Model-theoretic stability and superstability in classes of modules

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November 2020
Masaryk University Algebra Seminar

Background

- Abstract elementary classes were introduced by Shelah in the 80's.
- The abstract theory has developed rapidly.
- Abelian groups and modules
- Dividing lines: Stability and superstability.

Outline

1. **Basic notions**
2. Stability
3. Universal models
4. Superstability

Basic notions: Model theory of modules

pp-formulas and *pp*-types

Let R be a ring and $L_R = \{0, +, -\} \cup \{r\cdot : r \in R\}$ be the language of R-modules.

- ϕ is a positive primitive formula if it is an existentially quantified finite system of linear equations.
- $pp(\bar{b}/A, N)$ are the *pp*-formulas satisfied by $\bar{b}$ in N with parameters in A.

Pure submodules

$M \leq_p N$ if and only if $M \subseteq N$ and $pp(\bar{a}/\emptyset, M) = pp(\bar{a}/\emptyset, N)$ for every $\bar{a} \in M^{<\omega}$.

Fact

$M \leq_p N$ if and only if for every L right R-module $L \otimes M \to L \otimes N$ is a monomorphism.

Basic notions: AECs

An *abstract elementary class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where K is a class of τ-structures and $\leq_{\mathbf{K}}$ is a partial order on K.

Key axioms

1. Tarski-Vaught axioms.
2. Löwenheim-Skolem-Tarski axiom.

Intuitively

$\mathbf{K}$ is a "concrete" accessible category closed under directed colimit where all morphisms are monomorphism.

AECs of modules

$\mathbf{K} = (K, \leq_p)$ is an AEC with $K \subseteq R$-Mod for R an associative ring with unity.

Basic notions: AECs

Amalgamation property (AP)

If every $M \leq_{\mathbf{K}} N_1, N_2$ can be completed to a commutative square in $\mathbf{K}$.

Examples

- AP: R-Mod, locally injective R-modules.
- No AP: $\aleph_1$-free abelian groups.
- AP?: Finitely Butler groups.

Some properties

1. $\mathbf{K}$ has the *joint embedding property* (JEP): if every $M, N \in K$ can be $\mathbf{K}$-embedded to a model in $\mathbf{K}$.
2. $\mathbf{K}$ has *no maximal models* (NMM): if for every $M \in K$ can be properly extended in $\mathbf{K}$.

Outline

Stability: Galois-types

We assume amalgamation for simplicity.

Galois-types

1. $\mathbf{K}^3 = \{(a, M, N) : M, N \in \mathbf{K}, M \leq_{\mathbf{K}} N \text{ and } a \in N\}$.
2. For $(a_1, M_1, N_1), (a_2, M_2, N_2) \in \mathbf{K}^3$, we say $(a_1, M_1, N_1)E(a_2, M_2, N_2)$ if $M := M_1 = M_2$, and there exists $f_\ell : N_\ell \underset{M}{\longrightarrow} N$ such that $f_1(a_1) = f_2(a_2)$.
3. For $(a, M, N) \in \mathbf{K}^3$, let $\mathbf{gtp_K}(a/M; N) := [(a, M, N)]_E$.
4. $\mathbf{gS}(M)$ is the set of Galois-types over M.

Stability

Stable

- $\mathbf{K}$ is λ-stable if $|\mathbf{gS}(M)| \leq \lambda$ for all $M \in \mathbf{K}$ of cardinality λ.
- $\mathbf{K}$ is stable if there is a λ such that $\mathbf{K}$ is λ-stable.

Theorem (Fisher-Bauer 70s)

If T is a complete first-order theory extending the theory of modules, then $(Mod(T), \leq_p)$ is stable.

Question 1

Let R be an associative ring with unity.
If $(K, \leq_p)$ is an AEC of modules, is $\mathbf{K}$ stable? Is this true if $R = \mathbb{Z}$? Under what conditions on R is this true?

Stability under Hypothesis 1

Hypothesis 1

Let $\mathbf{K} = (K, \leq_p)$ be an AEC of modules such that:

1. K is closed under direct sums.
2. K is closed under direct summands.
3. K is closed under pure-injective envelopes, i.e., if $M \in K$, then $PE(M) \in K$.

Examples

- R-modules.
- Absolutely pure modules: For all N, $M \subseteq_R N$ implies $M \leq_p N$.
- Locally injective modules.
- Locally pure-injective modules.

Stability under Hypothesis 1

Lemma (M.)

Let $M, N_1, N_2 \in \mathbf{K}$, $M \leq_p N_1, N_2$, $b_1 \in N_1$ and $b_2 \in N_2$. Then:

$$\mathbf{gtp}(b_1/M; N_1) = \mathbf{gtp}(b_2/M; N_2) \text{ iff } \mathrm{pp}(b_1/M, N_1) = \mathrm{pp}(b_2/M, N_2).$$

Theorem (M.)

Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If $\lambda^{|R|+\aleph_0} = \lambda$, then $\mathbf{K}$ is λ-stable.

Proof Sketch.

- Let $\{\mathbf{gtp}(a_i/M; N) : i < \alpha\}$ be an enumeration of $\mathbf{gS}(M)$.
- Let $\Phi : \mathbf{gS}(M) \to S_{pp}^{Th(N)}(M)$ be such that $\phi(\mathbf{gtp}(a_i/M; N)) = pp(a_i/M, N)$.
- Φ is an injective function and use first-order stability.

Stability under Hypothesis 2

Hypothesis 2

Let $\mathbf{K} = (K, \leq_p)$ be an AEC of modules such that:

1. K is closed under direct sums.
2. K is closed under pure submodules.
3. K is closed under pure epimorphic images: $ker(f) \leq_p dom(f)$.

Examples

1. R-modules.
2. Flat modules.
3. Abelian p -groups: every element $g \neq 0$ has order p^n for some $n \in \mathbb{N}$.
4. Injective Torsion modules: A generalization of torsion abelian groups.

Stability under Hypothesis 2

Galois-types=*pp*-types?

- I do not know.
- We can identify Galois-types and *pp*-types in the class of *p*-groups.

Lemma (M.)

Assume K is closed under pure submodules, direct sums and isomorphism. TFAE:

- K is closed under pure epimorphic images.
- K is closed under pushouts of pure embeddings in R-Mod, i.e., if $M, N_1, N_2 \in K$, $f_1 : M \to N_1$ is a pure embedding, $f_2 : M \to N_2$ is a pure embeddings and P is the pushout of (f_1, f_2) in $R\text{-}Mod$, then $P \in K$.

Stability under Hypothesis 2

Lemma

There is a non-forking relation on Galois-types that satisfies:

1. (Lieberman-Rosický-Vasey) (Uniqueness) If $M \leq_p N$, $p, q \in \mathbf{gS}(N)$, p, q do not fork over M and $p \restriction_M = q \restriction_M$, then $p = q$.
2. (M.) (Local character) If $p \in \mathbf{gS}(M)$, then there is $N \leq_p M$ such that p does not fork over N and $\|N\| \leq |R| + \aleph_0$.

Proof sketch.

- $\mathbf{gtp}(a/M; N)$ does not fork over M_0 if there are $M_1, M_2, N' \in K$ such that $a \in M_1$, $M \leq_p M_2$, $N \leq_p N'$, $M_0 \leq_p M_1, M_2 \leq_p N'$ and $M_1 \overset{N'}{\underset{M_0}{\downarrow}} M_2$.
- Where $M_1 \overset{N'}{\underset{M_0}{\downarrow}} M_2$ if P is the pushout of $(i : M_0 \to M_1, j : M_0 \to M_2)$ in R-Mod, then the unique map $t : P \to N'$ is a pure embedding.

Stability under Hypothesis 2

Theorem (M.)

If $\lambda^{|R|+\aleph_0} = \lambda$, then **K** is λ-stable.

Stability for classes of torsion-free abelian groups

Hypothesis 3

Let $\mathbf{K} = (K, \leq_p)$ be an AEC with $K \subseteq TF$ such that:

1. K is has arbitrarily large models.
2. K is closed under pure submodules.

Examples

- Torsion-free abelian groups.
- $\aleph_1$-free abelian groups.
- Finitely Butler groups.

Stability for classes of torsion-free abelian groups

Lemma (M.)

If $\lambda^{|R|+\aleph_0} = \lambda$, then $\mathbf{K}$ is λ-stable.

Proof sketch. One identifies Galois-types with *pp*-types and then do the proof under Hypothesis 1.

Why does it works for torsion-free groups?

If $N_1, N_2 \leq_p N$, then $N_1 \cap N_2 \leq_p N$.

Theorem (M.)

Assume R is Von Neumann regular. If $\mathbf{K}$ is closed under submodules and has arbitrarily large models, then $\mathbf{K}$ is λ-stable for every λ such that $\lambda^{|R|+\aleph_0} = \lambda$.

Outline

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2. Stability
3. **Universal models**
4. Superstability

Universal models

Notation: $\mathbf{K}_\lambda = \{M \in K : M \text{ has cardinality } \lambda\}$

Universal model

- $M \in \mathbf{K}$ is a *universal model in* $\mathbf{K}_\lambda$ if $M \in \mathbf{K}_\lambda$ and if given any $N \in \mathbf{K}_\lambda$, there is a $\mathbf{K}$-embedding $f : N \to M$, i.e., $N \cong f[N] \leq_{\mathbf{K}} M$.
- We say that $\mathbf{K}$ has a universal model of cardinality λ if there is a universal model in $\mathbf{K}_\lambda$.

Examples

- $(\mathbb{Q}\text{-VS}, \subseteq_{\mathbb{Q}})$: For every λ, $\mathbb{Q}^{(\lambda)}$ is a universal model of size λ.
- $(p\text{-groups}, \subseteq)$: For every λ, $\mathbb{Z}(p^\infty)^{(\lambda)}$ is a universal model of size λ.

Question 2 (*Abelian groups* by L. Fuchs)

For which cardinals λ, does $(p\text{-groups}, \leq_p)$ has a universal model of cardinality λ? The same question for torsion-free abelian groups with pure embeddings.

Universal model in (p-groups, $\leq_p$)

Answer under GCH

There is a universal model for every uncountable cardinal.

Theorem (M.)

If $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda (\mu^{\aleph_0} < \lambda)$, then ($p$-groups, $\leq_p$) has a universal model of cardinality λ.

Proof sketch.

- If $\lambda^{\aleph_0} = \lambda$, then ($p$-groups, $\leq_p$) is λ-stable.
- (p-groups, $\leq_p$) has AP, JEP and NMM.
- (Kucera-M.) (p-groups, $\leq_p$) has a universal model of cardinality λ.

Lemma (Kucera-M.)

Let **K** be an AEC with AP, JEP and NMM. Assume there is a κ such if $\theta^\kappa = \theta$, then **K** is θ-stable.
If $\lambda^\kappa = \lambda$ or $\forall \mu < \lambda (\mu^\kappa < \lambda)$, then **K** has a universal model of size λ.

Universal models in (p-groups, $\leq_p$)

Lemma (M.)

Let λ be a regular cardinal and μ be an infinite cardinal. If $\mu^+ < \lambda < \mu^{\aleph_0}$, then ($p$-groups, $\leq_p$) does not have a universal model of cardinality λ.

An answer below $\aleph_\omega$ without $\aleph_0$ and $\aleph_1$

For $n \geq 2$, (p-groups, $\leq_p$) has a universal model of cardinality $\aleph_n$ if and only if $2^{\aleph_0} \leq \aleph_n$.

A partial answer for $\aleph_1$

1. If $2^{\aleph_0} = \aleph_1$, then (p-groups, $\leq_p$) has a universal model of cardinality $\aleph_1$.
2. If $2^{\aleph_0} > \aleph_1$ and the combinatorial principle $\clubsuit$ holds, then (p-groups, $\leq_p$) does not have a universal model of cardinality $\aleph_1$.

Universal models in (p-groups, $\leq_p$)

Open problems

- Does (p-groups, $\leq_p$) have a universal model of cardinality $\aleph_0$?
- Does (p-groups, $\leq_p$) have a universal model of cardinality $\aleph_\omega$?

Partial solution $\aleph_\omega$

If $2^{\aleph_0} < \aleph_\omega$, then ($p$-groups, $\leq_p$) has a universal model of cardinality $\aleph_\omega$.

Open problem

More generally, when does (p-groups, $\leq_p$) have a universal model of cardinality λ for λ singular?

Universal models in (TF, $\leq_p$)

Shelah has many results for this class, using the methods described above I was able to extend the positive results to any class of flat modules.

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4. **Superstability**

Superstability: Universal extensions and limit models

Universal extension (Kolman-Shelah)

M is *universal over* N if and only if $\|M\| = \|N\|$, $N \leq_{\mathbf{K}} M$ and for any $N^* \in \mathbf{K}$ of size $\|M\|$ such that $N \leq_{\mathbf{K}} N^*$, there is $f : N^* \underset{N}{\longrightarrow} M$.

Limit model (Kolman-Shelah)

Let λ be an infinite cardinal and $\alpha < \lambda^+$ a limit ordinal. M is a (λ, α)-*limit model over* N if and only if there is $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$ an increasing continuous chain such that:

1. $M_0 := N$.
2. $M = \bigcup_{i<\alpha} M_i$.
3. M_{i+1} is universal over M_i for each $i < \alpha$.

M is a λ-limit model if there is $\alpha < \lambda^+$ limit ordinal and $N \in K_\lambda$ such that M is a (λ, α)-limit model over N.

Superstability: Definition

Theorem (Shelah)

Let **K** be an AEC with JEP, AP and NMM. **K** is λ-stable if and only if there is a λ-limit model.

Uniqueness of limit models

K has *uniqueness of limit models of cardinality* λ if **K** has a λ-limit model and if given M, N λ-limit models, M and N are isomorphic.

Superstability

K is *superstable* if and only if **K** has uniqueness of limit models in a tail of cardinals.

Superstability

Question 3

If $(K, \leq_p)$ is an AEC of modules, under what condtions is **K** superstable? Is there an algebraic reason why this happens?

Key idea

Characterize limit models algebraically.

Superstability under Hypothesis 1

Hypothesis 1

Let $\mathbf{K} = (K, \leq_p)$ be an AEC of modules such that:

1. K is closed under direct sums.
2. K is closed under direct summands.
3. K is closed under pure-injective envelopes, i.e., if $M \in K$, then $PE(M) \in K$.

Long limit models (M.)

If M is a (λ, α)-limit model and $\mathrm{cf}(\alpha) \geq (|R| + \aleph_0)^+$, then M is pure-injective.

Superstability under Hypothesis 1

Very short limit models (M.)

If M is a (λ, ω)-limit model and N is a $(\lambda, (|R| + \aleph_0)^+)$-limit model, then M is isomorphic to $N^{(\aleph_0)}$.

Theorem (M.)

The following are equivalent.

1. $\mathbf{K}$ is superstable.
2. Every limit model in $\mathbf{K}$ is Σ-pure-injective.
3. Every model in $\mathbf{K}$ is pure-injective.
4. For every $\lambda \geq \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ has uniqueness of limit models of cardinality λ.

Noetherian rings

Noetherian rings ('22): Every absolutely pure module is injective.

Characterizing noetherian rings (M.)

Let R be a ring. The following are equivalent.

1. R is left noetherian.
2. $(R\text{-AbsP}, \leq_p)$ is superstable.
3. $(R\text{-l-inj}, \leq_p)$ is superstable.

More characterizations of noetherian rings(M.)

Let R be a ring. The following are equivalent.

1. R is left noetherian.
2. $(R\text{-Mod}, \subseteq_R)$ is superstable.

Pure-semisimple rings

Pure-semisimple rings ('77): Every R-module is pure-injective.

Characterizing pure-semisimple rings (M.)

Let R be a ring. The following are equivalent.

1. R is left pure-semisimple.
2. $(R\text{-Mod}, \leq_p)$ is superstable.
3. $(R\text{-l-pi}, \leq_p)$ is superstable.

Lemma (M.)

Let R be a ring. The following are equivalent.

1. R is left pure-semisimple.
2. Every AEC of modules, such that K is closed under direct sums and direct summands, is superstable.

Superstability under Hypothesis 2?

Hypothesis 2

Let $\mathbf{K} = (K, \leq_p)$ be an AEC of modules such that:

1. K is closed under direct sums.
2. K is closed under pure submodules.
3. K is closed under pure epimorphic images.

Characterize the limit models?

For most classes satisfying Hypothesis 2 limit models won't be pure-injective, so what are they?

Superstability in (R-Flat, $\leq_p$)

Long limit models (M.)

If M is a (λ, α)-limit model and $\mathrm{cf}(\alpha) \geq (|R| + \aleph_0)^+$, then M is cotorsion.

Perfect rings ('60): Every flat module is a projective module.

Characterizing perfect rings (M.)

For a ring R the following are equivalent.

1. R is left perfect.
2. $(R\text{-Flat}, \leq_p)$ is superstable.
3. Every limit model in $(R\text{-Flat}, \leq_p)$ is Σ-cotorsion.

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Thank you!