# Model categories of lcc categories and the gros model of dependent type theory

Martin E. Bidlingmaier

Aarhus University

# In this talk

An application of model category theory to the coherence problem of type theory.

1. Dependent type theory
   - Dependent type theory as essentially algebraic theory
   - Lcc categories and “gros” semantics
   - The coherence problem
2. Lcc sketches
   - Model categories of marked objects
   - Bousfield localization at “axioms”
3. Strict lcc categories
   - $\mathrm{sLcc}$ as category of algebraically fibrant objects
   - Partial interpretation of type theory in $\mathrm{sLcc}$
4. Algebraically cofibrant strict lcc categories
   - Strictification
   - A solution to the coherence problem
5. Conclusion & open problems

# Dependent type theory

Extensional (!) dependent type theory is an essentially algebraic theory (CwFs).
Sorts:

- contexts $\Gamma, \Delta$
- types $\sigma, \tau$
- terms $s, t$
- morphisms $f, g$.

Operations:

- Every type $\sigma$ is assigned its context: $\Gamma \vdash \sigma$.
- Every term $s$ is assigned its context & type: $\Gamma \vdash s : \sigma$.
- Contexts and morphisms form a category.
- We can *substitute* terms and types along morphisms:

$$f : \Delta \to \Gamma \text{ and } \Delta \vdash s : \sigma \implies \Gamma \vdash f(s) : f(\sigma)$$

- Context extension:

$$p : \Gamma \to \Gamma.\sigma \qquad\qquad \Gamma.\sigma \vdash v : p(\sigma)$$

- Type formers:

$$\Gamma \vdash s_1, s_2 : \sigma \implies \Gamma \vdash \mathrm{Eq}\, s_1\, s_2 \qquad\qquad \Gamma.\sigma \vdash \tau \implies \Gamma \vdash \Pi_\sigma\, \tau$$

# “Gros” semantics in lcc categories

## Definition

A category $\mathcal{C}$ is *locally cartesian closed (lcc)* if it has all finite limits and all pullback functors $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ have adjoints $\Sigma_f \dashv f^* \dashv \Pi_f$.

Examples: Elementary toposes, ex/lex completions thereof.

If $f : X \to 1$ and $\phi \hookrightarrow X$:

$$\Pi_f(\phi) = \forall x : X, \phi(x) \qquad\qquad \Sigma_f(\phi) \approx \exists x : X, \phi(x)$$

The category of all lcc categories should be model of type theory:

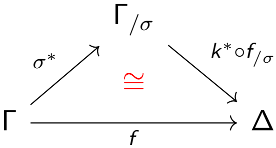
- Contexts $\Gamma$ are lcc categories
- Types $\Gamma \vdash \sigma$ are objects $\sigma \in \operatorname{Ob}\Gamma$
- Terms $\Gamma \vdash s : \sigma$ are morphisms $s : 1 \to \sigma$ in $\Gamma$.
- Morphisms $f : \Delta \to \Gamma$ are lcc functors.
- $\operatorname{Eq} s_1\, s_2$ is equalizer of $1 \overset{s_1}{\underset{s_2}{\rightrightarrows}} \sigma$.
- $\Gamma.\sigma = \Gamma_{/\sigma}$
- $\Pi_\sigma\, \tau$ is given by $\tau \in \operatorname{Ob}\Gamma_{/\sigma}$ and $\Pi_\sigma : \Gamma_{/\sigma} \to \Gamma$.

# Coherence problems

- Substitution does not commute *strictly* with type formers:

$$f(\mathrm{Eq}\, s_1\, s_2) \cong \mathrm{Eq}\, f(s_1)\, f(s_2)$$

- Context extension must have 1-categorical universal property, not bicategorical:




$$k : 1 \to f(\sigma) \text{ in } \Delta$$

# Model categories

## Definition

A *model category* is a category $\mathcal{M}$ equipped with three classes of maps *cofibrations, fibrations, weak equivalences* subject to a number of axioms.

- Presents higher localization $\mathcal{W}^{-1}\mathcal{M}$ at weak equivalences $\mathcal{W}$.
- Cofibrations are "good monos", fibrations "good epis".
- Can always factor as $X \overset{\sim}{\rightarrowtail} X' \twoheadrightarrow Y$ and $X \rightarrowtail Y' \overset{\sim}{\twoheadrightarrow} Y$.
- In particular: $X \to 1$ (fibrant replacement) and $0 \to X$ (cofibrant replacement).
- *Combinatorial* model category: Locally presentable, generated by *sets* of (trivial) cofibrations.

Idea:

- Lcc categories form a higher category, presented as model category.
- Coherence problems are about underlying 1-category.
- Different (but Quillen equivalent) presentations as model categories can vary in underlying 1-categories.
  $\to$ find good presentation!

# Marked objects

## Definition (Isaev)

Let $\mathcal{C}$ be a category and let $i : I \to \mathcal{C}$ be a diagram in $\mathcal{C}$. An *(i-)marked object* is given by an object $U(X)$ in $\mathcal{C}$ and a set of morphisms of the form $k : i(K) \to U(X)$, the *marked* morphisms, such that

$$k : i(K_2) \to U(X) \text{ is marked and } f : K_1 \to K_2 \implies k \circ i(f) \text{ is marked.}$$

A morphism of $i$-marked objects is a marking-preserving morphism in $\mathcal{C}$.

- For us: $I \subseteq \mathcal{C} = \mathrm{Cat}$ is subcategory, $\mathrm{Cat}^i = \mathrm{Cat}^I$.
- The forgetful functor $U : \mathcal{C}^I \to \mathcal{C}$ has both adjoints:

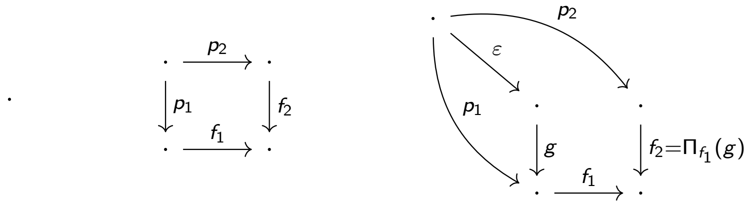
$$(-)^\flat \dashv U \dashv (-)^\sharp$$

  Minimal$^\flat$ and maximal$^\sharp$ markings.

# Lcc shapes

## Definition

$I_{\mathrm{Lcc}} \subseteq \mathrm{Cat}$ is given by the categories




$\mathrm{Tm}$ $\quad$ $\mathrm{Pb}$ $\quad$ $\mathrm{Pi}$

and the inclusion $\mathrm{Pb} \subseteq \mathrm{Pi}$.

Intuition:

- Marked maps $\mathrm{Tm} \to \mathcal{C}$: terminal objects,
- Marked maps $\mathrm{Pb} \to \mathcal{C}$: pullback squares,
- Marked maps $\mathrm{Pi} \to \mathcal{C}$: dependent products; $f_2 = \Pi_{f_1}(g)$.

# Model category structure on marked objects

## Theorem (Isaev)

*Let $\mathcal{M}$ be a combinatorial model category and let $i : I \to \mathcal{M}$ be a diagram in $\mathcal{M}$ such that every object in the image of $i$ is cofibrant. Then the following defines the structure of a combinatorial model category on $\mathcal{M}^i$:*

- *$f$ in $\mathcal{M}^i$ is a cofibration in $\mathcal{M}^i$ iff $U(f)$ is cofibration in $\mathcal{M}$.*
- *$f$ in $\mathcal{M}^i$ is a weak equivalence iff $U(f)$ is a weak equivalence in $\mathcal{M}$ and $f$ reflects markings up to homotopy.*

*A marked object $X$ is fibrant iff $U(X)$ is fibrant in $\mathcal{M}$ and the markings of $X$ are stable under homotopy. The adjunctions $(-)^\flat \dashv U$ and $U \dashv (-)^\sharp$ are Quillen adjunctions.*

$\mathrm{Cat}^{\mathrm{lcc}} := \mathrm{Cat}^{I_{\mathrm{Lcc}}}$ inherits model structure from canonical model structure on $\mathrm{Cat}$:

- Cofibrations are the functors that are injective on objects.
- Weak equivalences are marking-reflecting equivalences of categories.
- Fibrant objects are those where markings are stable under isomorphism of diagrams.

# Lcc sketches

Idea: Restrict fibrant objects to those that are actual lcc categories.
$\rightarrow$ add more trivial cofibrations.

## Definition

The model category $\mathrm{Lcc}$ of *lcc sketches* is the left Bousfield localization $S^{-1}\mathrm{Cat}^{\mathrm{lcc}}$, where $S$ consists of:

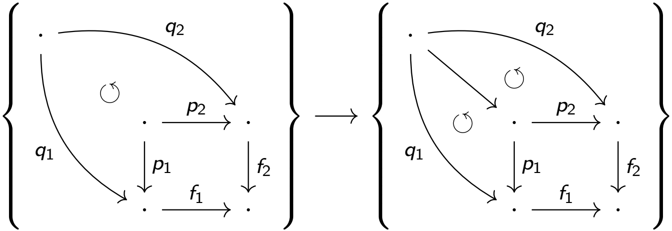
- $$\left\{ \begin{array}{ccc} \cdot & \xrightarrow{p_2} & \cdot \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f_2} \\ \cdot & \xrightarrow{f_1} & \cdot \end{array} \right\} \longrightarrow \left\{ \begin{array}{ccc} \cdot & \xrightarrow{p_2} & \cdot \\ \downarrow{\scriptstyle p_1} & \circlearrowleft & \downarrow{\scriptstyle f_2} \\ \cdot & \xrightarrow{f_1} & \cdot \end{array} \right\}$$

  *"Pullback squares commute."*

- $$\left\{ \begin{array}{ccc} & & \cdot \\ & & \downarrow{\scriptstyle f_2} \\ \cdot & \xrightarrow{f_1} & \cdot \end{array} \right\} \longrightarrow \left\{ \begin{array}{ccc} \cdot & \xrightarrow{p_2} & \cdot \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f_2} \\ \cdot & \xrightarrow{f_1} & \cdot \end{array} \right\}$$

  *"All pullbacks exist."*

# Lcc sketches (cont.)

- 


  *"Pullback squares satisfy the universal property."*
- ... and similar morphisms for terminal objects and dependent products.

Lifts against trivial cofibrations are *unique up to homotopy* $\implies$ uniqueness of factorization is automatic!

## Theorem

$\mathrm{Lcc}$ *presents the higher category of lcc categories.*

- *The fibrant objects are precisely lcc categories, with diagrams marked iff they satisfy universal property.*
- *Weak equivalences of fibrant objects are the equivalences of underlying categories.*
- $\mathrm{Ho}\,\mathrm{Lcc}$ *is the category of lcc categories and isomorphism classes of lcc functors.*
- *The homotopy function complexes of fibrant lcc sketches are the groupoids of lcc functors and their isomorphisms.* □

The subcategory of fibrant lcc sketches is what's usually called "category of lcc categories".
Interpretation of type theory would depend on AC, suffers from coherence issues.

## Question

Describe a set of generating (trivial) cofibrations for $\mathrm{Lcc}$.

# Algebraically fibrant objects

## Definition

Let $J$ be a suitable (e.g. generating) set of trivial cofibrations in a model category $\mathcal{M}$. The category $\operatorname{Alg}\mathcal{M}$ of *algebraically fibrant objects* of $\mathcal{M}$ (wrt. $J$) consists of object $G(X) \in \operatorname{Ob}\mathcal{M}$ with assigned lifts

$$\begin{array}{ccc} A & \xrightarrow{\ a\ } & G(X) \\ {\scriptstyle j}\big\downarrow & \nearrow_{\ell_X(j,a)} & \\ B & & \end{array}$$

against all $j \in J$.

# Model category structure on $\mathrm{Alg}\,\mathcal{M}$

## Theorem (Bourke)

*Let $\mathcal{M}$ be a combinatorial model category. Then $G : \mathrm{Alg}\,\mathcal{M} \to \mathcal{M}$ has a left adjoint $F$. The model structure of $\mathcal{M}$ can be transferred to $\mathrm{Alg}\,\mathcal{M}$, and $(F, G)$ is a Quillen equivalence.* □

- $G$ *reflects* weak equivalences and fibrations.
- Every object in $\mathrm{Alg}\,\mathcal{M}$ is fibrant.
- $GF$ is fibrant replacement monad on $\mathcal{M}$.
- If every $X \in \mathrm{Ob}\,\mathcal{M}$ is cofibrant: $FG$ is cofibrant replacement comonad on $\mathrm{Alg}\,\mathcal{M}$.
- *Duality of property and structure.*

## Question

If $\mathcal{M}$ is $\mathrm{Gpd}$-enriched, then $\mathrm{Alg}\,\mathcal{M}$ is $\mathrm{Gpd}$-enriched. What about $\mathrm{sSet}$-enrichment?

## Question

What are the least requirements on $J$? What if the lifts $\ell(j, a)$ are not specified for all $a$?

# Strict lcc categories

## Definition

The category of *strict lcc categories* is given by

$$\mathrm{sLcc} = \mathsf{Alg}\,\mathrm{Lcc}.$$

- Preservation of assigned lifts is trivial when lifts are unique $\implies$ only choice of pullback, terminal objects, dependent products matter (but not maps induced by universal properties)
- Morphisms are *strict* lcc functors: Preserve lcc structure on the nose.
- $FG$ is cofibrant replacement: Forget assigned lcc structure, freely adjoin new structure.

## Question

Does $\mathrm{sLcc}$ coincide with Lack's model category of algebras for a 2-monad $T$, instantiated with the free lcc category monad $T$ on $\mathrm{Cat}$?

# Partial interpretation of type theory in $\mathrm{sLcc}$

- Contexts are slcc categories $\Gamma$
- Morphisms are *strict* lcc functors $f : \Delta \to \Gamma$.
- Types $\Gamma \vdash \sigma$ are objects $\sigma \in \mathrm{Ob}\,\Gamma$.
- Terms $\Gamma \vdash s : \sigma$ are morphisms $s : 1 \to \sigma$ in $\Gamma$.
- Finite limit types are defined by *canonical* finite limits in $\Gamma$.
- Strict substitution, e.g.

$$f(\mathrm{Eq}\, s_1\, s_2) = \mathrm{Eq}\, f(s_1)\, f(s_2)$$

holds because morphisms in $\mathrm{sLcc}$ preserve lifts on the nose.

# Context extensions vs slice categories

- Context extension $p : \Gamma \to \Gamma.\sigma$ is pushout

$$\begin{array}{ccc} F(\{t, \sigma\}) & \longrightarrow & F(\{v : t \to \sigma\}) \\ {\scriptstyle t \mapsto 1}\downarrow & & \downarrow \\ \Gamma & \xrightarrow{p} & \Gamma.\sigma. \end{array}$$

- If $\tau \in \mathrm{Ob}\,\Gamma.\sigma$, how to define $\Pi_\sigma \tau \in \mathrm{Ob}\,\Gamma$?
- Want to apply $\Pi_\sigma : \Gamma_{/\sigma} \to \Gamma$.
- Do $\sigma^* : \Gamma \to \Gamma_{/\sigma}$ and diagonal $d : 1 = \mathrm{id}_\sigma \to \sigma^*(\sigma)$ induce map $\Gamma.\sigma \to \Gamma_{/\sigma}$?
- No, $\sigma^*$ is map in $\mathrm{Lcc}$, not in $\mathrm{sLcc}$ (not strict).

# Strictification

$\Gamma \in \mathrm{Ob}\,\mathrm{sLcc}$ is cofibrant $\iff$ $\varepsilon : F(G(\Gamma)) \to \Gamma$ is retraction:

$$\exists \lambda : \Gamma \to F(G(\Gamma)), \varepsilon\lambda = \mathrm{id}$$

Now:

$$\begin{aligned} & \sigma^* : G(\Gamma) \to G(\Gamma_{/\sigma}) \\ \rightsquigarrow \quad & \overline{\sigma^*} : F(G(\Gamma)) \to \Gamma_{/\sigma} \\ \rightsquigarrow \quad & (\sigma^*)^s := \overline{\sigma^*} \circ \lambda : \Gamma \to F(G(\Gamma)) \to \Gamma_{/\sigma} \end{aligned}$$

## Proposition

*If $\Gamma \in \mathrm{sLcc}$ is cofibrant, then for all $f : G(\Gamma) \to G(\Delta)$ there exists $f^s : \Gamma \to \Delta$ such that $f \cong G(f^s)$.* □

Good: Have map $a : \Gamma.\sigma \to \Gamma_{/\sigma}$.
Bad: $a$ is not compatible with morphisms in $\Gamma$! Need

$$g : E \to \Gamma \implies f^s \circ g = (f \circ G(g))^s$$

# Algebraically cofibrant objects

## Definition

Let $\mathcal{M}$ be a model category and let $C$ be a cofibrant replacement comonad on $\mathcal{M}$. An *algebraically cofibrant object* of $\mathcal{M}$ is a coalgebra for $C$; the category of such objects is denoted by $\operatorname{Coa}\mathcal{M}$.

- Structure map $\lambda : X \to C(X)$ is inclusion of retract $\implies$ coalgebras are cofibrant in $\mathrm{sLcc}$.
- Coalgebra morphisms preserve $\lambda$.

## Theorem (Ching & Riehl)

*Let $\mathcal{M}$ be a combinatorial and simplicial model category. Then there exists a suitable simplicial cofibrant replacement comonad. The model category structure of $\mathcal{M}$ can be transferred to $\operatorname{Coa}\mathcal{M}$ such that $\mathcal{M}$ and $\operatorname{Coa}\mathcal{M}$ are Quillen equivalent.*

# The interpretation of type theory in Coa $\mathrm{sLcc}$

- Category of contexts is Coa $\mathrm{sLcc}$.
- Types, terms, finite limit types are interpreted as in $\mathrm{sLcc}$.
- Coa $\mathrm{sLcc} \to \mathrm{sLcc}$ commutes with context extension.

## Lemma

*There is a natural transformation of functors* $(\text{Coa}\,\mathrm{sLcc})_* \to \mathrm{sLcc}$

$$((\lambda, \Gamma, \sigma) \mapsto \Gamma.\sigma) \Rightarrow ((\lambda, \Gamma, \sigma) \mapsto \Gamma_{/\sigma}).$$

*whose components are weak equivalences in* $\mathrm{sLcc}$

- Homotopy inverses can be found in $\mathrm{Lcc}$, also strictly natural.

## Theorem

*The opposite of* Coa $\mathrm{sLcc}$ *carries cwf structure (i.e. is a model of type theory) that supports finite limit,* $\Pi$ *and* $\Sigma$ *types.*

# Recap

Coherence problems are about the underlying 1-category of model categories.
Quillen equivalent model categories can vary in underlying 1-categories.

1. *Model category of sketches*:
   Universal objects *merely exist*, no canonical choice.
   $\rightarrow$ cannot even *state* substitution stability.
2. *Algebraically fibrant objects*:
   Have canonical universal objects preserved by morphisms/substitution.
   But: Context extension $\Gamma.\sigma$ is pushout, only "correct" when $\Gamma$ is cofibrant.
3. *Algebraically cofibrant objects*:
   Cofibrancy baked into structure.
   Can strictify maps $f : G(\Gamma) \to G(\Delta)$, strictification functorial.

# Some open problems

### Question

Can some very weak variant of HoTT be interpreted in lcc quasi-categories with this technique?

### Question

Is there a model category of sketches for every 2-monad $T$ on $\mathrm{Cat}$? What about $\mathrm{sSet}$-enriched monads on $\mathrm{sSet}$?

Marked objects probably only work when $T$-algebra structure is essentially unique ($\rightarrow$ $T$ is modality), but not e.g. for monoidal categories.

### Question

Quasi-categories/Kan complexes are weird: Composition is property and preserved up to homotopy, identities are structure and preserved up to equality. Is $\mathrm{sSet}$ of the form $\mathsf{Alg}\,\mathrm{ssSet}$ (or variation) for a model category of semi-simplicial sets $\mathrm{ssSet}$?

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