The universal exponentiable arrow

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Goal

To construct a universal exponentiable arrow.

- It is a pair $(\mathbb{G}, \partial)$ of a category with finite limits $\mathbb{G}$ and an exponentiable arrow ∂ in $\mathbb{G}$;
- It has the universal property that, for any such a pair $(\mathcal{C}, u)$, there exists a unique structure-preserving functor $\mathbb{G} \to \mathcal{C}$.

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Moreover, $\mathbb{G}$ is the category with finite limits such that

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- A syntactic proof is found in my preprint arXiv:2001.09940.
- Today I give a semantic proof, working with contextual categories.

Categories of theories

Consider (∞-)categories of *theories*, e.g.

- of $\wedge$-semilattices (propositional theories with $\top$ and $\wedge$);
- of clones (single-sorted algebraic theories);
- of categories with finite products (many-sorted algebraic theories);
- of categories with finite limits (essentially algebraic theories);
- of contextual categories (generalized algebraic theories);
- of ∞-categories with finite limits (dependent type theories with 1, Σ and $=$);
- of yet-to-be-defined elementary ∞-toposes (univalent type theories).

Question

What properties do categories of theories have in common?

Theories are essentially algebraic

Observation

A lot of categories of theories are essentially algebraic (equivalently, locally finitely presentable or compactly generated).

Then, by the *Gabriel-Ulmer duality*, for a category of theories $\mathcal{X}$, one can find a category with finite limits $\mathbb{T}_{\mathcal{X}}$ such that

$$\mathcal{X} \simeq \mathbf{Lex}(\mathbb{T}_{\mathcal{X}}, \mathbf{Set})$$

Question

What can we say about $\mathbb{T}_{\mathcal{X}}$?

Exponentiable arrows from categories of theories

Claim

Let $\mathcal{X}$ be a category of theories. Then $\mathbb{T}_{\mathcal{X}}$ is the free category with finite limits, with a special exponentiable arrow, and with some other arrows.

In this talk we see some selected cases:

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In this talk we see some selected cases:

- when $\mathfrak{X}$ is the category of $\bigwedge$-semilattices, $\mathbb{T}_{\mathfrak{X}}$ is the free category with finite limits and with an exponentiable monomorphism whose associated internal preorder is an internal $\bigwedge$-semilattice;

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- when $\mathfrak{X}$ is the category of $\bigwedge$-semilattices, $\mathbb{T}_{\mathfrak{X}}$ is the free category with finite limits and with an exponentiable monomorphism whose associated internal preorder is an internal $\bigwedge$-semilattice;
- when $\mathfrak{X}$ is the category of contextual categories, $\mathbb{T}_{\mathfrak{X}}$ is the free category with finite limits and with an exponentiable arrow.

Outline

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Exponentiable arrows

Definition

An arrow $u : a \to b$ in a category $\mathcal{C}$ with finite limits is exponentiable if the pullback functor $u^* : \mathcal{C}/b \to \mathcal{C}/a$ has a right adjoint $u_* : \mathcal{C}/a \to \mathcal{C}/b$ called the *pushforward along* u.

$$\mathcal{C}/b \underset{u_*}{\overset{u^*}{\rightleftarrows}} \mathcal{C}/a \qquad (u^* \dashv u_*)$$

Exponentiable arrows

Proposition

For an arrow $u : a \to b$ in a category $\mathcal{C}$ with finite limits, the following are equivalent:

1. *u is exponentiable;*
2. *the pushout functor*

$$\mathbf{y}_{\mathcal{C}^{\mathrm{op}}}(u)_! : \mathbf{y}_{\mathcal{C}^{\mathrm{op}}}(b)/\mathbf{Lex}(\mathcal{C}, \mathbf{Set}) \to \mathbf{y}_{\mathcal{C}^{\mathrm{op}}}(a)/\mathbf{Lex}(\mathcal{C}, \mathbf{Set})$$

   has a left adjoint;

3. *the pushout functor $\mathbf{y}_{\mathcal{C}^{\mathrm{op}}}(u)_!$ preserves limits,*

where $\mathbf{y}_{\mathcal{C}^{\mathrm{op}}} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Lex}(\mathcal{C}, \mathbf{Set})$ is the Yoneda embedding.

Proof.

The equivalence of Items 2 and 3 follows from the adjoint functor theorem. □

Essentially algebraic categories

A category $\mathcal{X}$ is *essentially algebraic* if it is the category of Σ-algebras for some essentially algebraic theory Σ. This is equivalent to that $\mathcal{X}$ is locally finitely presentable (Adámek and Rosický 1994), so we have

$$\mathcal{X} \simeq \mathbf{Lex}(\mathbb{T}_{\mathcal{X}}, \mathbf{Set})$$

where $\mathbb{T}_{\mathcal{X}}^{\mathrm{op}} \subset \mathcal{X}$ is the full subcategory spanned by the *finitely presentable objects*. $\mathbb{T}_{\mathcal{X}}$ contains the *universal internal Σ-algebra*: for any category with finite limits $\mathcal{D}$, we have

$$\{\text{internal } \Sigma\text{-algebras in } \mathcal{D}\} \simeq \mathbf{Lex}(\mathbb{T}_{\mathcal{X}}, \mathcal{D}).$$

Proof scheme for exponentiability

Let $\mathcal{X} \simeq \mathbf{Lex}(\mathbb{T}_{\mathcal{X}}, \boldsymbol{Set})$ be a category of theories. To construct an exponentiable arrow in $\mathbb{T}_{\mathcal{X}}$:

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3. show that a "slice construction" fits into a pushout diagram in $\mathcal{X}$

$$\begin{array}{ccc} \langle \sigma \rangle & \xrightarrow{a} & C \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle a^*} \\ \langle \sigma, \theta : \top \to \sigma \rangle & \xrightarrow[(a, \delta_a)]{} & C/a; \end{array}$$

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4. since $\langle \sigma \rangle$ and $\langle \sigma, \theta : \top \to \sigma \rangle$ are finitely presentable and since the slice construction preserves limits, ι determines an exponentiable arrow $\partial : E \to U$ in $\mathbb{T}_{\mathcal{X}}$.

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2. observe that the internal Σ-algebra associated to ∂ coincides with the universal internal Σ-algebra;

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2. observe that the internal Σ-algebra associated to ∂ coincides with the universal internal Σ-algebra;
3. then it follows that the lex functor $\mathbb{T}_{\chi} \to \mathcal{D}$ corresponding to the internal Σ-algebra associated to u commutes with pushforwards.

Outline

$\wedge$-semilattices

Definition

A $\wedge$-*semilattice* is a poset that has finite limits as a category.

Let $\mathbf{Lat}_\wedge$ denote the category of $\wedge$-semilattices and let $\mathbb{L}$ be such that $\mathbf{Lat}_\wedge \simeq \mathbf{Lex}(\mathbb{L}, \mathbf{Set})$.

Some free $\wedge$-semilattices

We define some free $\wedge$-semilattices:

- $\langle \sigma \rangle$ the $\wedge$-semilattice freely generated by an object σ;
- $\langle \sigma, \top \leq \sigma \rangle$ the $\wedge$-semilattice obtained from $\langle \sigma \rangle$ by forcing $\top \leq \sigma$ (so it is just the trivial $\wedge$-semilattice).

We will show the following.

Proposition

The canonical morphism $\iota : \langle \sigma \rangle \to \langle \sigma, \top \leq \sigma \rangle$ determines an exponentiable arrow in $\mathbb{L}$.

Slice $\bigwedge$-semilattices

For a $\bigwedge$-semilattice C and an object $a \in C$, let

$$C/a = \{b \in C \mid b \leq a\}.$$

We define a morphism $a^* : C \to C/a$ by $b \mapsto b \wedge a$ such that $\top \leq a^* a$ in C/a.

Proposition

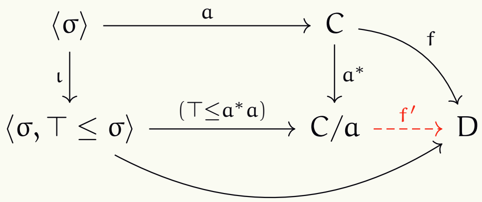
The diagram

$$\begin{array}{ccc} \langle \sigma \rangle & \xrightarrow{\quad a \quad} & C \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle a^*} \\ \langle \sigma, \top \leq \sigma \rangle & \xrightarrow[(\top \leq a^* a)]{} & C/a \end{array}$$

is a pushout in $\mathbf{Lat}_\wedge$.

Universal property of a slice $\wedge$-semilattice

A unique filler



is given by the restriction $C/a \subset C \xrightarrow{f} D$. (Since $\top \leq f(a)$, this restriction is indeed a morphism of $\wedge$-semilattices.)

An exponentiable monomorphism

Observe:

- $\langle \sigma \rangle$ and $\langle \sigma, \top \leq \sigma \rangle$ are finitely presentable;
- the slice construction preserves limits.

Then the morphism $\iota : \langle \sigma \rangle \to \langle \sigma, \top \leq \sigma \rangle$ determines an exponentiable arrow $\partial : E \to U$ in $\mathbb{L}$, where $\mathbf{Lat}_{\wedge} \simeq \mathbf{Lex}(\mathbb{L}, \mathbf{Set})$.
Since ι is epic, ∂ is monic.

The associated internal preorder

By the exponentiability of ∂, we can construct the associated internal preorder $\leq \subset U \times U$: an arrow $(u, v) : x \to U \times U$ factors through $\leq$ iff there exists a (necessarily unique) arrow $u^* E \to v^* E$ over x. Observe:

Proposition

The internal preorder associated to ∂ coincides with the universal internal $\bigwedge$-semilattice.

Then we get:

Theorem

$\mathbb{L}$ is the free category with finite limits and with an exponentiable monomorphism whose associated internal preorder is an internal $\bigwedge$-semilattice.

Outline

Contextual categories (Cartmell 1978)

A contextual category consists of the following data:

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- for any $a \triangleleft b$, an arrow $p_b : b \twoheadrightarrow a$;

Contextual categories (Cartmell 1978)

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- a category $\mathcal{C}$;
- a tree structure $\lhd$ on the set of objects;
- for any $a \lhd b$, an arrow $p_b : b \rightarrowtriangle a$;
- for any $a \lhd b$ and arrow $u : a' \to a$, an object $u^* b$ and an arrow $q_{u,b} : u^* b \to b$ such that $a' \lhd u^* b$ and the square

$$\begin{array}{ccc} u^* b & \xrightarrow{q_{u,b}} & b \\ {\scriptstyle p_{u^* b}} \downarrow & & \downarrow {\scriptstyle p_b} \\ a' & \xrightarrow[u]{} & a \end{array}$$

  is a pullback

satisfying ...

Contextual categories, continued

A contextual category consists of ... satisfying:

1. the root $\top$ of the tree is a terminal object;
2. the operator $(u, b) \mapsto (u^* b, q_{u,b})$ is strictly functorial.

A morphism of contextual categories is a functor that strictly preserves the structure. Let $\mathbf{CtxCat}$ denote the category of contextual categories.

Example

Given a dependent type theory, the category of contexts and context morphisms is a contextual category with the relation

$$\Gamma \triangleleft (\Gamma, x : A).$$

Cartmell showed that $\mathbf{CtxCat} \simeq \mathbf{GAT}$.

The essentially algebraic theory of contextual categories

A tree is equivalent to a chain of sets

$$\{\top\} \leftarrow A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \dots,$$

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where Obj_n is the sort of objects of depth n;
- for any n and m, a span
$$\mathrm{Obj}_n \leftarrow \mathrm{Hom}_{n,m} \rightarrow \mathrm{Obj}_m;$$
- a bunch of partial operators and equations.

Let $\mathbb{G}$ be such that $\mathbf{CtxCat} \simeq \mathbf{Lex}(\mathbb{G}, \mathbf{Set})$.

Some free contextual categories

We define some free contextual categories:

- $\langle\sigma\rangle$ the contextual category freely generated by an object σ of depth 1;
- $\langle\sigma, \theta : \top \to \sigma\rangle$ the contextual category obtained from $\langle\sigma\rangle$ by adding a global section $\theta : \top \to \sigma$.

We will show the following.

Proposition

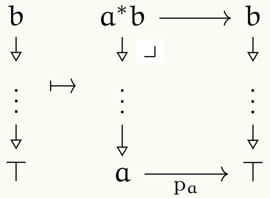
The canonical morphism $\iota : \langle\sigma\rangle \to \langle\sigma, \theta : \top \to \sigma\rangle$ *determines an exponentiable morphism in* $\mathbb{G}$.

Slice contextual categories

For a contextual category C and an object $a \in C$ of depth 1, the objects above a with respect to the tree structure form a contextual category C/a. An object of depth n in C/a is an object b of depth $n + 1$ in C such that

$$a = b_0 \triangleleft b_1 \triangleleft \ldots \triangleleft b_n = b.$$

The pullback along $p_a : a \to \top$ yields a morphism $a^* : C \to C/a$.



The diagonal arrow $a \to a \times_\top a$ determines a global section $\delta_a : \top \to a^* a$ in C/a.

Universal property of a slice contextual category

Proposition

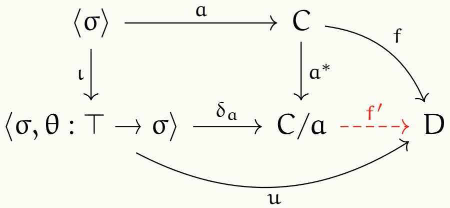
For any contextual category C *and object* $a \in C$ *of depth* 1*, the diagram*

$$\begin{array}{ccc} \langle \sigma \rangle & \xrightarrow{a} & C \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle a^*} \\ \langle \sigma, \theta : \top \to \sigma \rangle & \xrightarrow[\delta_a]{} & C/a \end{array}$$

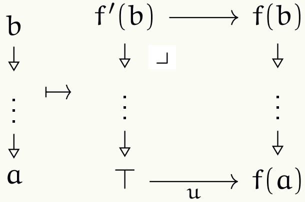
is a pushout in $\mathbf{CtxCat}$*.*

Proof of the universal property

A unique filler



is given a morphism $f' : C/a \to D$ defined by by $b \mapsto u^* f(b)$.



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Observe:

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Then the morphism $\iota : \langle \sigma \rangle \to \langle \sigma, \theta : \top \to \sigma \rangle$ determines an exponentiable arrow $\partial : E \to U$ in $\mathbb{G}$, where $\mathbf{CtxCat} \simeq \mathbf{Lex}(\mathbb{G}, \mathbf{Set})$.

The associated internal contextual category

Since $\partial : \mathrm{E} \to \mathrm{U}$ in $\mathbb{G}$ is exponentiable, we have the *polynomial functor* P_∂

$$\mathbb{G} \xrightarrow{\mathrm{E}^*} \mathbb{G}/\mathrm{E} \xrightarrow{\partial_*} \mathbb{G}/\mathrm{U} \xrightarrow{\mathrm{U}_!} \mathbb{G}.$$

We have the chain

$$\top \leftarrow \mathrm{P}^1_\partial(\top) \leftarrow \mathrm{P}^2_\partial(\top) \leftarrow \mathrm{P}^3_\partial(\top) \leftarrow \dots$$

which is part of an internal contextual category in $\mathbb{G}$.

The universal exponentiable arrow

Observe:

Proposition

The internal contextual category associated to ∂ coincides with the universal internal contextual category.

Then we get:

Theorem

$\mathbb{G}$ *is the free category with finite limits and with an exponentiable arrow.*

Further results

We can obtain similar results for a variety of categories of theories. The slice construction works for the following categories of theories:

- of (∞-)categories with finite limits;
- of locally cartesian closed (∞-)categories;
- of regular (coherent) categories;
- of elementary toposes;

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- of Lawvere theories.

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Fiore and Mahmoud (2014) constructed the *universal exponentiable object* and gave a connection to the theory of clones.

References I

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Marcelo Fiore and Ola Mahmoud (2014). *Functorial Semantics of Second-Order Algebraic Theories*. arXiv: 1401.4697v1.

The universal internal $\bigwedge$-semilattice

The universal internal $\bigwedge$-semilattice is determined by the cospan

$$\langle \sigma \rangle \longrightarrow \langle \sigma, \tau, \sigma \leq \tau \rangle \longleftarrow \langle \tau \rangle$$

in $\mathbf{Lat}_\wedge$. A morphism $\langle \sigma, \tau, \sigma \leq \tau \rangle \to C$ corresponds to a pair of objects $a, b \in C$ such that $a \leq b$.

Lemma

$a \leq b$ *iff there exists a (necessarily unique) morphism* $C/b \to C/a$ *under* C.

This lemma and the universal property of slices implies that $\langle \sigma, \tau, \sigma \leq \tau \rangle$ in $\mathbb{L}$ has the same universal property as the internal preorder associated to ∂.

The universal internal contextual category

Let $\langle \sigma_n \rangle$ denotes the contextual category freely generated by an object σ_n of depth n. The universal internal contextual category is determined by

$$\langle \sigma_1 \rangle \longrightarrow \langle \sigma_2 \rangle \longrightarrow \langle \sigma_3 \rangle \longrightarrow \ldots$$

and some other structure in $\mathbf{CtxCat}$. An object $a \in C$ of depth $n+1$ corresponds to a chain $\top \triangleleft a_1 \triangleleft \ldots \triangleleft a_{n+1} = a$ fitting into the diagram

$$\begin{array}{ccccc}
\langle \sigma \rangle & \xrightarrow{\quad a_1 \quad} & C & & \\
{\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle a_1^*} & & \\
\langle \sigma, \theta : \top \to \sigma \rangle & \xrightarrow[(a_1, \delta_{a_1})]{} & C/a_1 & \xleftarrow[(a_2 \triangleleft \ldots a_{n+1})]{} & \langle \sigma_n \rangle .
\end{array}$$

Then $\langle a_{n+1} \rangle \in \mathbb{G}$ has the same universal property as $P_\partial(\langle a_n \rangle)$, and thus $\langle a_n \rangle \simeq P_\partial^n(\top)$ in $\mathbb{G}$.