Categoricity in infinite quantifier theories

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However, it is categorical in every λ with respect to the notion of internal size $|A|$ defined as follows. If $r(A)$ is the least regular cardinal λ such that A is λ-presentable, then:

$$|A| = \begin{cases} \kappa & \text{if } r(A) = \kappa^+ \\ r(A) & \text{if } r(A) \text{ is limit} \end{cases}$$

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Assuming *GCH*, we have:

$$\lambda^\mu = \begin{cases} \lambda & \text{if } cof(\lambda) > \mu \text{ and } 2^\mu < \lambda \\ \lambda^+ & \text{if } cof(\lambda) \leq \mu \text{ and } 2^\mu < \lambda \end{cases}$$

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As a result, eventually there is exactly one μ-Hilbert space of cardinality λ regular not a successor of a cardinal of cofinality $\leq \mu$, but there are two μ-Hilbert spaces (of internal sizes λ and λ^+) if it is such a successor.On the other hand, it is categorical in every λ with respect to internal size.

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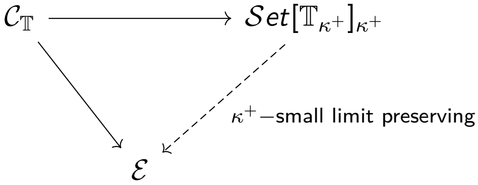
Recall that the κ^+-classifying topos of $\mathbb{T}_{\kappa^+}$ (Espindola 2017), $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ is defined through the following universal property:



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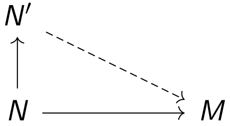

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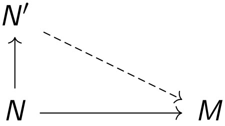

Recall that the category of κ^+-saturated models $Sat_{\kappa^+}(\mathcal{K})$ is axiomatizable by some theory $\mathbb{T}^{sat}_{\kappa^+}$ and that if τ_D is the dense (alternatively, atomic) Grothendieck topology on $\mathcal{K}_\kappa^{op}$ (where $\mathcal{K}_\kappa$ is the subcategory of objects of internal size κ), we have

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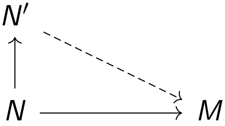

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Theorem

Let $\mathbb{T}_\kappa$ axiomatize $\mathcal{K}_{\geq\kappa}$. Then the κ^+-classifying topos of $\mathbb{T}_\kappa$ is equivalent to the presheaf topos $\mathcal{S}et^{\mathcal{K}_\kappa}$. Moreover, the canonical embedding of the syntactic category is given by (note that $\mathcal{K}_\kappa \ni M : \mathcal{C}_{\mathbb{T}_\kappa} \to \mathcal{S}et$):

$$\mathcal{C}_{\mathbb{T}_\kappa} \xrightarrow{\quad ev \quad} \mathcal{S}et^{\mathcal{K}_\kappa}$$

$$X \longmapsto M \mapsto M(X)$$

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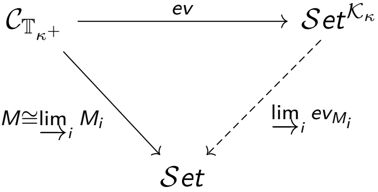

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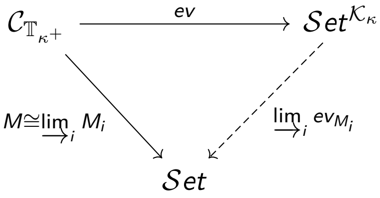

This proves the universal property when $\mathcal{E} = \mathcal{S}et$.

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Let now $\mathcal{E}$ be the κ^+-classifying topos of $\mathbb{T}_\kappa$. Then it has enough κ^+-points by the infinitary Deligne completeness theorems (Espindola 2017).

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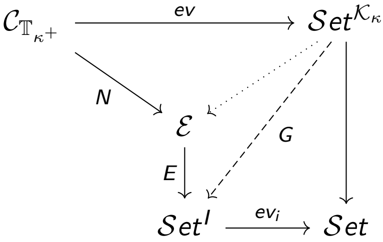

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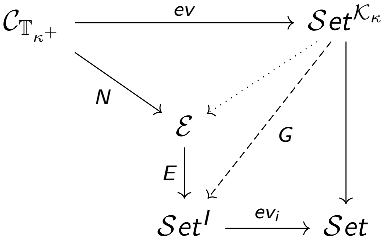

Now every object F in $\mathcal{S}et^{\mathcal{K}_\kappa}$ can be written as $F \cong \varinjlim_i [M_i, -]_{Mod_\lambda(\mathbb{T})}$, i.e.:

$$\varinjlim_i [\varinjlim_j [\phi_{ij}, -]_{\mathcal{C}_\mathbb{T}}, -]_{Mod_\lambda(\mathbb{T})} \cong \varinjlim_i \varprojlim_j [[\phi_{ij}, -]_{\mathcal{C}_\mathbb{T}}, -]_{Mod_\lambda(\mathbb{T})} \cong \varinjlim_i \varprojlim_j ev(\phi_{ij})$$

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$$G(F) \cong \varinjlim_{i} \varprojlim_{j} G \circ ev(\phi_{ij})$$

and hence G is completely determined by its values on $ev(\mathcal{C}_{\mathbb{T}_\kappa})$, which land in $\mathcal{E}$. Since E also preserves colimits and κ^+-small limits, the whole G lands in $\mathcal{E}$.

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This proves the universal property when $\mathcal{E}$ is κ^+-classifying topos of $\mathbb{T}_\kappa$. Since this later satisfies the same universal property, we must have $\mathcal{E} \cong \mathcal{S}et^{\mathcal{K}_\kappa}$. This finishes the proof.

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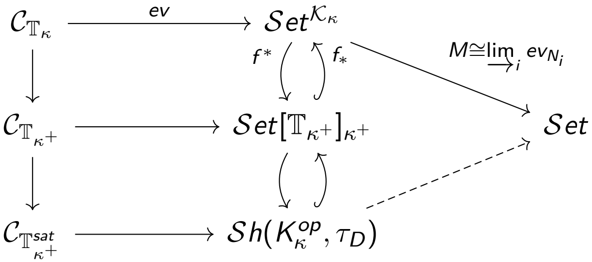

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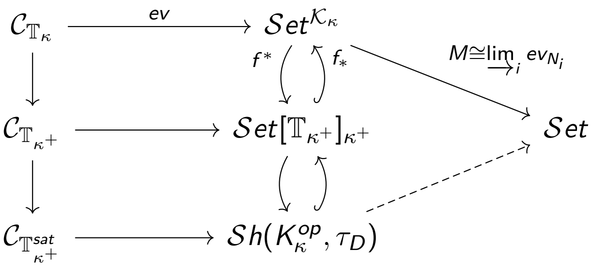

If M is a model corresponding to a categoricity cardinal λ, then we can prove that M is λ-closed and therefore λ-saturated, which induces the dotted morphism above.

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We put a topology on G_0 whose basic opens consist of sets of models of the form:

$$(\phi(\mathbf{x}), \mathbf{c}) := \{M \in G_0 : \mathbf{c} \in [[\phi(\mathbf{x})]]^M\}$$

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We now consider the least topology on G_1 which makes the source and target $s, t : G_1 \to G_0$ continuous and that contains all sets of the form:

$$(\mathbf{a} \mapsto \mathbf{b}) := \{f \in G_1 : \mathbf{a} \in s(f) \text{ and } f(\mathbf{a}) = \mathbf{b}\}$$

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Recall that the topos of equivariant sheaves on our topological groupoid has as objects pairs $(a : A \to G_0, \alpha)$, where a is a local homeomorphism and $\alpha : G_1 \times_{G_0} A \to A$ satisfies the conditions:

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- $a(\alpha(f, x)) = t(f)$
- $\alpha(1, x) = x$
- $\alpha(g, \alpha(f, x)) = \alpha(gf, x)$

Note that $\alpha(f, -)$ is a bijective function from the fiber over $s(f)$ to the fiber over $t(f)$.

Classifying topos for internal size κ^+

Recall that the topos of equivariant sheaves on our topological groupoid has as objects pairs $(a : A \to G_0, \alpha)$, where a is a local homeomorphism and $\alpha : G_1 \times_{G_0} A \to A$ satisfies the conditions:

- $a(\alpha(f, x)) = t(f)$
- $\alpha(1, x) = x$
- $\alpha(g, \alpha(f, x)) = \alpha(gf, x)$

Note that $\alpha(f, -)$ is a bijective function from the fiber over $s(f)$ to the fiber over $t(f)$.

Theorem

Then the κ^+-classifying topos for objects of internal size κ^+ is precisely the topos of equivariant sheaves $\mathcal{S}h_{G_1}(G_0)$

Large cardinals imply amalgamation

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Theorem

Let κ be a strongly compact cardinal and consider an accessible category $\mathcal{K}$ equivalent to the category of models of some $\mathcal{L}_{\kappa,\kappa}$ theory $\mathbb{T}$. If $\mathcal{K}$ is categorical at $\lambda \geq \kappa$, then $\mathcal{K}_{\geq\kappa}$ has the amalgamation property.

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Proof.

We have a conservative functor preserving universal quantification:

$$ev : (\mathcal{C}_{\mathbb{T}_\kappa})_\kappa \to \mathcal{S}et^{\mathcal{K}_{\geq\kappa,\leq\lambda}}$$

The model N_λ of size λ forces the sentence $\forall \mathbf{x}(\sigma(\mathbf{x}) \vee \neg\sigma(\mathbf{x}))$ (for σ in $\mathcal{L}_{\kappa,\kappa}$), so that $\mathcal{S}et^{\mathcal{K}_{\geq\kappa,\leq\lambda}}$ forces its double negation. Since $(\mathcal{C}_{\mathbb{T}_\kappa})_\kappa$ is two-valued (by categoricity in λ), $(\mathcal{C}_{\mathbb{T}_\kappa})_\kappa$ satisfies $\forall \mathbf{x}(\sigma(\mathbf{x}) \vee \neg\sigma(\mathbf{x}))$. This implies that all morphisms in $\mathcal{K}_{\geq\kappa}$ are κ-pure, so that the amalgamation property follows as in the finitary case. □

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Theorem

(Shelah's eventual categoricity conjecture for accessible categories with directed colimits). Assume GCH and that there is a proper class of strongly compact cardinals. Let $\mathcal{K}$ be an accessible category with directed colimits. Then there exists a cardinal μ_0 such that if $\mathcal{K}$ is categorical in some $\lambda \geq \mu_0$, it is categorical in all $\lambda' \geq \mu_0$.

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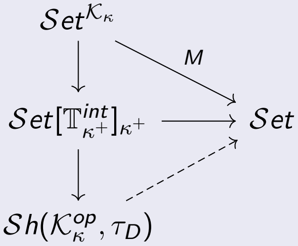
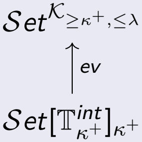
Proof.





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Let κ be in S and assume $(\kappa^+)^\kappa = \kappa^+$. Then $\mathcal{K}_\kappa$ has the amalgamation property if and only if $\mathcal{S}et^{\mathcal{K}_\kappa}$ is a De Morgan topos (it satisfies $\top \vdash_{\mathbf{x}} \neg\phi \vee \neg\neg\phi$ for κ^+-coherent ϕ).

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Theorem

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Proof.

$$\begin{array}{ccc} & & \mathcal{S}et[\mathbb{T}_\kappa]_{\kappa^+} \cong \mathcal{S}et^{\mathcal{K}_\kappa} \\ & \nearrow & \downarrow \\ \mathcal{S}et[\mathbb{T}_\kappa]_\kappa & \longrightarrow & \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \end{array}$$

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Thank you!