Semiholonomní jety ve stochastickém kalkulu

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AMathNet síť pro transfer znalostí v aplikované matematice

Obsah

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Martingaly

A random process $\{X_t; t \in T\}$ is called the $\mathcal{F}_t$-*martingale*, if $\mathrm{E}(X_t|\mathcal{F}_s) = X_s$. A random variable $S\colon \Omega \to T$ is called the *stopping time*, if $\{\omega \in \Omega; S(\omega) \leq t\} \in \mathcal{F}_t$. A random process $\{X_t; t \in T\}$ is called the *local* $\mathcal{F}_t$-*martingale* if there is a sequence $\{S_n; n \in \mathbb{N}\}$ of stopping times ($\lim S_n = \infty$) such that $X_t^{S_n} = X_{\min\{t,S_n\}}$ is a $\mathcal{F}_t$-martingale for all $n \in \mathbb{N}$. (Of course, every $\mathcal{F}_t$-martingale is a local $\mathcal{F}_t$-martingale, too.)

Semimartingaly

A random process $\{X_t; t \in T\}$ is called the $\mathcal{F}_t$-*semimartingale*, if it is expressible for all $t \geq 0$ in a form $X_t = X_0 + M_t + V_t$ where M_t is a local $\mathcal{F}_t$-martingale and V_t is a $\mathcal{F}_t$-adapted process with trajectories of local finite variations (we assume $M_0 = 0$ and $V_0 = 0$). So, we consider V as a signal (trend) and M as a noise.

Změna souřadnic

Classical differential calculus cannot be applied to semimartingales. An extension of differential calculus is available, namely the stochastic calculus. In the stochastic calculus, the coordinate change is not tensorial.

Rozklady

Let $t, m \in \mathbb{N}$. The *multiindex of length t with entries up to m* is a t-tuple

$$I = (i_1, \dots, i_t) \in N(m)^t = \underbrace{N(m) \times \cdots \times N(m)}_{t\text{-times}},$$

where $N(m) = \{1, \dots, m\}$.

Rozklady

The *standard ordered partition* of a multiindex I to u multiindexes $I_1, \dots, I_u$, $u \leq t$ is a map

$$\pi|_u^{t(m)} \colon N(m)^t \to N(m)^{t_1} \times \cdots \times N(m)^{t_u},$$

$$\pi|_u^{t(m)}(I) = (I_1, \dots, I_u) = \Big((i_{s_1^1}, \dots, i_{s_{t_1}^1}), \dots, (i_{s_1^u}, \dots, i_{s_{t_u}^u}) \Big)$$

(where $t_1 + \cdots + t_u = t$, $s_1^1, \dots, s_{t_u}^u \in N(m)$) satisfying

(i) $s_1^1 < \cdots < s_{t_1}^1, \dots, s_u^1 < \cdots < s_{t_u}^u$,

(ii) $s_1^1 < \cdots < s_1^u$.

Rozklady

The set of all standard ordered partitions of a multiindex I of the length t with entries up to m to u multiindexes $I_1, \dots, I_u$ will be denoted by $\Pi|_u^{t(m)}$. Further, we put $\Pi|^{t(m)} = \Pi|_1^{t(m)} \cup \dots \cup \Pi|_t^{t(m)}$.

Rozklady

Let M be an m-dimensional manifold. Given two systems $(x^1, \dots, x^m)$ and $(y^1, \dots, y^m)$ of local coordinates on M and a multiindex $I \in N(m)^t$, we define the functions $y^j_{/I}$ ($j \in N(m)$) by

$$y^j_{/I} = \frac{\partial^t y^j}{\partial x^{i_1} \dots \partial x^{i_t}}.$$

For $J = (j_1, \dots, j_u) \in N(m)^u$, we put

$$y^J_{/I} = \sum_{\Pi|_u^{t(m)}} y^{j_1}_{/I_1} \dots y^{j_u}_{/I_u},$$

where the sum runs over $\Pi|_u^{t(m)}$, i.e. over all standard ordered partitions of I to u multiindexes.

Rozklady - příklad

Suppose $J = (j_1, j_2)$ and $I = (i_1, i_2, i_3)$. Then

$$y^J_{/I} = \sum_{\Pi|_2^{3(m)}} y^{j_1}_{/I_1} y^{j_2}_{/I_2} = y^{j_1}_{/(i_1)} y^{j_2}_{/(i_2,i_3)} + y^{j_1}_{/(i_1,i_2)} y^{j_2}_{/(i_3)} + y^{j_1}_{/(i_1,i_3)} y^{j_2}_{/(i_2)}.$$

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Bandl derivačních řetězců

Let M be a manifold. Given a point x in M, a *derivative string* at x of degree (r, s) and length (T, U) assigns to each local coordinate system round x a set of real-valued arrays H^{AD}_{BC} indexed by multiindexes A, B, C, D with $|A| = r$, $|B| = s$, $\min\{1, T\} \leq |C| \leq T + 1$, $\min\{1, U\} \leq |D| \leq U$ which transform under coordinate change from $(y^1 \dots, y^m)$ to $(x^1 \dots, x^m)$ by

$$H^{AD}_{BC} = x^A_{/I} y^J_{/B} H^{IJ}_{KL} y^K_{/C} x^D_{/L}.$$

If we glue spaces of derivative strings of degree (r, s) and length (T, U) over all points of M, we obtain a fiber bundle which we denote by $\mathcal{S}^{rU}_{sT} M$. Sections of this bundle will be called *derivative string fields*.

Jety zobrazení

Two maps $f, g\colon M \to N$ are said to *determine the same r–jet* at $x \in M$, if for every curve $\gamma\colon \mathbb{R} \to M$ with $\gamma(0) = x$ the curves $f \circ \gamma$ and $g \circ \gamma$ have the r–th order contact at zero. In such a case we write $j^r_x f = j^r_x g$ and an equivalence class of this relation is called an *r–jet* of M into N.
(f, g have the same r-th order Taylor development at x)

Fibrované variety

Fibered manifolds are obtained by a glueing:

$$TM = \bigcup_{x\in M} T_x M \;\text{ or }\; \bigcup_{x\in M} J_0^1(\mathbb{R}, M)_x,$$

etc. In general, fibered manifolds have a form

$$(Y, \pi, M, S),$$

Y, M, S smooth manifolds, $\pi\colon Y \to M$ the canonical projection, each $x \in M$ has a neighborhood $U \subset M$ such that $\pi^{-1}(U)$ is diffeomorphic to $U \times S$ (so-called *local trivialization*).

Jetové prostory

$J^1_0(\mathbb{R}, M)_x = T_x M$ the *tangent space* in $x \in M$;
$J^1_0(\mathbb{R}, M) = TM$ the *tangent bundle* over M; local coordinates of an element of TM are

$$\begin{aligned} x^i &= \gamma^i(0) \\ y^i &= \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}(0) \end{aligned}$$

$i = 1, \dots, m$;
$J^r_0(\mathbb{R}^n, M) = T^r_n M$ the *bundle of n-dimensional velocities of order r* over M; local coordinates
$x^i, y^i_1, \dots, y^i_n, \dots, y^i_{n\dots n}$, $i = 1, \dots, m$.
The functor $T^r_n \colon \mathcal{M}f \to \mathcal{FM}$.

Jetové prostory

On the other hand,
$J^1_x(M,\mathbb{R})_0 = T^*_x M$ the *cotangent space* in $x \in M$;
$J^1(M,\mathbb{R})_0 = T^* M$ the *cotangent bundle* over M; local coordinates x^i, p_i;
$J^r(M,\mathbb{R}^n)_0 = T^{r*}_n M$ the *bundle of n-dimensional covelocities of order r* over M; local coordinates $x^i, p_{i,1}, \dots, p_{i,n}, \dots, p_{i,n\dots n}$, $i = 1, \dots, m$.
The functor $T^{r*}_n \colon \mathcal{M}f \to \mathcal{FM}_m$.

$T^{r*}_n M$ is a vector bundle for all r, $T^r_n M$ is a vector bundle only for $r = 1$.

Neholonomní jety

We recall how to define nonholonomic jets:

$$\tilde{J}^1(M,N) := J^1(M,N);$$

by induction, let

$$\alpha\colon \tilde{J}^{r-1}(M,N) \to M$$

denote the source projection and

$$\beta\colon \tilde{J}^{r-1}(M,N) \to N$$

the target projection of $(r-1)$–th nonholonomic jets. Then X is said to be a *nonholonomic r–jet* with the source $x \in M$ and the target $\bar{x} \in N$, if there is a local section

$$\sigma\colon M \to \tilde{J}^{r-1}(M,N)$$

such that $X = j^1_x\sigma$ and $\beta(\sigma(x)) = \bar{x}$.

Neholonomní rychlosti

We define the *nonholonomic bundle of k–dimensional velocities of order r* by

$$\tilde{T}^r_k M = \tilde{J}^r_0(\mathbb{R}^k, M).$$

(C. Ehresmann).
There is a natural equivalence $\epsilon_r : \tilde{T}^r_k \to \underbrace{T^1_k \dots T^1_k}_{r\text{–times}}$ defined by the following induction. For $r = 1$, ϵ_1 is the identity. Let t_u denote the translation on $\mathbb{R}^k$ transforming 0 into u. If $\sigma : \mathbb{R}^k \to \tilde{J}^{r-1}(\mathbb{R}^k, M)$ is a section, then $\sigma(u) \circ j^{r-1}_0(t_u) \in \tilde{T}^{r-1}_k M$. Hence $\epsilon_{r-1}\left(\sigma(u) \circ j^{r-1}_0 t_u\right) \in \underbrace{T^1_k \dots T^1_k}_{r-1\text{–times}} M$ and we define $\epsilon_r(j^1_0 \sigma) = j^1_0 \epsilon_{r-1}\left(\sigma(u) \circ j^{r-1}_0 t_u\right)$.

Projekce

For every s, $0 \leq s \leq r$, we denote by $\pi^s \colon \tilde{T}^s_k M \to M$ the canonical projection to the base. Further, we denote $\pi^s_b := \pi^s_{\tilde{T}^b_k M} \colon \tilde{T}^s_k(\tilde{T}^b_k M) \to \tilde{T}^b_k M$ projection with $\tilde{T}^b_k M$ as the base space, ${}_a\pi^s := \tilde{T}^a_k \pi^s \colon \tilde{T}^a_k(\tilde{T}^s_k M) \to \tilde{T}^a_k M$ induced projection originating by the posterior application of the functor $\tilde{T}^a_k$ and ${}_a\pi^s_b := \tilde{T}^a \pi^s_{\tilde{T}^b M}$ the general case containing both previous cases.

Semiholonomní rychlosti

In general, we can obtain subbundles of $\tilde{T}^r_k M$ by equalizations of some projections to $\tilde{T}^r_k M$. Especially, An element $Z \in \tilde{T}^r_k M$ is called the *semiholonomic* k–dimensional velocity of r–th order, if for all $q = 1, \dots, r$

$$\pi^1_{r-1}(Z) = {}_{q-1}\pi^r_{r-q}(Z)$$

is satisfied. We denote by $\bar{T}^r_k$ the bundle of semiholonomic k–dimensional velocities of r–th order.

Semiholonomní rychlosti

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Identifikační teorém

There is an isomorphism of vector bundles

$$\mathcal{S}^{rU}_{sT}M = \bigotimes^{r} TM \otimes \bigotimes^{s} T^*M \otimes \bar{T}^{T*}M \otimes (\bar{T}^{U*}M)^*.$$

Literatura

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Děkuji za pozornost.

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