Distributive Laws for Relative Monads

Algebra Seminar
Masaryk University, Brno

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University of Leeds

Introduction

This talk is based on the preprint:

Distributive laws for relative monads, `arXiv:2007.12982`, 2020.
(G. Lobbia)

Theorem

Let $\mathcal{K}$ be a 2-category, (X, I, T) a relative monad in $\mathcal{K}$ and (S, S_0) a compatible monad with I. The following are equivalent:

- *a relative distributive law of T over (S, S_0);*
- *a lifting $\hat{T}$ of T to the algebras of (S, S_0);*
- *a lifting $\tilde{S}$ of S to the relative right modules of T.*

Today: $\mathcal{K} = \mathbf{Cat}, \mathbf{CAT}$

Outline of the talk

1. **Distributive Laws** (for monads);
2. **Relative Monads**;
3. **Relative Distributive Laws**
   (i.e. distributive laws between a monad and a relative monad).

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Distributive Laws

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Definition (Beck)

Let (S, m, s) and (T, n, t) be monads on $\mathbb{C}$. A **distributive law** *of T over S consists of a natural transformation $d\colon ST \to TS$ such that:*

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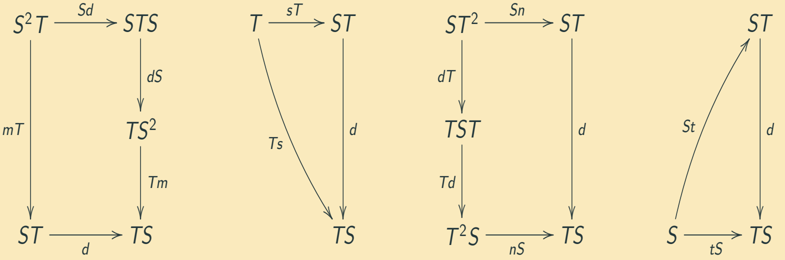

▸ Relative Distributive Laws

$d\colon ST \to TS$ distributive law

$\Rightarrow$ TS has a monad structure given by

$$TSTS \xrightarrow{TdS} T^2S^2 \xrightarrow{nS} TS^2 \xrightarrow{Tm} TS \qquad 1_{\mathbb{C}} \xrightarrow{s} S \xrightarrow{tS} TS$$

Theorem (Beck)

Let S and T be two monads on $\mathbb{C}$. TFAE

- *A* **distributive law** *$d\colon ST \to TS$;*
- *A* **lifting** *of T to S-algebras $\hat{T}\colon S\text{-}\mathcal{A}lg \to S\text{-}\mathcal{A}lg$;*
- *An* **extension** *$\tilde{S}\colon \mathcal{K}l(T) \to \mathcal{K}l(T)$ of S to the Kleisli category $\mathcal{K}l(T)$;*
- *A monad structure on TS that is* **compatible** *with S and T.*

Usual strategy: find a lifting to algebras $\Rightarrow$ get distributive law
Then extensions to Kleisli and composite monad.

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Example 1

$P =$ power set monad, $S =$ monad of monoids.
$\Rightarrow$ $S\text{-Alg} =$ **Mon** and $\mathrm{Kl}(P) =$ **Rel** category of sets and relations

$$\begin{aligned}\hat{P}\colon \mathbf{Mon} &\longrightarrow \mathbf{Mon}\\ M &\longmapsto PM = \{A \mid A \subseteq M\}\end{aligned}$$

where PM has the monoid structure:

$$A \cdot B := \{a \cdot b \mid a \in A \,\mathrm{and}\, b \in B\}$$

$\Rightarrow$ There is a distributive law $d\colon SP \to PS$, an extension $\tilde{S}\colon \mathbf{Rel} \to \mathbf{Rel}$ and a monad structure on PS.

$$\begin{aligned}d_X : SPX &\longrightarrow PSX\\ A_1 \dots A_n &\longmapsto \{a_1 \dots a_n \mid a_i \in A_i\}\end{aligned}.$$

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$T =$ monad of abelian groups, $S =$ monad of monoids.

For $X \in \mathbf{Set}$

$$SX = \{x_1 \cdots x_n \mid x_i \in X\}$$

$$TX = \left\{ \sum_{i=1}^{n} a_i x_i \mid a_i \in \mathbb{Z} \text{ and } x_i \in X \right\}$$

$\Rightarrow$ a distributive law $d : ST \to TS$ of T on S:

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Relative Monads

Why Relative Monads?

Problem: Let $\mathbb{C}$ be a small category, then $P(\mathbb{C}) := \mathbf{Cat}(\mathbb{C}^{op}, \mathbf{Set})$ is just locally small.

$$P\colon \mathbf{Cat} \longrightarrow \mathbf{CAT}$$

Relative Monads generalise the concept of monad to functors defined on a subcategory.

Aim: Have a new version of distributive laws describing the lifting $\hat{P}$ given by Day's convolution product.

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$P =$ power set monad.

- $\mathrm{Kl}(P) =$ category of sets and relations;
- P-Alg$=$ sup-semilatices.

What if we want to consider relations/sup-semilatices with an *upper bound* on cardinality of sets? Or even a set theory where PX is a class?

Problem: $P \colon \mathbf{Set}_{\leq\kappa} \to \mathbf{Set}$ is not an endofunctor.

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Relative Monads

Definition

A **relative monad** T *over* $I : \mathbb{C}_0 \to \mathbb{C}$ *consists of:*

- $TX \in \mathbb{C}$, *for every* $X \in \mathbb{C}_0$;
- *functions* $(-)^{\dagger}_{X,Y} : \mathbb{C}(IX, TY) \to \mathbb{C}(TX, TY)$ *for* $X, Y \in \mathbb{C}_0$;
- *morphisms* $t_X : IX \to TX$ *in* $\mathbb{C}$ *for* $X \in \mathbb{C}_0$;

such that:

Associativity: $(g^{\dagger} \cdot f)^{\dagger} = g^{\dagger} \cdot f^{\dagger}$ *(for* $f : IX \to TY$, $g : IY \to TZ$*)*;

Left Unity: $f = f^{\dagger} \cdot t_X$ *(for* $f : IX \to TY$*)*;

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Examples

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$$\begin{array}{rccc} t_X : & IX & \longrightarrow & PX \\ & x & \longmapsto & \{x\} \end{array} \qquad \begin{array}{rccc} f : & IX & \longrightarrow & PY \\ \hline f^\dagger : & PX & \longrightarrow & PY \\ & J & \longmapsto & \bigcup_{j\in J} f(j) \end{array}$$

2. $I\colon \mathbf{Fin} \hookrightarrow \mathbf{Set}$ inclusion, $Tn := \mathbf{Set}(In, R)$ with R ring,

$$\begin{array}{rccc} t_n : & In & \longrightarrow & \mathbf{Set}(In, R) \\ & i & \longmapsto & \delta_i \end{array} \qquad \begin{array}{rcl} f : In & \longrightarrow & \mathbf{Set}(Im, R) \\ \hline f^\dagger : Tn & \longrightarrow & \mathbf{Set}(Im, R) \\ \alpha & \longmapsto & \sum_{i\in n} \alpha(i)\cdot f(i)(-) \end{array}$$

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Relative Monads generalise Monads

Relative Monads with $I = 1$	Monads
$(-)^{\dagger}_{X,Y} \colon \mathbb{C}(X, SY) \to \mathbb{C}(SX, SY)$	$m \colon S^2 \to S$
$(g^{\dagger} \cdot f)^{\dagger} = g^{\dagger} \cdot f^{\dagger}$	Associativity
$f = f^{\dagger} \cdot s_X$ and $s_X^{\dagger} = 1_{SX}$	Left/Right Unit Law

Proof.

(Manes)

($\Leftarrow$) For any $f \colon X \to SY$, we define $f^{\dagger}$ as $m_Y \cdot Sf \colon SX \to SY$;

($\Rightarrow$) Given an extension $(-)^{\dagger}$ we define m_X as $(1_{SX})^{\dagger} \colon S^2X \to SX$;

Using unity, and axioms for a relative monad we can prove that these constructions are inverse of each other. □

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Relative Algebras

Definition

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with maps $(-)^A_X \colon \mathbb{C}(IX, A) \to \mathbb{C}(TX, A)$

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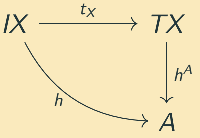

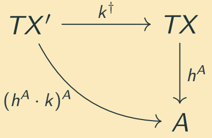

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Relative Distributive Laws

When can we talk about relative distributive laws?

We want a relative monad I, $T\colon \mathbb{C}_0 \to \mathbb{C}$ and a monad $S\colon \mathbb{C} \to \mathbb{C}$ that *restrict nicely* to $\mathbb{C}_0$, i.e.

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Let $I\colon \mathbb{C}_0 \to \mathbb{C}$ be a functor. We define a **compatible monad with** I *as a pair of monads $S_0\colon \mathbb{C}_0 \to \mathbb{C}_0$ and $S\colon \mathbb{C} \to \mathbb{C}$ such that $SI = IS_0$, $mI = Im_0$ and $sI = Is_0$.*

We will define a relative distributive law of a relative monad T on a compatible monad (S, S_0) with I.

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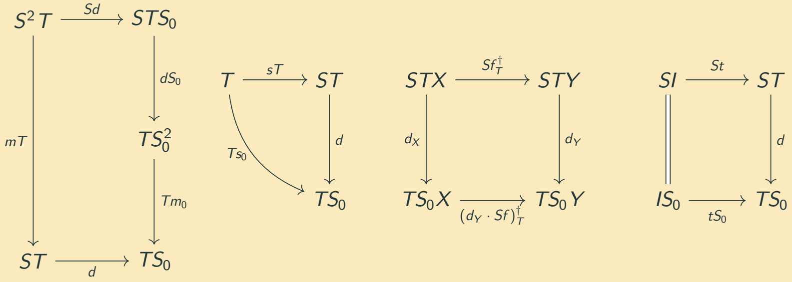
Relative Distributive Laws

▸ Distributive Laws

Definition

$I, T \colon \mathbb{C}_0 \to \mathbb{C}$ relative monad, (S, S_0) compatible with I. A **relative distributive law** *of T over (S, S_0) is a transformation $d \colon ST \to TS_0$ satisfying four axioms (for any $f \colon IX \to TY$):*



Beck-like Theorem

Theorem (Lobbia)

Given a relative monad $I, T: \mathbb{C}_0 \to \mathbb{C}$ and a compatible monad (S, S_0) with I, TFAE:

(1) *A* **relative distributive law** $d : ST \to TS_0$;

(2) *A* **lifting** $\hat{T}: S_0\text{-}Alg \to S\text{-}Alg$ *of* T *to the algebras of* S_0 *and* S;

(3) *An* **extension** $\tilde{S}: Kl(T) \to Kl(T)$ *of* S *to the Kleisli of* T.

Proof.

(1) $\Leftrightarrow$ (2) direct proof.
(1) $\Leftrightarrow$ (3) similar to *The formal theory of monads* by Street (see next slide). □

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Relative distributive laws ⇔ Extension to Kleisli

Proof.

(Sketch) Define the 2-category of relative monads $\mathbf{Rel}(\mathbf{Cat})$.

Define another 2-category $\mathbf{Ext}(\mathbf{Cat})$ with relative monads as objects and extensions to Kleisli as morphisms.

$$\mathbf{Rel}(\mathbf{Cat}) \cong \mathbf{Ext}(\mathbf{Cat})$$

$$\Rightarrow \ \mathbf{Mnd}(\mathbf{Rel}(\mathbf{Cat})) \cong \mathbf{Mnd}(\mathbf{Ext}(\mathbf{Cat}))$$

Objects of $\mathbf{Mnd}(\mathbf{Rel}(\mathbf{Cat}))$ = relative distributive laws;

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Differences with distributive laws

- Duality does not hold: relative monads in $\mathcal{K}^{op}$ are not the same as relative monads in $\mathcal{K}$;
- Extensions to Kleisli need new axioms: we require that $U\colon \mathrm{Kl}(T) \to \mathbb{C}$ is a monad morphism and that $t\colon I \to UJ_0$ is a monad transformation (and another axiom).

	Relative Algebras (left modules)	Relative Kleisli (right modules)
Relative adjunction	$\surd$	X
Equivalence with **Rel**($\mathcal{K}$)	X	$\surd$
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Example

$T =$ power set relative monad, $S =$ monad of monoids, $S_\kappa = S \upharpoonright \mathbf{Set}_{\leq\kappa}$.

For $X \in \mathbf{Set}_{\leq\kappa}$ and $Y \in \mathbf{Set}$

$$SY = \{y_1 \cdots y_n \mid y_i \in Y,\, n \in \mathbb{N}\}$$

$$S_\kappa X = \{x_1 \cdots x_n \mid x_i \in X,\, n \in \mathbb{N}\}$$

$$TX = P(X) = \{A \mid A \subseteq X\}$$

$\Rightarrow$ a *relative* distributive law $d : ST \to TS$ of T on (S, S_k):

$$\begin{aligned} d_X : STX &\longrightarrow TS_\kappa X \\ A_1 \ldots A_n &\longmapsto \{a_1 \ldots a_n \mid a_i \in A_i\} \end{aligned}.$$

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Applications

- There is a lifting of the power set relative monad to $\mathbf{Mon}_{\leq \kappa} \hookrightarrow \mathbf{Mon}$.

- There exists an extension of the free monoid monad to the category of relations over sets with cardinality $\leq \kappa$.

- The monad S of pointed sets is compatible with $I\colon \mathbf{Fin} \to \mathbf{Set}$. The relative monad $Vn := \mathbf{Set}(In, K)$ has a lifting $\hat{V}\colon \mathbf{Fin}_* \to \mathbf{Set}_*$.

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Future Work

- Prove that a relative distributive law of T over (S, S_0) is equivalent to a relative monad structure on TS_0 *compatible* with T and (S, S_0);
- Extend this work to relative **pseudo**monads, define distributive laws between a relative pseudomonad and a 2-monad;
- Possible connection with Lawvere Theories,
  MEMO: Lawvere Theories are equivalent to finitary monads.

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