}[\mathbb{T}_{\kappa^{++}}]_{\kappa^{++}} & \longrightarrow & \dots & \longrightarrow & \text{Set}[\mathbb{T}_{\mu}]_{\mu} \\ \downarrow & & \downarrow & & & & \vdots \\ \text{Sh}(K_{\kappa}^{\text{op}}, \mathcal{T}_D) & \longrightarrow & \text{Sh}(K_{\kappa^+}^{\text{op}}, \mathcal{T}_D) & \longrightarrow & \dots & \longrightarrow & \mathcal{E} \end{array}$$

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This also serves for the case in which λ is limit.

Wrapping up

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Proof of Shelah's eventual categoricity conjecture.

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Remark

Assume *GCH*. Let \mathcal{K} be an accessible category with all morphisms monomorphisms, directed bounds and amalgamation. Then the same proof outlined also proves that there is a cardinal κ such that if \mathcal{K} is λ -categorical for some $\lambda \geq \kappa$ (i.e., it has only one object of internal size λ up to isomorphism) then it is λ' -categorical for every $\lambda' \geq \kappa$. The hypotheses on \mathcal{K} can be spared assuming instead a proper class of strongly compact cardinals.

Thank you!