Accessible infinity cosmoi (w/ Steve Lack)

Intro: How do we know "alg cats" are cocomplete? Eg. Grp, Rng...

One way: Grp has limits (easy - lims well behaved) + Grp is accessible (ok - filt cols well behaved)

Thm) Accessible + complete $\Rightarrow$ cocomplete. (still some work)

Why do "alg functors" have left adjoints? Eg. Rng $\longrightarrow$ Grp

- Clearly pres lims (easy), filt colims (easy)

Thm) $U: A \longrightarrow B$ accessible (pres λ-filt cols for some λ) lim pres bet. acc. complete cats, then U has left adj.

Pass from sets w structure $\Rightarrow$ cats w/ str, ∞-cats w/ str?

- What about 2-cat of monoidal cats & strong monoidal functors?
- ∞-cosmos QCat prod of quasicats w/ finite prods?

Goal: answer these sorts of questions, focusing on ∞-cosmos world.

Overview:

- ∞-cosmoi
- Access. ∞-cosmoi
- Adj. Fun. theorems & cocompl...

∞-Cosmoi (Riehl - Verity)

- ∞-cosmos $\mathcal{C}$ simplicially enr. cat w class of "isofibrations" $A \twoheadrightarrow B$.
- $\mathcal{C}(a,b)$ a quasicat.
- $\mathcal{C}$ has some lims – eg prods, powers, pullbacks of isofibrations, ... + some further props.

- Basic examples:
  - $q\mathcal{C}at$: simp. enr. cat of quasicats...
  - CSS : complete Segal spaces
  - $\mathcal{C}at$: cats & functors

  $\left.\begin{array}{l}q\mathcal{C}at\\ \text{CSS}\end{array}\right\}$ $(\infty,1)$-cats

- Each quasicat X has homotopy cat $\pi_1 X$: same 0-cells.
- So each ∞-cosmos $\mathcal{C}$ has homotopy 2-cat $h\mathcal{C}$ w/ same obs as $\mathcal{C}$ & $(h\mathcal{C})(a,b) = h(\mathcal{C}(a,b))$.
  So $h\mathcal{C}$ has same obs & arrows as $\mathcal{C}$, 2-cells are e-classes.
- Define $f : X \to Y \in \mathcal{C}$ to be an equiv / l. or r. adjoint $\Leftrightarrow$ it is one in $h\mathcal{C}$ (2-cat).
- Surprisingly: captures correct

∞-categorical notions, using 2-cat. theory

- Can talk about limits:
$A \in \mathcal{C}$ has lims of shape X if diagonal $A \xrightarrow{A^!} A^X$ ← power
has right adjoint.

- Then can prove eg. "rapl" using usual 2-cat proof.
Applies to many notions of $(\infty,1)$-cat: this is what model independent means.

- Enriched functor $F : A \to B$ between ∞-cosmoi if it pres. isofibs & defining limits.

called a cosmological functor.

Constructions of ∞-cosmoi

- $\mathcal{C}$ model str enriched over Joyal model str on SSet, with all Fib ob are cofib., then $\mathcal{C}_{fib}$ is ∞-cosmos.

$\Rightarrow$ QCat, CSS, Cat ∞-cosmoi

- $\mathcal{C}$ ∞-cosmos $\Rightarrow$ $\mathcal{C}^{\twoheadrightarrow}$ ∞-cosmos of isofibs
- $\Rightarrow$ $\mathcal{C}^{\twoheadrightarrow}/A$ isofibs over A.
- $\mathrm{Rari}(\mathcal{C}) \hookrightarrow \mathcal{C}^{\twoheadrightarrow}$ sub ∞-cosmos of maps having a right adj. right inverse (i.e. counit invertible) morphisms of adjunctions ("mate invertible")
NOT full!
- If
$$\begin{tikzcd} \mathcal{K}' \arrow[r, "F'"] \arrow[d, "i'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{K} \arrow[d, "i"] \\ K' \arrow[r, "F"] & K \end{tikzcd}$$
is pb of simp. cats w' $\mathcal{K}, K, K'$ are ∞-cosmoi, & F cosmo., i cosmo. embedding,
then $\mathcal{K}'$ is ∞-cosmos, F' cosmo & i' cosm. embedding.

- From these, can construct $\mathcal{C}_{X} \hookrightarrow \mathcal{C}$ ∞-cosmos of ∞-cats w' lims or cols of shape X
- $\mathrm{Cart}(\mathcal{C}) \hookrightarrow \mathcal{C}^{\twoheadrightarrow}$ ∞-cosmos of cart. fibs, ---.

Accessible infinity cosmoi - w S. Lack

Def) ∞-cosmos $\mathcal{C}$ is acc. if

(1) ~~ϕ~~ $\mathcal{C}$ acc. as sSet-enr. cat:
- this means: $\mathcal{C}_0$ is accessible:
- each powering functor $(-)^X : \mathcal{C}_0 \to \mathcal{C}$ by a sset X is accessible.

(2) Full subcats of $\mathrm{Arr}(\mathcal{C})$ containing isofibs, equivalences are accessible & accessibly embedded.

Remark: - Isofibs accessible $\Rightarrow$ $\mathcal{C}^{\natural}$ accessible.

- Equivs are accessible implies (in fact, is equiv. to)

$\exists \lambda$ st λ-filt. colimits are homotopy colimits:

if J λ-filt cat & $D : J \to \mathcal{C}$, let $Q \to \Delta 1 \in [J, \mathrm{sSet}]$ be cofib. rep. of $\Delta 1$:

then

$$\mathcal{C}(\mathrm{colim}\, D, X) \cong [J, \mathrm{sSet}](\Delta 1, \mathcal{C}(D-, X)) \longrightarrow [J, \mathrm{sSet}](Q, \mathcal{C}(D-, X))$$

is an equiv. of q-cats.

IMPORTANT!

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Constructions of a ∞-cosmoi

- $\mathcal{C}$ combi model str enriched over Joyal model str on SSet, with all Fib obs are cofib, then $\mathcal{C}_{fib}$ acc is ∞-cosmos.

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- $\mathrm{Rari}(\mathcal{C}) \hookrightarrow \mathcal{C}^{\twoheadrightarrow}$ sub acc. ∞-cosmos of maps having a right adj. right inverse (i.e. counit invertible) morphisms of adjunctions ('mate invertible') NOT full! Tricky!

- If
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Makkai-Paré limth.
is pb of simp. cats w' $\mathcal{X}, \mathcal{K}, \mathcal{K}'$ are acc. ∞-cosmoi, & F acc. cosmo., i acc cosmo. then $\mathcal{X}'$ is acc. ∞-cosmos, F' acc. embedding acc. cosmo & i' cosm. embedding.

- From these, can construct $\mathcal{C}_{\cdots,X} \longrightarrow \mathcal{C}$ ∞-cosmos of ∞-cats w' lims or cols of shape X
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Key result

$\mathcal{C}$ acc ∞-cosmos $\Rightarrow$ $Ravi(\mathcal{C})$ acc.

Idea of proof

- Easier case (w/ sim. proof!)
- let $Disc(\mathcal{C}) \hookrightarrow \mathcal{C}$ be ∞-cosmos of discrete obs:

  X is disc. $\Leftrightarrow$ $\mathcal{C}(A,X)$ Kan complex $\forall A$.

- Claim X discrete $\Leftrightarrow$
- $\mathcal{C}(B,X)$ is disc for λ-pres B for some λ
- $\exists \lambda$ st λ-filt colims are h. colims.

  Write $A = \operatorname{colim}_{i \in I} A_i$, I λ-filt.

- Then $\mathcal{C}(A,X) = \lim_{i \in I} \mathcal{C}(A_i, X)$

  homotopy lim of disc obs in Q-cat.

  $Disc(Q\text{-cat}) \hookrightarrow Q\text{-cat}$ cosm.

  embedding $\Rightarrow$ closed under homotopy limits

  $\Rightarrow$ as $\mathcal{C}(A_i, X) \in Disc(Q\text{-cat})$

  $\Rightarrow$ $\mathcal{C}(A,X)$ discrete $\Rightarrow$ X disc.

So

$$\begin{tikzcd} \mathrm{Disc}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{C}_\lambda, \mathrm{Disc}(\mathcal{Q}\text{-}\mathcal{C}at)]} \arrow[d] \\ \mathcal{C} \arrow[r] & {[\mathcal{C}_\lambda, \mathcal{Q}\text{-}\mathcal{C}at]} \end{tikzcd}$$

is pb of acc. λ-pres cats $\Rightarrow$ by MP th. so an acc ∞-cosmos.

Applications

- Direct app of "Adjoint Fun. Ths" (2020) w/ Steve & L. Vokřínek.
- Done for monoidal model cat V.
- Results interp. for SSets w/ Joyal model str, spec. from acc. simp. cats w/ cof w. lims $\Rightarrow$ acc. inf. cosmoi.

Th 1) Let A be an acc. ∞-cosmos. Then A has flex weighted htpy colimits (sense of Riehl-Verity)

Th 2) Let A, B be acc. ∞-cosmoi & $U: A \longrightarrow B$ acc. cosmological functor. Then U has homotopical l. adj in sense:
For each $b \in B$ $\exists$ Fb & $b \xrightarrow{\eta_b} UFb$ st $\forall a \in A$

$$A(Fb, a) \longrightarrow B(b, Ua)$$

$$Fb \xrightarrow{\alpha} a \longmapsto b \xrightarrow{\eta} UFb \xrightarrow{\alpha} Ua$$

is a surj. equiv. of quasicats.

Summary:

- Acc. ∞-cosmoi
- Most, if not all, ∞-cosmoi are accessible
- Hence admit homotopical colimits & simple htpical adjoint functor.

Thanks!