The stable homotopy hypothesis

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joint work with
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Outline

- The Homotopy hypothesis
  - What is it about?
  - The Tamsamani model
- The Stable homotopy hypothesis
  - What is it about?
  - Modeling the categorical side
  - Proof of the SHH

The homotopy hypothesis

Homotopy hypothesis

Homotopy Hypothesis (Grothendieck ’83)

Topological spaces are “the same” as ∞-groupoids

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More refined version:
n-types are “the same” as n-groupoids

$$\mathrm{Ho}(\mathrm{Top}_{[0,n]}) \simeq \mathrm{Ho}(\mathrm{Gpd}^n)$$

Definitions

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- n**-groupoids** are...different things, depending on whom you ask!

n-groupoids

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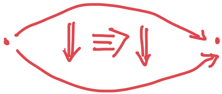

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Since all models of n-groupoids satisfy the HH, they are all equivalent for homotopy theory purposes.

Homotopy hypothesis: the idea

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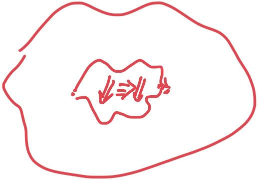

Think about the points of a space as objects, paths between them as 1-cells, homotopies between paths as 2-cells, homotopies between homotopies between paths as 3-cells, and so on.

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- $n = 1$: we have the correspondence between 1-types and groupoids given by the fundamental groupoid functor, and the realization. Eilenberg-MacLane

$$|G| \mapsfrom G$$

$$\Pi_1 : \mathrm{Top}_{[0,1]} \leftrightarrow \mathrm{Gpd}^1 : |-|$$

$X \longmapsto \pi_1 X$ is a cat with
obj = points in X
maps = htpy classes of paths in X

The Tamsamani model

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Instead, we need a more general (weaker) type of higher structure, where associativity and unitality of composites works up to higher data.

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- The sets of cells in dimension 0 up to n
- The behavior of the compositions
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- The sets of cells in dimension 0 up to n
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A natural way to do this is to use **multisimplicial sets**, since we can encode compositions via the Segal maps.

The Tamsamani model

$X : \Delta^{op} \to \mathcal{C}$

Let $X \in [\Delta^{\mathrm{op}}, \mathcal{C}]$ be a simplicial object in a category $\mathcal{C}$ with pullbacks.

Definition: Segal maps

For each $k \geq 2$, let $\nu_j : X_k \to X_1$ be induced by the map $\nu^j : [1] \to [k]$ in Δ sending 0 to $j-1$ and 1 to j.

$$\begin{tikzcd}
 & & & X_k \arrow[dll, "\nu_1"'] \arrow[d, "\nu_2"] \arrow[drrrr, "\nu_k"] & & & & & \\
 & X_1 \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & & X_1 \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & & \dots & & X_1 \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \\
X_0 & & X_0 & & \dots & & X_0 & & X_0
\end{tikzcd}$$

The **k-th Segal map** is $S_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$

$k = 2$

X_0 = "points"

X_1 = "segments" = $\overset{d_1}{\bullet} \longrightarrow \overset{d_0}{\bullet}$

X_2 = "triangles"

$X_2 \longrightarrow X_1 \times_{X_0} X_1$

picks out the composites

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Definition: Tam^n

- $\mathrm{Tam}^0 = \mathrm{Set}$, 0-equivalences $=$ bijections
- $\mathrm{Tam}^1 = \mathrm{Cat}$, 1-equivalences $=$ equivalences of categories
- for $n > 1$, Tam^n are the functors $X \in [(\Delta^{\mathrm{op}})^{n-1}, \mathrm{Cat}] \subseteq [(\Delta^{\mathrm{op}})^n, \mathrm{Set}]$ such that
  - X_0 is discrete
  - $X_k \in \mathrm{Tam}^{n-1}$ for all $k > 0$
  - for all $k \geq 2$, the Segal map $X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ is an $(n-1)$-equivalence

The Tamsamani model

$X : (\Delta^{op})^{n-1} \longrightarrow Cat$

Intuition:

- they are multi-simplicial objects, with Segal maps in all the simplicial directions
- X_0 (resp. $X_{\underset{r}{1\dots1}0}$) is the set of 0-cells (resp. r-cells for $1 \leq r \leq n-2$)
- the set of $(n-1)$ (resp. n)-cells is given by $obX_{\underset{n-1}{1\dots1}}$ (resp. $morX_{\underset{n-1}{1\dots1}}$)
- we compose cells using the Segal maps

$$X_1 \times_{X_0} X_1 \xleftarrow{\simeq} X_2 \xrightarrow{d_1} X_1$$

where $d_1 \colon [2] \to [1]$ is the face map in Δ^{op}

The Tamsamani model

Equivalences: a higher dimensional version of “fully faithful and essentially surjective”

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Definition: n-equivalences in Tam^n

- 0-equivs are bijections
- 1-equivs are equivs of categories
- for $n > 1$, an n-equivalence is a map $f\colon X \to Y$ in Tam^n such that
  - For all $a, b \in X_0$, the induced map

    $$f(a,b)\colon X(a,b) \to Y(fa, fb)$$

    hom (n-1) cats

    is an $(n-1)$-equivalence
  - $p^{(n-1)}f$ is an $(n-1)$-equivalence

truncation Functor that "[illegible] divides out" by the highest invertible cells.

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Definition: GTam^n

- $\mathrm{GTam}^0 = \mathrm{Set} \subseteq \mathrm{Set}$
- $\mathrm{GTam}^1 = \mathrm{Gpd} \subseteq \mathrm{Cat}$
- for $n > 1$, $\mathrm{GTam}^n \subseteq \mathrm{Tam}^n$ are the functors $X \in \mathrm{Tam}^n$ such that
    - $X_k \in \mathrm{GTam}^{n-1}$ for all $k > 0$
    - $p^{(n-1)}X \in \mathrm{GTam}^{n-1}$

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Homotopy hypothesis (Tamsamani)

Geometric realization $|-|: \mathrm{GTam}^n \to \mathrm{Top}_{[0,n]}$ induces an equivalence on homotopy categories.

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- contain the infinite loop spaces, and through these, characterize all (co)homology theories by Brown's Representability Theorem,
- can be interpreted as an abelianization of spaces by May's Recognition Theorem.

Spectra: homotopy groups

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Definition: homotopy groups of spectra

The i**th homotopy group** of a spectrum X is defined as

$$\pi_i X = \operatorname{colim}_j \pi_{i+j} X^j,$$

where the colimit is taken over the maps

$$[S^{i+j}, X^j] \longrightarrow [\Sigma S^{i+j}, \underline{\underline{\Sigma X^j}}] \longrightarrow [S^{i+j+1}, X^{j+1}]$$

$$\pi_{i+j} X^j \xrightarrow{\Sigma} \pi_{\underline{i+j+1}} \Sigma X^j \xrightarrow{\sigma_*^j} \pi_{i+j+1} X^{j+1} \longrightarrow \cdots$$

and runs over all $j \geq 0$ when $i \geq 0$, and over $j + i \geq 0$ for $i < 0$.

Spectra: model structure

Definition: weak equivalence of spectra

A map of spectra $f\colon X \to Y$ is a **stable weak equivalence** if it induces isomorphisms $f_*\colon \pi_n X \to \pi_n Y$ for all n.

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Theorem (Bousfield–Friedlander)

The category of spectra admits a model structure with the stable weak equivalences.

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0

abelian group

$g_1, g_2, \ldots$

product $g_1 \cdot g_2$

$\boxed{g_1 \cdot g_2 = g_2 \cdot g_1}$

1

"Categorify"

Cat. G with:

obj: $*$ (just one)

maps: $$\begin{tikzcd} * \arrow[loop above, "g_1"] \arrow[loop right, "g_2"] \end{tikzcd}$$ $g_1 \circ g_2 = g_1 \cdot g_2$

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- $n = 1$: Stable 1-types correspond to Picard categories (Patel): groupoids with a symmetric monoidal structure and invertible objects; alternatively, groupoidal bicategories with one object $+$ symmetry.
- $n = 2$: Stable 2-types correspond to Picard Bicategories (Gurski–Johnson–Osorno).

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Instead, we will use the change in perspective outlined above.

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A Tamsamani $(n+1)$-groupoid with one object is a functor

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We want the composition to be commutative.

Modeling Picard n-categories

$$X\colon (\Delta^{\mathrm{op}})^{n+1} \to \mathrm{Set}$$

How do we compose? Using the Segal condition!

$$X_1 \times_{X_0} X_1 \overset{\longrightarrow}{\xleftarrow{\sim}} X_2 \xrightarrow{d_1} X_1$$

where $d_1\colon [2] \to [1]$ is the face map in Δ^{op}.

Modeling Picard n-categories

$$\begin{tikzcd} X : \Delta^{op} \times (\Delta^{op})^{n} \arrow[r] \arrow[d] & Set \\ X : (\Delta^{\mathrm{op}})^{n+1} \to \mathrm{Set} & \end{tikzcd}$$

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where $d_1 \colon [2] \to [1]$ is the face map in Δ^{op}.

$(\text{fin. sets. and order-preserving maps})^{op}$

To introduce the symmetry, we extend the first Δ^{op} to a Γ.

fin. pointed sets and basepoint-preserving functions.

Modeling Picard n-categories

$$X \in [(\Delta^{\mathrm{op}}) \times (\Delta^{\mathrm{op}})^n, \mathrm{Set}] \hookrightarrow [(\Gamma) \times (\Delta^{\mathrm{op}})^n, \mathrm{Set}]$$

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Since $\phi d_1 = \phi d_1 \circ t$, they must induce the same map $X_2 \to X_1$, and this makes composition commutative.

Modeling Picard n-categories

Our proposed model:

Definition: Picard n-category (MOPSV)

A Picard n-category is a functor $X\colon \Gamma \times (\Delta^{\mathrm{op}})^n \to \mathrm{Set}$ such that the restriction to $(\Delta^{\mathrm{op}})^{n+1}$ is a Tamsamani $(n+1)$-groupoid with one object.

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Definition: weak equivalences (MOPSV)

A map of Picard n-categories $f\colon X \to Y$ is a weak equivalence if it's an equivalence of Tamsamani $(n+1)$-groupoids after restricting to $(\Delta^{\mathrm{op}})^{n+1}$.

Γ-objects

How do we connect $\mathrm{Ho}(\mathrm{PicTam}^n)$ and $\mathrm{Ho}(\mathrm{Sp}_{[0,n]})$?

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Let $\mathcal{C}$ be a pointed category. A Γ-*object* in $\mathcal{C}$ is a functor $\Gamma \to \mathcal{C}$ mapping $\langle 0 \rangle$ to $*$. Notation: $\Gamma\mathcal{C}$.

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When in addition $\mathcal{C}$ has weak equivalences, we can define **special** Γ-objects.

Definition: special Γ-object

A Γ-object A in $\mathcal{C}$ is special if, for each $k \geq 0$, the map

$$A\langle k \rangle \to A\langle 1 \rangle \times \cdots \times A\langle 1 \rangle = A\langle 1 \rangle^{\times k}$$

is a weak equivalence.

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For a special Γ-object A in one of these categories, the set $\pi_0 A\langle 1\rangle$ becomes an abelian monoid, with multiplication

$$\pi_0 A\langle 1\rangle \times \pi_0 A\langle 1\rangle \underset{\longrightarrow}{\xleftarrow{(\nu_1,\nu_2)}} \pi_0 A\langle 2\rangle \xrightarrow{m} \pi_0 A\langle 1\rangle$$

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Definition: very special Γ-object

A Γ-object A is very special if it is special, and the abelian monoid $\pi_0 A\langle 1\rangle$ is a group.

Proving the SHH

Theorem: SHH (MOPSV)

Picard n-categories are “the same” as stable n-types.

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$$\text{v.s.}\underbrace{\Gamma\underbrace{s\mathrm{Set}_*}_{}}_{} \xrightarrow[\text{BF}]{\simeq_{\text{Ho}}} \mathrm{Sp}_{\geq 0}$$

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$$\begin{tikzcd}
\mathrm{v.s.}\Gamma s\mathrm{Set}_* \arrow[rr, "\simeq_{\mathsf{Ho}}", "\mathsf{BF}"'] & {} \arrow[d, rightsquigarrow] & \mathrm{Sp}_{\geq 0} \\
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 & {} \arrow[u, squiggly] & & & \\
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$\Gamma \longrightarrow [(\Delta^{op})^n, Set]$

Theorem (MOPSV)

A functor $X : \Gamma \times (\Delta^{\mathrm{op}})^n \to \mathrm{Set}$ is a Picard–Tamsamani n-category if and only if it’s a very special Γ-object in Tamsamani n-groupoids.

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$$\begin{tikzcd}
 & & \mathrm{v.s.}\Gamma s\mathrm{Set}_* \arrow[r, "\simeq_{\mathsf{Ho}}", "\mathsf{BF}"' {name=BF}] & \mathrm{Sp}_{\geq 0} \\
\mathrm{PicTam}^n \simeq \arrow[rrr, bend left, red] & \mathrm{v.s.}\Gamma\mathrm{GTam}^n \arrow[r, "\simeq_{\mathsf{Ho}}" {name=M1}, "\mathsf{MOPSV}"' {name=M1b}] & \mathrm{v.s.}\Gamma s\mathrm{Set}_{*[0,n]} \arrow[r, "\simeq_{\mathsf{Ho}}" {name=M2}, "\mathsf{MOPSV}"'] & \mathrm{Sp}_{[0,n]} \\
 & \mathrm{GTam}^n \arrow[r, "\simeq_{\mathsf{Ho}}" {name=HH}, "\mathsf{HH\ (Tamsamani)}"'] & s\mathrm{Set}_{*[0,n]} &
\arrow[from=BF, to=M2, squiggly]
\arrow[from=HH, to=M1b, squiggly]
\end{tikzcd}$$

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Thanks for your time!