Injectivity in metric enriched categories

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Met the category of (generalized) metric spaces and nonexpanding maps. ***CMet*** its full subcategory consisting of complete metric spaces.

Both ***Met*** and ***CMet*** are locally $\aleph_1$-presentable and symmetric monoidal closed. $A \otimes B$ is $A \times B$ with the $+$-metric: $d((a,b),(a',b')) = d(a,a') + d(b,b')$.

For an uncountable regular cardinal λ, A is λ-presentable in ***Met*** iff $|A| < \lambda$ and A is λ-presentable in ***CMet*** iff $\mathrm{dc}(A) < \lambda$. The only $\aleph_0$-presentable object in ***Met*** or ***CMet*** is the empty space.

Met-enriched categories: continuous logic (functional analysis), quantitative algebras (probabilistic programming).

Ban the category of Banach spaces and linear maps of norm ≤ 1.

$f, g : A \to B$, $f \sim_\varepsilon g$ means that $d(f, g) \leq \varepsilon$ in $\mathcal{K}(A, B)$

$f : A \to B$, K is ε-*injective* to f if, for every $g : A \to K$, there exists $h : B \to K$ with $hf \sim_\varepsilon g$.

K is *approximately injective* to f if it is ε-injective to f for every $\varepsilon > 0$.

If K is injective to f then it is approximately injective to f.

No separable Banach space is injective to isometries between finite-dimensional Banach spaces. But there is a separable Banach space approximately injective to these isometries. Moreover, this space is approximately saturated to these isometries which means that if g is an isometry then h is an isometry too. It is called a Gurarii space and it is the unique separable Banach space approximately saturated to isometries between finite-dimensional Banach spaces.

[RT2018] has started a category-theory approach to this.

$f : K \to L$ is called an *isometry* if for every pair $u, v : M \to K$ we have $d(u, v) = d(fu, fv)$.

Dually, f called a *coisometry* if for every pair $u, v : L \to M$ we have $d(u, v) = d(uf, vf)$.

In ***Met***, ***CMet***, or ***Ban***, isometries have the usual meaning. Coisometries are dense morphisms.

If $\mathcal{K}$ has conical limits and $\mathcal{K}_0$ is wellpowered then isometries form a right part of a factorization system. Dually, if $\mathcal{K}$ has conical colimits and $\mathcal{K}_0$ is cowellpowered then coisometries form a left part of a factorization system.

In ***Met*** these factorization systems are (surjective, isometry) and (coisometry, closed isometry). Both in ***CMet*** and ***Ban*** they coincide and we get (coisometry, isometry) factorization system.

In an ordinary category treated as discretely enriched, isometry = mono and coisometry = epi.

Lemma 1. Let $\mathcal{K}$ be enriched over ***CMet***. Then K is approximately injective to f iff $\mathcal{K}(f, K) : \mathcal{K}(B, K) \to \mathcal{K}(A, K)$ is a coisometry.

In the terminology of [LR2012], K is coisometry-injective.

Given a class $\mathcal{F}$ of morphisms, $\mathcal{F}^{\triangle}$ denotes the class of all objects approximately injective to every $f \in \mathcal{F}$ and $\mathcal{F}^{\blacktriangle}$ the class of all objects injective to every $f \in \mathcal{F}$.
Given a class of objects $\mathcal{L}$, ${}^{\triangle}\mathcal{L}$ denotes the class of all morphism f such that every $L \in \mathcal{L}$ is approximately injective to f. Analogously, ${}^{\blacktriangle}\mathcal{L}$. We have ${}^{\blacktriangle}\mathcal{L} \subseteq {}^{\triangle}\mathcal{L}$.

${}^{\blacktriangle}\mathcal{L}$ is left cancellable and closed under pushouts and transfinite compositions.

Lemma 2. ${}^{\triangle}\mathcal{L}$ is *approximately left cancellable*: if $f \in {}^{\triangle}\mathcal{L}$ and for every $\varepsilon > 0$ there is h_ε with $h_\varepsilon g \sim_\varepsilon f$ then $g \in {}^{\triangle}\mathcal{L}$.
In particular, ${}^{\triangle}\mathcal{L}$ is left cancellable.

An ε-*pushout* of morphisms $f : A \to B,\ g : A \to C$ in a is given by an ε-commutative square

$$\begin{tikzcd} B \arrow[r, "\overline{g}"] & D \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "\overline{f}"'] \end{tikzcd}$$

such that, for any ε-commutative square

$$\begin{tikzcd} B \arrow[r, "g'"] & D' \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "f'"'] \end{tikzcd},$$

there is a unique morphism $t : D \to D'$ such that $t\overline{f} = f'$ and $t\overline{g} = g'$.

Lemma 3. In an ε-pushout, $f \in {}^{\triangle}\mathcal{L}$ implies that $\overline{f} \in {}^{\blacktriangle}\mathcal{L}$. In particular, $\overline{f} \in {}^{\triangle}\mathcal{L}$.

Corollary 1. ${}^{\triangle}\mathcal{L}$ is closed under transfinite compositions of ε-pushouts.

It does not mean that ${}^{\triangle}\mathcal{L}$ is closed under transfinite compositions.

In fact, an ε-pushout

$$\begin{tikzcd} B \arrow[r, "\overline{\mathrm{id}_A}"] & D \\ A \arrow[u, "f"] \arrow[r, "\mathrm{id}_A"'] & A \arrow[u, "\overline{f}"'] \end{tikzcd}$$

does not need to be a pushout.

The closure of $\mathcal{F}$ under transfinite compositions of ε-pushouts is denoted by $\mathrm{cell}_{\mathrm{ap}}(\mathcal{F})$. Its members are called *approximately cellular* for $\mathcal{F}$.

An object A is called *approximately λ-small* w.r.t. $\mathcal{F}$ if for every λ-directed transfinite composition $(k_{ij} : K_i \to K_j)_{i \leq j \leq \mu}$ of morphisms approximately cellular for $\mathcal{F}$ and every $\varepsilon > 0$ we have that

(a) every morphism $f : A \to K_\mu$ has an ε-factorization $f' : A \to K_i$ through $k_{i\mu}$ for some $i < \mu$, i.e. $f \sim_\varepsilon k_{i\mu} f'$,

(b) if $k_{i\mu} f' \sim_\varepsilon k_{i\mu} f''$ for $f', f'' : A \to K_i$ and $i < \mu$, then $k_{ij} f' \sim_\varepsilon k_{ij} f''$ for some $i \leq j < \mu$.

If $\mathcal{K}$ is enriched over ***CMet*** then A is approximately λ-small w.r.t. $\mathcal{F}$ iff $\mathcal{K}(K, -) : \mathcal{K} \to$ ***CMet*** preserves λ-directed transfinite compositions of approximately cellular morphisms for $\mathcal{F}$.

Theorem 1. Let morphisms from $\mathcal{F}$ have the domains approximately small w.r.t. $\mathcal{F}$. Then $\mathcal{F}^\triangle$ is weakly reflective with approximately cellular weak reflections.
Moreover, ${}^\triangle(\mathcal{F}^\triangle)$ is the approximately cancellable closure of $\mathrm{cell}_{\mathrm{ap}}(\mathcal{F})$.

A colimit of

$$\begin{tikzcd} B & \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \end{tikzcd}$$

weighted by

$$\begin{tikzcd} & 1 \arrow[d] \\ 1 \arrow[r] & 2_\varepsilon \end{tikzcd}$$

is the ε-pushout

$$\begin{tikzcd} B \arrow[r, "\overline{g}"] & D \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "\overline{f}"'] \end{tikzcd}$$

such that $\overline{f}$ and $\overline{g}$ are jointly coisometric.

Theorem 2. $\mathcal{K}$ has weighted limits and colimits iff $\mathcal{K}_0$ has limits and colimits and $\mathcal{K}$ has coisometric ε-pushouts and isometric ε-pullbacks for all $\varepsilon > 0$.

Tensor $2_\varepsilon \otimes A$ is the ε-pushout

$$\begin{tikzcd} A \arrow[r, "\overline{g}"] & D \\ A \arrow[u, "\mathrm{id}_A"] \arrow[r, "\mathrm{id}_A"'] & A \arrow[u, "\overline{f}"'] \end{tikzcd}$$

Since every metric space is a colimit of 2_ε's, this yields tensor $M \otimes A$ as a colimit of ε-pushouts.

An ε-pushout in ***Met*** is the coproduct $B + C$ where we change the distance $d(fa, ga)$ from ∞ to ε and generated the metric, i.e., force the triagle inequality.

An ε-pullback in ***Met*** is the subspace of $B \times C$ consiting of all pairs (b, c) such that $d(fb, gc) \leq \varepsilon$.

Ban is locally $\aleph_1$-presentable and the unit ball functor $U : \boldsymbol{Ban} \to \boldsymbol{Met}$ preserves limits and $\aleph_1$-directed colimits. Thus it has a left adjoint $F : \boldsymbol{Met} \to \boldsymbol{Ban}$. $\aleph_1$-presentable Banach spaces are separable Banach spaces.

The category $\boldsymbol{Ban}_b$ of Banach spaces an bounded linear maps is symmetric monoidal closed where $\otimes$ is the projective tensor product and internal hom $\{K, L\}$ is the Banach space of bounded linear maps $K \to L$.

Tensors $M \otimes K$ in ***Ban*** are projective tensor products $FM \otimes K$ and cotensors $[M, K]$ are internal homs $\{FM, K\}$.

Thus ***Ban*** has coisometric ε-pushouts and isometric ε-pullbacks.

An ε-pushout is the coproduct $B \oplus C$ endowed with the norm generated by $\| (fa, -ga) \| = \varepsilon \| a \|$.

Proposition 1. Finite-dimensional Banach spaces are approximately $\aleph_0$-small w.r.t. isometries.

A Banach space is *approximately $\aleph_0$-saturated* if it is approximately injective w.r.t. isometries between finite-dimensional Banach spaces in the category of Banach spaces and isometries.

Theorem 3. There is a unique (up to isomorphism) separable approximately $\aleph_0$-saturated Banach space.

Its existence follows from Theorem 1. For uniqueness, we use the approximate back-and-forth argument. Given two such spaces K and L, we express them as unions od finite-dimensional subspaces $K = \mathrm{colim}_{i<\omega} K_i$ and $L = \mathrm{colim}_{j<\omega} L_j$ where $K_0 = L_0 = 0$ and with the connecting isometries $k_{i_1 i_2}$ and $l_{j_1 j_2}$.

Since L is approximately $\aleph_0$-saturated, there is an isometry $f : K_1 \to L$. Following Proposition 1, there is $f_0 : K_1 \to L_j$ such that $l_j f_0 \sim_{\varepsilon_0} f$. To get an isometry $g : L_j \to K$ with $g f_0 \sim_{\varepsilon_1} k_1$, we would need that f_0 is an isometry. But isometries are not left ε-cancellable.

A morphism $f : A \to B$ is called an ε-*isometry* provided there are isometries $g : B \to C$ and $h : A \to C$ such that $gf \sim_\varepsilon h$.

In ***Ban***, f is an ε-isometry if $\| a \| - \| fa \| \leq \varepsilon$ for every $a \in A$ with $\| a \| \leq 1$.

Lemma 2. $f : A \to B$ is an ε-isometry iff its ε-pushout along id_A is an isometry.

Since finite-dimensional Banach spaces are closed under ε-pushouts, for every isometry $f : A \to B$ between finite-dimensional Banach spaces, C above can be taken finite-dimensional.

Using this, we can run an approximate back-and-forth argument giving Theorem 3.

Gurarii constructed his space in 1965. Its unicity remained open for some time and was proved by Lusky in 1976 using deep Banach space theory. Kubiś and Solecki gave an elementary proof in 2011 and our approach puts it to a category theory framework.

Garbulińska-Węgrzyn and Kubiś (2015) found approximately $\aleph_0$-saturated objects in $\boldsymbol{Ban}^{\rightarrow}$, i.e., in operators on Banach spaces. It is easy to see that the resulting approximately $\aleph_0$-saturated operator $t : K_1 \to K_2$ with K_1 and K_2 separable has both K_1 and K_2 Gurarii, hence isomorphic. Thus t is an operator on the Gurarii space.

It does not seem that operators between finite-dimensional Banach spaces are approximately $\aleph_0$-small w.r.t. isometries in $\boldsymbol{Ban}^{\rightarrow}$. But they behave well w.r.t. transfinite compositions of isometries giving approximately $\aleph_0$-saturated objects. Hence, we can put it to our context.

Are finite-dimensional Banach algebras approximately $\aleph_0$-small w.r.t. isometries? One can show it for free Banach algebras over finite-dimensional Banach spaces. The same for C^*-algebras.