Masaryk University
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Symmetries of Parabolic Geometries

Ph.D. Thesis

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Abstrakt. V této práci se budeme zabývat symetriemi parabolických geometrií. Naší hlavní motivací jsou afinní lokálně symetrické prostory. Varieta s afinní konexí je lokálně symetrická právě tehdy, když je její torze nulová a křivost kovariantně konstantní. Na tyto geometrie lze nahlížet jako na konkrétní případ reduktivních Cartanových geometrií. Parabolické geometrie jsou jiným speciálním případem Cartanových geometrií. Nejsou reduktivní a jsou druhého řádu.

Budeme se zabývat zejména |1|–gradovanými parabolickými geometriemi. Pro tyto geometrie je definice symetrie intuitivní a odpovídá klasickému přístupu. Dokážeme analogii klasických výsledků a s využitím existující teorie parabolických geometrií ukážeme nějaká další omezení na křivost.

Většina symetrických |1|–gradovaných geometrií musí být lokálně plochá. Existují však i některé 'zajímavější' případy geometrií, které mohou mít symetrii v bodě ve kterém je nenulová křivost. S využitím teorie Weylových struktur budeme hledat další omezení na Weylovu křivost těchto případů. Nakonec ukážeme nějaké další důsledky pro projektivní a konformní geometrie.

Abstract. We introduce and discuss symmetries for the so called parabolic geometries. Our motivation comes from affine locally symmetric spaces. The manifold with affine connection is locally symmetric if and only if the torison vanishes and the curvature is covariantly constant. These geometries can be understood as the special case of reductive Cartan geometries. The parabolic geometries represent another special case of the general Cartan geometries. They are of second order and never reductive.

We are interested in |1|–graded geometries. In this case, the definition of the symmetry is a generalization of the clasical one and follows the intuitive idea. We show an analogy of the results from the affine locally symmetric spaces and we get more curvature restrictions, which come from the general theory of parabolic geometries.

Many types of symmetric |1|–graded geometries have to be locally flat. There are also some 'interesting' types, which can carry a symmetry in the point and still allow some nonzero curvature at this point. Our main tool to study these examples is the theory of Weyl structures. We study the Weyl curvature of |1|–graded geometries to get more restriction for the 'interesting' ones. Finally we show some corollaries in the projective and conformal geometries.

Contents

Preface

Locally symmetric spaces are the well known and studied examples of manifolds with rich additional structures. There mostly exists some restriction on the geometry coming from existence of some special morphisms of the structure. The best known examples are affine locally symmetric spaces and Riemannian locally symmetric spaces.

The affine locally symmetric space is a manifold such that at each point, there is a locally defined affine transformation which preserves the point and its differential reverts whole tangent space in the point. Riemannian locally symmetric spaces are affine locally symmetric spaces such that the latter affine transformations are isometries of the metric. These spaces can be equivalently characterized via curvature and torsion of the corresponding connection. The aim of this work is to extend the classical concepts for a wider class of geometries.

In the Chapter 1, we remind basic facts on the affine locally symmetric spaces. We give basic definitions in Section 1 and show the basic property of these spaces in a different way in Section 3.

One can consult [**13**] for more detailed study of affine and Riemannian locally symmetric spaces. The classical approach to the symmetric and locally symmetric spaces can be also found in [**10**]. Further examples of general geometries allowing symmetries have been studied only rarely. See [**11**] for study of symmetries on Cauchy–Riemann manifolds and [**16**] for discussion on projective symmetries on a manifold.

We would like to generalize the notion on symmetries for a manifold, which carry a Cartan connection. These Cartan geometries involve many types of geometries, affine connections and Riemannian manifolds are special cases of them. We give basic facts about Cartan geometries in Section 2. Our main reference will be [**18**].

There is no universal definition of symmetry for arbitrary Cartan geometry, see Section 3. We are interested mainly in $|1|$–graded parabolic geometries, which involve many types of interesting examples. The best known ones are projective and conformal structures.

Parabolic geometries are special cases of Cartan geometries. The existence of the filtration and the associated gradation of the tangent bundle is typical for them and we define the symmetry as an automorphism, which reverts the smallest part of this filtration. The $|1|$–graded geometries are the easiest ones. The smallest part of the filtration is equal to the whole tangent space and the definition of symmetries coincides with the classical one.

We summarize basic facts about parabolic geometries in Chapter 2. More detailed discussion and proofs can be found in [**2, 5, 6, 19**]. In Section 4, we give basic definitions and describe general properties. In Section 5, we introduce our main tool to study the $|1|$–graded geometries – Weyl structures. They exist for all parabolic geometries and allow us to describe the underlying structure on the manifold. We describe Weyl structures only in the $|1|$–graded case, general theory can be found in [**6**]. In the Section 6 we give the main definition. We define symmetric parabolic geometry and present the easiest examples – the homogeneous models.

In Chapter 3 we discuss the symmetric $|1|$–graded geometries. Motivated by the affine case we would like to find some restriction on the curvature of $|1|$–graded geometry carrying some symmetry. The crucial observation is that all symmetric $|1|$–graded geometries are torsion free. One of the useful features of the parabolic geometries is the unique normalization of the Cartan connections providing full information on the curvature in terms of its harmonic components. Moreover, the latter components are easily computable via Lie algebra cohomologies, and in most of our examples the harmonic part of the curvature appears in the torsion only. But there are also some 'interesting' geometries such that this theory gives us no more information. We describe this in Section 8. For more information on harmonic curvature see e.g. [**5, 6, 21**].

In Section 9, we return to the Weyl structures. We study the action of symmetries on the Weyl structures. We show, that there exist so called fixed Weyl structures, which are crucial for us. We also study the consequences of the latter results for effective geometries.

In the Section 10 we show some stronger curvature restriction coming from the existence of more then one symmetry in a point. We apply the results on some of the 'interesting' geometries. Namely, we deal with projective and conformal geometry.

Our general approach in the Sections 9 and 10 was inspired by [**16**] who studied the projective case in a classical setup of affine connections. Our methods work for all $|1|$–graded parabolic geometries.

Acknowledgments. I would like to thank to my supervisor Jan Slovák for his leading in the topic and the other discussion. I have learned most of the topics from the forthcoming monography [**7**] by Andreas Čap and Jan Slovák, as well as discussions with many nice colleagues, Vojtěch Žádník in particular. The financial support by the grant GACR 201/05/H005 has been essential too.

CHAPTER 1

Cartan geometries

In this chapter we first remind the classical definition of affine locally symmetric spaces. In the next section we summarize basic facts about Cartan geometries and we establish the notation. Finally we return to the locally symmetric spaces and we discuss them via the language of Cartan geometries. More on the classical theory of affine symmetric spaces and affine locally symmetric spaces can be found in [**13, 10**]. For more detailed discussion of Cartan geometries see [**18**].

1. Affine locally symmetric spaces

Let M be a manifold with a linear connection on the tangent bundle given by a covariant derivative

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M), \\ (\xi, \eta) &\mapsto \nabla_\xi \eta. \end{aligned}$$

We call a linear connection on the tangent bundle an *affine connection.*

Let us remind that the *curvature* R of the affine connection and its *torsion* T are given by

$$\begin{aligned} R(\xi,\eta)(\mu) &= \nabla_\xi \nabla_\eta(\mu) - \nabla_\eta \nabla_\xi(\mu) - \nabla_{[\xi,\eta]}(\mu), \\ T(\xi,\eta) &= \nabla_\xi(\eta) - \nabla_\eta(\xi) - [\xi,\eta] \end{aligned}$$

for all $\xi, \eta, \mu \in \mathfrak{X}(M)$. A smooth diffeomorphism f of M is called an *affine transformation* of M if f preserves the affine connection, i.e.

$$(f^*\nabla)_\xi(\eta) = f_*^{-1}(\nabla_{f_*\xi}(f_*\eta)) = \nabla_\xi(\eta).$$

If f is an affine transformation, then R and T are invariant with respect to f, i.e. $f^*R = R$ and $f^*T = T$. Now we remind the well known definition:

Definition 1.1. Let M be a manifold with an affine connection ∇. A *symmetry* at $x \in M$ is an affine transformation s_x defined on some neighborhood $U \subseteq M,\ x \in U$ such that:

(i) $s_x(x) = x$
(ii) $T_x s_x = -\mathrm{id}_{T_xM}$.

A manifold M with an affine connection ∇ is called *affine locally symmetric space* if there is some symmetry in each $x \in M$.

There is the well known description of an affine locally symmetric space given by its curvature and torsion, see [**13**]:

Proposition 1.2. *The manifold M with an affine connection ∇ is affine locally symmetric if and only if $T = 0$ and $\nabla R = 0$.*

There is an equivalent way how to understand affine geometries. We take the first order frame bundle $p : \mathcal{P}^1M \to M$ with structure group $Gl(n,\mathbb{R})$, where n is the dimension of M. There is the *canonical form* $\theta \in \Omega^1(\mathcal{P}^1M, \mathbb{R}^n)$ given on the frame bundle, its value $\theta(u)(\xi)$ is given as the coordinate of the projection $Tp.\xi \in T_{p(u)}M$ in the frame u. This form is strictly horizontal and $Gl(n,\mathbb{R})$–equivariant. In fact, we get a (first order) G–structure with structure group $Gl(n,\mathbb{R})$.

Next, there is a well known one to one correspondence between affine connections ∇ on M and principal connections $\gamma \in \Omega^1(\mathcal{P}^1M, \mathfrak{gl}(n,\mathbb{R}))$ on the $Gl(n,\mathbb{R})$–structure. For fixed principal connection γ we take the induced connection on TM. The tangent bundle is identified with $\mathcal{P}^1M \times_{Gl(n,\mathbb{R})} \mathbb{R}^n$ such that the pair $[\![u, X]\!]$ represents a vector with coordinates X in the frame u. The principal connection γ together with the canonical form θ give a 1–form

$$\omega = \theta + \gamma \in \Omega^1(\mathcal{P}^1M, \mathfrak{a}(n,\mathbb{R})),$$

where $\mathfrak{a}(n,\mathbb{R}) = \mathbb{R}^n \oplus \mathfrak{gl}(n,\mathbb{R})$. This form satisfies conditions on Cartan connection and we will call it affine Cartan connection.

Further, there is the automorphism $\mathcal{P}^1f$ of $\mathcal{P}^1M$ uniquely corresponding to the automorphism f of M and $\mathcal{P}^1f$ automatically preserves the canonical form θ. If f is an affine transformation, then $\mathcal{P}^1f$ preserves the connection γ and we get the automorphism of the affine Cartan connection.

It is easy to reformulate notions from affine symmetric spaces in the language of affine Cartan connection. The general concept of Cartan connections covers all first order G–structures with chosen compatible connections. We return to the affine locally symmetric spaces in Section 3.

2. Introduction to Cartan geometries

Let G be a Lie group, $P \subset G$ a Lie subgroup, and write $\mathfrak{g}$ and $\mathfrak{p}$ for their Lie algebras. A *Cartan geometry* of type (G, P) on a smooth manifold M is a principal fiber bundle $p : \mathcal{G} \to M$ with structure group P, together with a 1–form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ called a *Cartan connection* such that:

(1) $(r^h)^*\omega = \mathrm{Ad}_{h^{-1}} \circ \omega$ for each $h \in P$
(2) $\omega(\zeta_X(u)) = X$ for each $X \in \mathfrak{p}$
(3) $\omega(u) : T_u\mathcal{G} \longrightarrow \mathfrak{g}$ is a linear isomorphism for each $u \in \mathcal{G}$.

Here $\zeta_X \in \mathfrak{X}(\mathcal{G})$ denotes the fundamental vector field generated by $X \in \mathfrak{p}$.

A Cartan geometry is called *split* if and only if there is a fixed Lie subalgebra $\mathfrak{g}_- \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$ as a vector space. A Cartan geometry is called *reductive* if and only if $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$ is a fixed decomposition on P–modules with respect to the adjoint action.

The *homogeneous model* for Cartan geometries of type (G, P) is the canonical P–bundle $p : G \longrightarrow G/P$ endowed with the left Maurer–Cartan form $\omega_G \in \Omega^1(G, \mathfrak{g})$.

Example 2.1. *Affine geometry.* Let G be the affine group $A(n,\mathbb{R}) = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ v & A \end{smallmatrix}\right) \mid v \in \mathbb{R}^n, A \in Gl(n,\mathbb{R}) \right\}$. This group acts on every $u \in \mathbb{R}^n$ such that $u \mapsto Au + v$. It can be realized using matrix multiplication as follows: If we identify $\mathbb{R}^n$ with $\{ \left(\begin{smallmatrix} 1 \\ u \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid u \in \mathbb{R}^n \}$ we have

$$\left(\begin{smallmatrix} 1 & 0 \\ v & A \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 \\ u \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ Au+v \end{smallmatrix}\right).$$

The subgroup P is taken as the stabilizer of $0 \in \mathbb{R}^n$. It is isomorphic to the general lineal group $Gl(n, \mathbb{R})$ and the elements are of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right) \in A(n, \mathbb{R})$. The Lie algebra of G is $\mathfrak{a}(n, \mathbb{R}) = \left\{ \left(\begin{smallmatrix} 0 & 0 \\ w & B \end{smallmatrix}\right) \mid w \in \mathbb{R}^n, B \in \mathfrak{gl}(n, \mathbb{R}) \right\}$ and $\mathfrak{gl}(n, \mathbb{R}) = \left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & B \end{smallmatrix}\right) \mid B \in \mathfrak{gl}(n, \mathbb{R}) \right\}$ in this case. We can easily compute that $\mathfrak{g} = \mathfrak{a}(n, \mathbb{R})$ is split as $\mathbb{R}^n \oplus \mathfrak{gl}(n, \mathbb{R})$, where $\mathfrak{g}_- = \mathbb{R}^n$ and $\mathfrak{p} = \mathfrak{gl}(n, \mathbb{R})$ are $Gl(n, \mathbb{R})$–modules and geometries of this type are reductive. We have

$$\mathrm{Ad}_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right)} \left(\begin{smallmatrix} 0 & 0 \\ w & B \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 0 \\ w & B \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & A^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ Aw & ABA^{-1} \end{smallmatrix}\right).$$

Homogeneous model is the affine plane $\mathbb{R}^n \simeq A(n, \mathbb{R})/Gl(n, \mathbb{R})$.

The curvature of a Cartan geometry is given by the *curvature form* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined by the structure equation

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)].$$

The third property of ω defines the so called *constant vector fields* $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{G})$ for every element $X \in \mathfrak{g}$. These generate the tangent bundle $T\mathcal{G}$ and the curvature can be equivalently described by the *curvature function* $\kappa : \mathcal{G} \to \wedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$, where

$$\begin{aligned} \kappa(u)(X, Y) &= K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)) = \\ &= [X, Y] - \omega([\omega^{-1}(X), \omega^{-1}(Y)](u)). \end{aligned}$$

If at least one of the arguments is vertical, then the curvature vanishes and the curvature function may be viewed as $\kappa : \mathcal{G} \to \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. This function is also right–equivariant, i.e. $\kappa \circ r^g = g^{-1} \cdot \kappa$ for all $g \in P$, where $\cdot$ is the tensor product of the adjoint actions $\underline{\mathrm{Ad}}^*$ on $(\mathfrak{g}/\mathfrak{p})^*$ induced from adjoint action on $\mathfrak{g}$ and Ad on $\mathfrak{g}$.

The *torsion* of the Cartan geometry is defined by the composition of the values of the curvature function with the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$. If the torsion is zero, i.e. the values of κ are in $\wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$, we call the Cartan geometry *torsion free.*

A *morphism of Cartan geometries* of the same type from $(\mathcal{G} \to M, \omega)$ to $(\mathcal{G}' \to M', \omega')$ is a principal bundle morphism $\varphi : \mathcal{G} \to \mathcal{G}'$ such that $\varphi^*\omega' = \omega$. Further we will denote the base morphism of φ induced on M as $\underline{\varphi} : M \to M$. Every morphism of Cartan geometries preserves constant vector fields and then it preserves flows of constant vector fields, i.e. we have

$$\begin{aligned} T\varphi \circ \omega^{-1}(X) &= \omega'^{-1}(X) \circ \varphi, \\ \varphi \circ \mathrm{Fl}_t^{\omega^{-1}(X)}(u) &= \mathrm{Fl}_t^{\omega'^{-1}(X)}(\varphi(u)) \end{aligned}$$

for all $X \in \mathfrak{g}$. In addition, the curvatures of the geometries K and K' are φ–related and their curvature functions κ and κ' satisfy $\kappa = \kappa' \circ \varphi$.

The Maurer–Cartan equation says that the curvature of homogeneous model is zero. It can be proved, see e.g. [**18**]:

THEOREM 2.2. *If the curvature of a Cartan geometry of type (G, P) vanishes, then the geometry is locally isomorphic with the homogeneous model $(G \to G/P, \omega_G)$.*

Cartan geometry is called *locally flat* if the curvature κ vanishes. Homogeneous models are sometimes called *flat* models.

We shall deal with the automorphisms of Cartan geometries. In the homogeneous case, there is the famous Liouville theorem, see [**18**]:

THEOREM 2.3. *All (locally defined) automorphisms of the homogeneous model $(G \to G/P, \omega_G)$ are left multiplications by elements of G.*

We define the *kernel* K of the geometry of type (G, P) as the maximal normal subgroup of G that is contained in P. The geometry is called *effective* if the kernel is trivial and the geometry is called *infinitesimally effective* if the kernel is discrete. Clearly the kernel of the geometry is discrete if and only if there is no ideal of $\mathfrak{g}$ contained in $\mathfrak{p}$.

We can explain the meaning of the elements from the kernel in the following way: On the homogeneous model, the automorphism given by the left multiplication by element g induces the base automorphism $\ell_g : G/H \to G/H$. It can be shown that the element g belongs to the kernel if and only if $\ell_g = \mathrm{id}_{G/H}$. Thus multiplication by elements from K covers identity on G/P.

More generally, we can nicely describe the morphisms covering the fixed base morphism via the kernel. It can be proved, see [**18**]:

THEOREM 2.4. *Let $(\mathcal{G} \to M, \omega)$ and $(\mathcal{G}' \to M', \omega')$ be Cartan geometries of type (G, P) and let us denote by K the kernel. Let φ_1 and φ_2 be morphisms of these Cartan geometries such that they have the same base morphism $\underline{\varphi} : M \to M'$. Then there exists smooth function $f : \mathcal{G} \to K$ such that $\varphi_1(u) = \varphi_2(u) \cdot f(u)$ for all $u \in \mathcal{G}$.*

In particular, if the geometry is effective, then $\varphi_1 = \varphi_2$ and f is constant on connected components of M for infinitesimally effective geometries.

The existence of the Cartan connection allows us to describe nicely the tangent bundle TM of the base manifold. We have $TM \simeq \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, where the action of P on $\mathfrak{g}/\mathfrak{p}$ is the action $\underline{\mathrm{Ad}}$ induced by the action Ad on $\mathfrak{g}$. The identification is provided by the mapping

$$[\![u, X + \mathfrak{p}]\!] \mapsto Tp.\omega^{-1}(X)(u).$$

The tangent vector is equivalently given by so called *frame form*, the P–equivariant mapping

$$s : \mathcal{G} \to \mathfrak{g}/\mathfrak{p}, \quad s(u) = X + \mathfrak{p},$$

which is exactly the vector $[\![u, s(u)]\!] \in T_{p(u)}M$.

In the case of split geometry we can write $TM \simeq \mathcal{G} \times_P \mathfrak{g}_-$ for the $\underline{\mathrm{Ad}}$–action on $\mathfrak{g}_- \simeq \mathfrak{g}/\mathfrak{p}$. We have similar identifications for the cotangent bundle and arbitrary tensor bundles.

The morphism φ of Cartan geometries by means of its base morphism $\underline{\varphi}$ induces uniquely the tangent morphism $T\underline{\varphi} : TM \to TM'$ and it can be nicely written using the previous identification. We have

$$\begin{aligned} T\underline{\varphi}([\![u, X + \mathfrak{p}]\!]) &= T\underline{\varphi} \circ Tp \circ \omega^{-1}(X)(u) = Tp \circ T\varphi \circ \omega^{-1}(X)(u) = \\ &= Tp \circ \omega'^{-1}(X)(\varphi(u)) = [\![\varphi(u), X + \mathfrak{p}]\!]. \end{aligned}$$

Again, similar computation works for the cotangent bundle and arbitrary tensor bundles and can be rewritten in the language of frame forms. In the sequel, we will use both descriptions of tensors and tensor fields.

3. Symmetries on Cartan geometries

Let us return to the case of some manifold M with affine connection ∇. We can easily reformulate the definition of affine locally symmetric space in the language of affine Cartan geometries. We know that the manifold M of dimension n with the affine connection ∇ corresponds to the Cartan geometry $(\mathcal{G} \to M, \omega)$ of type $(A(n,\mathbb{R}), Gl(n,\mathbb{R}))$. Geometries of this type are reductive and first order, we just have $\mathcal{G} \simeq \mathcal{P}^1 M$ and the Cartan connection $\omega \in \Omega^1(\mathcal{P}^1 M, \mathfrak{a}(n,\mathbb{R}))$ naturally divides itself into the canonical form $\theta \in \Omega^1(\mathcal{P}^1 M, \mathbb{R}^n)$ and the principal connection $\gamma \in \Omega^1(\mathcal{P}^1 M, \mathfrak{gl}(n,\mathbb{R}))$. We get the following definition:

Definition 3.1. The *symmetry in* $x \in M$ on the affine Cartan geometry, i.e. on the Cartan geometry of type $(A(n,\mathbb{R}), Gl(n,\mathbb{R}))$, is the locally defined diffeomorphism s_x on M satisfying following conditions:

(i) $s_x(x) = x$
(ii) $T_x s_x = -\mathrm{id}_{T_x M}$
(iii) s_x is covered by an automorphism φ of the affine Cartan geometry, i.e. $s_x = \underline{\varphi}$ on a suitable neighborhood of x.

The affine Cartan geometry is called *locally symmetric* if there exists some symmetry in each $x \in M$.

Obviously, the only possible covering φ of s_x is $\mathcal{P}^1 s_x$. Our definition clearly corresponds to the definition of the affine symmetry. It allows us to discuss the proof of the Theorem 1.2 in the following way: Suppose that there is a symmetry s_x in x covered by an automorphism φ of the affine Cartan geometry. The curvature of the geometry is described by the $Gl(n,\mathbb{R})$–equivariant function

$$\kappa : \mathcal{P}^1 M \to \wedge^2 \mathbb{R}^{n*} \otimes \mathfrak{a}(n,\mathbb{R}).$$

It naturally splits into two $Gl(n,\mathbb{R})$–equivariant parts corresponding to the torsion and curvature of the connection γ. We have

$$\begin{aligned} \tau &: \mathcal{P}^1 M \to \wedge^2 \mathbb{R}^{n*} \otimes \mathbb{R}^n, \\ \rho &: \mathcal{P}^1 M \to \wedge^2 \mathbb{R}^{n*} \otimes \mathfrak{gl}(n,\mathbb{R}). \end{aligned}$$

In fact, functions τ and ρ are the frame forms of the torsion $T \in \Gamma(\wedge^2 T^* M \otimes TM)$ and the curvature $R \in \Gamma(\wedge^2 T^* M \otimes TM^* \otimes TM)$ of ∇. We use the identification $\mathfrak{gl}(n,\mathbb{R}) \simeq \mathbb{R}^{n*} \otimes \mathbb{R}^n$.

The morphism φ in the point x satisfies

$$T\underline{\varphi}([\![u, X]\!]) = [\![\varphi(u), X]\!] = [\![uA, X]\!] = [\![u, A^{-1}X]\!]$$

for suitable transition matrix $A \in Gl(n,\mathbb{R})$, where u is some fixed frame in the fiber over x and the coordinates $X \in \mathbb{R}^n$ are arbitrary. The left multiplication coincides in the affine case exactly with the adjoint action. At the same time we have that $T\varphi([\![u, X]\!]) = [\![u, -X]\!]$ holds in the frame u and we get $A^{-1}X = -X$.

The torsion function τ satisfies

$$\tau(\varphi(u))(X, Y) = \tau(uA)(X, Y) = A(\tau(u)(A^{-1}X, A^{-1}Y))$$

and we know that left multiplication on $\mathbb{R}^n$ by the element A^{-1} changes the sign of each $Z \in \mathbb{R}^n$. Clearly, the left multiplication by element A changes the sign too. The properties of κ give us

$$\tau(u)(X,Y) = \tau(\varphi(u))(X,Y) = -\tau(u)(-X,-Y) = -\tau(u)(X,Y)$$

and the torsion has to vanish.

The same argument does not work for the curvature ρ. The values are in $\wedge^2\mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^n$ and we get trivial identity

$$\rho(u)(X,Y)(Z) = \rho(\varphi(u))(X,Y)(Z) = -\rho(u)(-X,-Y)(-Z).$$

In the case of affine geometry, we can simply take the affine connection and study covariant derivatives of the curvature. The derivative ∇R is described by a frame form

$$\nabla\rho : \mathcal{P}^1 M \to \mathbb{R}^{n*} \otimes \wedge^2\mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^n$$

such that $\nabla_\xi R$ is simply given by the function $\xi^{hor} \cdot \rho : \mathcal{P}^1 M \to \wedge^2\mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^n$. Here ξ^{hor} is the horizontal lift of $\xi \in \mathfrak{X}(M)$ with respect to γ.

For the function $\nabla\rho$ we have

$$\nabla_U\rho(u)(X,Y)(Z) = \nabla_U\rho(\varphi(u))(X,Y)(Z) = -\nabla_{-U}\rho(u)(-X,-Y)(-Z) = \\ = -\nabla_U\rho(u)(X,Y)(Z)$$

and the covariant derivative of the curvature vanishes. In fact, we have got one direction of the proof of the Proposition 1.2.

The opposite direction of the proof is more complicated and uses the concept of normal coordinates for affine geometries. It is possible to extend the symmetry from one point (in which is it uniquely given) to the neighborhood of this point as the solution of the system of first order ordinary differential equations. The choice of normal coordinates allows us to write explicitly this system, which is a system of partial differential equations in general. It uses a nice fact that in the normal coordinates, the geodesics are only straight lines and symmetry turns them. The proof can be found in [**13**].

Remark on the general case

We would like to formulate the definition of the symmetry for a Cartan geometry of arbitrary type. We could generalize the definition for affine Cartan geometries and define the symmetry in x on the Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G,P) as the base morphism $\underline{\varphi}$ of some automorphism φ of $\mathcal{G}$ such that $\underline{\varphi}(x) = x$ and $T_x\underline{\varphi}|_{T_xM} = -\mathrm{id}_{T_xM}$. It seems to be nice and simple, but it is not reasonable for general geometries.

As an example we take the contact geometries. We again start with the classical approach and we describe the corresponding structure on the manifold. A contact structure on the manifold M of dimension $2n+1$ is a distribution $HM \subset TM$ of codimension 1 such that the mapping

$$\mathcal{L} : \wedge^2 HM \to TM/HM$$

induced by the Lie bracket is nondegenerate.

Suppose that there is some automorphism s_x of M respecting the contact structure such that $s_x(x) = x$ and $T_x s_x = -\mathrm{id}_{T_xM}$, i.e. some 'symmetry' in $x \in M$. We then have $T_x s_x^H(\mathcal{L}(\xi,\eta)) = \mathcal{L}(T_x s_x(\xi), T_x s_x(\eta))$ for all $\xi, \eta \in \mathfrak{X}(M)$, where $T_x s_x^H$ denotes the morphism induced from $T_x s_x$ on $T_x M/H_x M$. Next, for any $\xi, \eta \in \Gamma(HM) \subset \Gamma(TM)$ we have

$$\begin{aligned}\mathcal{L}(T_x s_x(\xi), T_x s_x(\eta)) &= \mathcal{L}(-\mathrm{id}(\xi), -\mathrm{id}(\eta)) = \mathcal{L}(-\xi,-\eta) = \\ &= q([-\xi,-\eta]) = q([\xi,\eta])\end{aligned}$$

in x, where $q : TM \to TM/HM$ is the projection. For the same vector fields we simultaneously have

$$T_x s_x^H(\mathcal{L}(\xi,\eta)) = -\mathrm{id}^H(\mathcal{L}(\xi,\eta)) = -\mathrm{id}^H(q([\xi,\eta])) = -q([\xi,\eta]),$$

where $-\mathrm{id}^H$ is an automorphism of $T_x M/H_x M$ induced from $-\mathrm{id}$ on $T_x M$ and it is again only sign change. Thus our assumptions on s_x imply $q([\xi,\eta]) = -q([\xi,\eta])$ and the non–degeneracy of $\mathcal{L}$ gives us that $q([\xi,\eta])$ is not identically zero. The latter simple computation shows that we cannot require the differential to be minus identity everywhere.

More generally, we encounter similar behavior on all filtered manifolds, i.e. on manifolds with fixed sequence of distributions $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ such that the Lie brackets of vector fields satisfy $[\xi,\eta] \in T^{i+j}M$ for all $\xi \in T^i M$ and $\eta \in T^j M$. The morphisms of filtered manifold respect the filtrations and so their differentials may be required to be minus identity only on the smallest subspace $T^{-1}M$.

All parabolic geometries are geometrically defined as filtered manifolds with some non–degeneracy conditions and often also some more structural information. It turns out, that the concept of symmetries as above makes good sense in general.

In this work, however, we shall be mainly interested in the special case of parabolic geometries where the filtration is trivial, the so called $|1|$–graded geometries. Then the definition completely reduces to the classical one. We return to the definition of the symmetry after the introduction to parabolic geometries, see Section 6.

We belive that our approach will be fruitful also for more general parabolic geometries. In particular, the contact parabolic geometries enjoy a filtration of contact type and so all morphisms of such Cartan geometries are morphism of the corresponding contact structures. Such symmetries have not been studied much in the literature yet, see [**11**] for an example. We shall come back to such problems elsewhere.

CHAPTER 2

Basic facts on parabolic geometries

In this chapter we summarize basic facts on parabolic geometries. At first, we remind definitions and results of the general theory. We give here a complete list of $|1|$–graded geometries. Next, we recall our main tool for studying $|1|$–graded geometries – the Weyl structures. Finally, we formulate the necessary condition for existence of a symmetry on the $|1|$–graded geometry. We finish this chapter with the discussion of homogeneous models. For more detailed discussion of parabolic geometries see e.g. [**6**].

4. Parabolic geometries

Let $\mathfrak{g}$ be a semisimple Lie algebra and $k > 0$. The $|k|$*–grading on* $\mathfrak{g}$ is the vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i and j (we understand $\mathfrak{g}_r = 0$ for $|r| > k$) and such that the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$. We will suppose that there is no simple ideal of $\mathfrak{g}$ contained in $\mathfrak{g}_0$. Each gradation of $\mathfrak{g}$ defines the *filtration*

$$\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \cdots \supset \mathfrak{g}^k = \mathfrak{g}_k,$$

where $\mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$. In particular $\mathfrak{g}_0$ and $\mathfrak{g}^0 =: \mathfrak{p}$ are subalgebras of $\mathfrak{g}$ and $\mathfrak{g}^1 =: \mathfrak{p}_+$ is a nilpotent ideal in $\mathfrak{p}$.

Let G be a semisimple Lie group with the Lie algebra $\mathfrak{g}$. The choice of the group G and also the choice of the subgroups $G_0 \subset P \subset G$ (with the prescribed subalgebras $\mathfrak{p}$ and $\mathfrak{g}_0$) impact the properties of the resulting geometries. The obvious choice is this one:

$$\begin{aligned} G_0 &:= \{g \in G \mid \mathrm{Ad}_g(\mathfrak{g}_i) \subset \mathfrak{g}_i,\ \forall i = -k, \dots, k\}, \\ P &:= \{g \in G \mid \mathrm{Ad}_g(\mathfrak{g}^i) \subset \mathfrak{g}^i,\ \forall i = -k, \dots, k\}. \end{aligned}$$

It is the maximal possible choice, but we may also take the connected component of the unit in these subgroups or anything between these two extremes. It is not difficult to show for these subgroups, see [**21**]:

Proposition 4.1. *Let $\mathfrak{g}$ be a $|k|$–graded semisimple Lie algebra and G be a Lie group with Lie algebra $\mathfrak{g}$.*

(1) *$G_0 \subset P \subset G$ are closed subgroups with Lie algebras $\mathfrak{g}_0$ and $\mathfrak{p}$, respectively.*
(2) *The map $(g_0, Z) \mapsto g_0 \exp Z$ defines a diffeomorphism $G_0 \times \mathfrak{p}_+ \to P$.*

The group P is a semidirect product of the reductive subgroup G_0 and the nilpotent normal subgroup $P_+ := \exp \mathfrak{p}_+$ of P.

A *parabolic geometry* is a Cartan geometry of type (G,P), where G and P are as above. If the length of the gradation of $\mathfrak{g}$ is k, then the geometry is called $|k|$–*graded*.

These geometries are always split, but never reductive. Furthermore, under the assumption that there is no simple ideal of $\mathfrak{g}$ contained in $\mathfrak{g}_0$, parabolic geometries are infinitesimally effective, but they are not effective in general.

EXAMPLE 4.2. *Conformal Riemannian structures.* We take the Cartan geometry of the type (G,P) where $G = O(p+1,q+1)$ is the orthogonal group and P is the Poincaré conformal subgroup. The group G exactly looks like

$$G = \left\{ A \;\middle|\; A \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & J & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) A^{-1} = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & J & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) \right\},$$

where $J = \left(\begin{smallmatrix} E & 0 \\ 0 & -E \end{smallmatrix}\right)$ is the standard product of signature (p,q). Its Lie algebra is of the form

$$\mathfrak{g} = \left\{ \left(\begin{smallmatrix} a & Z & 0 \\ X & A & -JZ^T \\ 0 & -X^TJ & -a \end{smallmatrix}\right) \;\middle|\; a \in \mathbb{R},\ X, Y^T \in \mathbb{R}^n,\ A \in \mathfrak{o}(p,q) \right\}.$$

It can be written as the sum of three parts $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{-1} \simeq \mathbb{R}^n$, $\mathfrak{g}_0 \simeq \mathbb{R} \oplus \mathfrak{o}(p,q)$ and $\mathfrak{g}_1 \simeq \mathbb{R}^{n*}$. These parts correspond to the block lower triangular part, block diagonal part and block upper triangular part and give exactly the gradation of $\mathfrak{g}$ of the length $|1|$. The elements from the subgroup

$$G_0 = \left\{ \left(\begin{smallmatrix} \lambda & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & \lambda^{-1} \end{smallmatrix}\right) \;\middle|\; \lambda \in \mathbb{R} - \{0\},\ C \in O(p,q) \right\}$$

preserve this gradation. Elements from subgroup $P = G_0 \rtimes \exp \mathfrak{g}_1$ preserving the filtration then look like

$$\left(\begin{smallmatrix} \lambda & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & \lambda^{-1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & Z & -\frac{1}{2}ZJZ^T \\ 0 & E & -JZ^T \\ 0 & 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \lambda & \lambda Z & -\frac{\lambda}{2}ZJZ^T \\ 0 & C & -CJZ^T \\ 0 & 0 & \lambda^{-1} \end{smallmatrix}\right).$$

We get the $|1|$–graded geometry and its homogeneous model is the conformal pseudosphere of the corresponding signature.

THE CURVATURE OF PARABOLIC GEOMETRIES

The curvature function $\kappa : \mathcal{G} \to \wedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$ is valued in the cochains for the second cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$. This group can be also computed as the homology of the codifferential $\partial^* : \wedge^{k+1}\mathfrak{g}_-^* \otimes \mathfrak{g} \to \wedge^k \mathfrak{g}_-^* \otimes \mathfrak{g}$, where

$$\begin{aligned} \partial^*(Z_0 \wedge \cdots \wedge Z_k \otimes W) = & \sum_{i=0}^{k} (-1)^{i+1} Z_0 \wedge \cdots \hat{i} \cdots \wedge Z_k \otimes Z_i \cdot W + \\ & \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge \cdots \hat{i} \cdots \hat{j} \cdots \wedge Z_k \otimes W \end{aligned}$$

for all $Z_0, \dots, Z_k \in \mathfrak{g}_-^* \simeq \mathfrak{p}_+$ and $W \in \mathfrak{g}$. We use here the duality between P–modules $\mathfrak{g}_- \simeq \mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}_+$ given by the Killing form on $\mathfrak{g}$. The symbol $\hat{i}$ denotes omission of corresponding Z_i.

The parabolic geometry is called *normal* if the curvature satisfies

$$\partial^* \circ \kappa = 0.$$

If the geometry is normal, we can define the *harmonic part of curvature* $\kappa_H : \mathcal{G} \to H^2(\mathfrak{g}_-, \mathfrak{g})$ as the composition of the curvature function and the projection to the second cohomology group.

Thanks to the gradation of $\mathfrak{g}$, there are several decompositions of the curvature of the parabolic geometry. One of the possibilities is the decomposition into *homogeneous components*, which is of the form

$$\kappa = \sum_{i=-k+2}^{3k} \kappa^{(i)},$$

where $\kappa^{(i)}(u)(X,Y) \in \mathfrak{g}_{p+q+i}$ for all $X \in \mathfrak{g}_p, Y \in \mathfrak{g}_q$ and $u \in \mathcal{G}$.

The parabolic geometry is called *regular* if the curvature function κ satisfies $\kappa^{(r)} = 0$ for all $r \leq 0$. The crucial structural description of the curvature is provided by the following Theorem, see [**21**]:

THEOREM 4.3. *The curvature κ of a regular normal geometry vanishes if and only if its harmonic part κ_H vanishes.*

Moreover, if all homogeneous components of κ of degrees less than j vanish identically and there is no cohomology $H^2_j(\mathfrak{g}_-, \mathfrak{g})$, then also the curvature component of degree j vanishes.

Another possibility is the decomposition of the curvature according to the values:

$$\kappa = \sum_{j=-k}^{k} \kappa_j$$

and in an arbitrary frame u we have $\kappa_j(u) \in \mathfrak{g}_- \wedge \mathfrak{g}_- \to \mathfrak{g}_j$. In fact, the component κ_- valued in $\mathfrak{g}_-$ is the torsion of the geometry.

In the case of $|1|$–graded geometries the decomposition by the homogeneity corresponds to the decomposition according to the values. The homogeneous component of degree 1 corresponds to the torsion while the homogeneous components of degrees 2 and 3 coincide with κ_0 and κ_1.

THE CURVATURE AS A TRACTOR VALUED FORM

Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P). We can define the *adjoint tractor bundle* $\mathcal{A}M$ as the associated bundle $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$. The Lie bracket on $\mathfrak{g}$ defines a bundle map

$$\{\ ,\ \} : \mathcal{A}M \otimes \mathcal{A}M \to \mathcal{A}M$$

which makes any fiber of $\mathcal{A}M$ isomorphic with the Lie algebra $\mathfrak{g}$. For all $u \in \mathcal{G}$ and $X, Y \in \mathfrak{g}$ it is defined by

$$\{[\![u, X]\!], [\![u, Y]\!]\} = [\![u, [X, Y]]\!].$$

More generally, let $\lambda : G \to Gl(V)$ be a linear representation. We define the *tractor bundle* $\mathcal{V}M$ as the associated bundle $\mathcal{V}M := \mathcal{G} \times_P V$ with respect to the restriction of the action λ to the subgroup P. In the case of adjoint representation $\mathrm{Ad} : G \to Gl(\mathfrak{g})$ we get exactly the adjoint bundle.

Elements of the associated bundle and also the sections of the bundle are called (*adjoint*) *tractors*. We have to start with the G–representation and not only with the P–representation, see [**3, 4**] for explanation and complete discussion of the theory of tractors and tractor calculi.

The adjoint tractor bundle $\mathcal{A}M$ acts on an arbitrary tractor bundle $\mathcal{V}M = \mathcal{G} \times_P V$. For all $u \in \mathcal{G}$, $X \in \mathfrak{g}$ and $v \in V$ we define the *algebraic action*

$$\bullet : \mathcal{A}M \otimes \mathcal{V}M \to \mathcal{V}M$$
$$[\![u, X]\!] \bullet [\![u, v]\!] = [\![u, \lambda'_X(v)]\!],$$

where $\lambda' : \mathfrak{g} \to \mathfrak{gl}(V)$ is the infinitesimal representation given by $\lambda : G \to Gl(V)$. If we take $\lambda = \mathrm{Ad}$ and $\lambda' = \mathrm{ad}$, we get exactly the bracket $\{\ ,\ \}$. It is sometimes called *algebraic bracket*.

The action is correctly defined because the G–action λ' is equivariant for the action Ad on $\mathfrak{g}$ and for the action on $L(V, V)$ induced by λ. In fact, any G–representation λ gives the natural bundle $\mathcal{V}$ on Cartan geometries of fixed type and all P–invariant operations on representations give rise to geometric operations on the corresponding natural bundles. Similarly, the P–equivariant morphisms of representations give corresponding bundle morphisms.

In this way, the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ naturally induces the bundle projection

$$\Pi : \mathcal{G} \times_P \mathfrak{g} = \mathcal{A}M \longrightarrow TM = \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$$

and we can see the adjoint tractor bundle as an extension of the tangent bundle. We can easily see:

Proposition 4.4. *The curvature function κ of a Cartan geometry can be viewed as a smooth section of associated bundle $\mathcal{G} \times_P (\wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g})$ with respect to the action coming from adjoint action of G, and equivalently as a $\mathcal{A}M$–valued two form $\kappa \in \Omega^2(M, \mathcal{A}M)$.*

We get the torsion as the composition $T := \Pi \circ \kappa \in \Omega^2(M, TM)$.

Now, let us suppose that we have a $|k|$–graded parabolic geometry. The filtration of $\mathfrak{g}$ is P–invariant and gives us the filtration of the adjoint subbundles

$$\mathcal{A}M = \mathcal{A}^{-k}M \supset \mathcal{A}^{-k+1}M \supset \cdots \supset \mathcal{A}^kM,$$

where $\mathcal{A}^iM := \mathcal{G} \times_P \mathfrak{g}^i$. The bundle $\mathcal{A}^0M$ is exactly the kernel of the projection Π. At the same time, we get the *associated graded bundle*

$$\mathcal{G} \times_P \mathrm{gr}(\mathfrak{g}) = \mathrm{gr}(\mathcal{A}M) = \mathcal{A}_{-k}M \oplus \mathcal{A}_{-k+1}M \oplus \cdots \oplus \mathcal{A}_kM,$$

where $\mathcal{A}_iM = \mathcal{A}^iM/\mathcal{A}^{i+1}M \simeq \mathcal{G} \times_P \mathfrak{g}^i/\mathfrak{g}^{i+1}$. Since the Lie bracket on $\mathrm{gr}(\mathfrak{g})$ is $\exp \mathfrak{g}_1$–invariant, there is the *algebraic bracket* on $\mathrm{gr}(\mathcal{A}M)$ defined by means of the Lie bracket. The latter bracket is compatible with the algebraic bracket on the tractor bundle and we denote both by the same symbol. We have

$$\{\ ,\ \} : \mathcal{A}_iM \times \mathcal{A}_jM \to \mathcal{A}_{i+j}M.$$

For each parabolic geometry we have the identification $TM \simeq \mathcal{G} \times_P \mathfrak{g}_-$, where the action of P on $\mathfrak{g}_- \simeq \mathfrak{g}/\mathfrak{p}$ is coming from adjoint action. The Killing form on $\mathfrak{g}$ induces the duality between this P–module and $\mathfrak{p}_+ \simeq \mathfrak{g}^1$ and we get $\mathcal{A}^1M \simeq \mathcal{G} \times_P \mathfrak{p}_+ \simeq T^*M$. In fact, the Killing form gives the pairing on $\mathrm{gr}(\mathcal{A}M)$ such that $\mathcal{A}_i^* \simeq \mathcal{A}_{-i}$.

We see that $TM \simeq \mathcal{A}M/\mathcal{A}^0M$ and we obtain the induced filtration of the tangent bundle

$$TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M,$$

where $T^iM \simeq \mathcal{A}^iM/\mathcal{A}^0M$. The filtration is compatible with the bracket of vector fields if and only if $\kappa(T^iM, T^jM) \subset \mathcal{A}^{i+j}M$ for all $i, j < 0$. Next, we can easily see:

Proposition 4.5. *The curvature κ of parabolic geometry is a section of $\wedge^2(\mathcal{A}M/\mathcal{A}^0M)^* \otimes \mathcal{A}M$ and then $\mathcal{A}M$–valued 2–form on M.*

The geometry is regular if and only if $\kappa(T^iM, T^jM) \subset \mathcal{A}^{i+j+1}M$ for all $i, j < 0$. The geometry is torsion free if and only if $\kappa(TM, TM) \subset \mathcal{A}^0M$. The Lie algebra codifferential defines natural mapping

$$\partial^* : \wedge^{k+1}\mathcal{A}^1M \otimes \mathcal{A}M \to \wedge^k\mathcal{A}^1M \otimes \mathcal{A}M$$

which is homogeneous of degree zero and the geometry is normal if and only if $\partial^*(\kappa) = 0$.

Again, the filtration of TM gives us the associated graded bundle

$$\mathrm{gr}(TM) = \mathrm{gr}_{-k}(TM) \oplus \cdots \oplus \mathrm{gr}_{-1}(TM),$$

where $\mathrm{gr}_i(TM) = T^iM/T^{i+1}M \simeq \mathcal{A}_iM$. The action of P_+ on $\mathcal{G}$ is free and the quotient $\mathcal{G}/P_+ =: \mathcal{G}_0 \to M$ is principal bundle with structure group $G_0 = P/P_+$. The action of P_+ on $\mathfrak{g}^i/\mathfrak{g}^{i+1}$ is trivial and we have $\mathfrak{g}^i/\mathfrak{g}^{i+1} \simeq \mathfrak{g}_i$ as G_0–modules. We get $\mathrm{gr}_i(TM) \simeq \mathcal{G}_0 \times_{G_0} \mathfrak{g}_i$ and $\mathrm{gr}(TM) \simeq \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$. For each $x \in M$, the space $\mathrm{gr}(T_xM)$ is the nilpotent graded Lie algebra isomorphic to the algebra $\mathfrak{g}_-$ with the bracket given by Lie bracket of vector fields.

Underlying structures

In the first section, we described the well known correspondence between all affine Cartan geometries and the affine connections on the base manifold. In fact, there is the similar correspondence between all types of Cartan geometries of first order and corresponding geometrical structures on base manifolds. It can be described via theory of G–structures and can be found in [**12**].

Parabolic geometries are not first order structures and the concept does not work here, but there is a concept of underlying structures. It gives nice description of structures on base manifolds coming from parabolic geometries.

We define in general a *filtered manifold* as a manifold M together with a filtration $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ such that for sections ξ of T^iM and η of T^jM the Lie bracket $[\xi, \eta]$ is a section of $T^{i+j}M$ and we get an associated graded bundle $\mathrm{gr}(TM)$ again.

On the associated graded bundle we obtain so called the *Levi bracket*

$$\mathcal{L} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM).$$

It is induced by $\Gamma(T^iM) \times \Gamma(T^jM) \to \Gamma(\mathrm{gr}_{i+j}(TM))$ which is the composition of Lie bracket of vector fields with the projection $T_iM \to T_iM/T_{i+1}M$. It depends only on the class in $\mathrm{gr}_i(TM)$ and gives

$$\mathrm{gr}_i(TM) \times \mathrm{gr}_j(TM) \to \mathrm{gr}_{i+j}(TM).$$

For each $x \in M$, this makes $\mathrm{gr}(T_xM)$ into the nilpotent graded Lie algebra which is called the *symbol algebra* of the filtered manifold at the point x.

There is equivalent description of regular parabolic geometry, see [**6**]:

Proposition 4.6. *Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P). The geometry is regular if and only if the induced filtration of TM makes M into the filtered manifold such that the bracket on each symbol algebra coincides with algebraic bracket.*

In particular, each symbol algebra is isomorphic to $\mathfrak{g}_-$.

If we start with a regular parabolic geometry, we get exactly these data on the base manifold:

- A filtration $\{T^iM\}$ of the tangent bundle such that each symbol algebra is isomorphic to $\mathfrak{g}_-$
- A reduction of structure group of the associated graded bundle $\mathrm{gr}(TM)$ with respect to $\mathrm{Ad} : G_0 \to \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$ (the reduction is trivial in the case $G_0 = \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$).

We call these data the underlying *infinitesimal flag structure*.

The proof of the following equivalence between such infinitesimal flag structures and regular normal parabolic geometries can be found in [**21, 5**]:

Theorem 4.7. *Let M be a filtered manifold such that each symbol algebra is isomorphic to $\mathfrak{g}_-$ and let $\mathcal{G}_0 \to M$ be a reduction of the frame bundle of $\mathrm{gr}(TM)$ to the structure group G_0. Then there is a regular normal parabolic geometry $(p : \mathcal{G} \to M, \omega)$ inducing the given data.*

If $H^1_\ell(\mathfrak{g}_-, \mathfrak{g})$ are trivial for all $\ell > 0$, then the normal regular geometry is unique up to isomorphism.

The construction is functorial and the latter Theorem describes an equivalence of categories.

$|1|$–graded geometries

In this case, the filtration of the tangent bundle is trivial. We need only the reduction of $\mathrm{gr}(TM)$ to the structure group G_0. The $|1|$–graded geometries are automatically regular and we get the correspondence between normal $|1|$–graded parabolic geometries and first order G–structures whose structure groups G_0 appear as the reductive part of a parabolic subgroup $P \subset G$ and $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

We give here survey on all $|1|$–graded geometries with the simple group G. The classification of semisimple Lie algebras in terms of simple roots is well known and for a given $\mathfrak{g}$ there is a complete description of all parabolic subalgebras, see [**21, 6**] for more details. The latter description allows to classify all corresponding geometries. We list them all here together with their non–zero components of the harmonic curvature (notice some overlaps in low dimensions).

- A_ℓ **:** the split form, $\ell \geq 2$, the almost Grassmannian structures, $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{R})$, $\mathfrak{g}_0 = \mathfrak{s}(\mathfrak{gl}(p, \mathbb{R}) \times \mathfrak{gl}(q, \mathbb{R}))$, $p + q = \ell + 1$. Moreover
  - $p = 1,\ q = 2$ or $p = 2,\ q = 1$ **:** the projective structures dim $= 2$, one curvature of homogeneity 3
  - $p = 1,\ q > 2$ or $p > 2,\ q = 1$ **:** the projective structures dim > 2, one curvature of homogeneity 2
  - $p = 2,\ q = 2$ **:** dim $= 4$, two curvatures of homogeneity 2
  - $p = 2,\ q > 2$ or $p > 2,\ q = 2$ **:** dim $= pq$, one torsion, one curvature of homogeneity 2

$p > 2$, $q > 2$ **:** dim $= pq$, two torsions

A_ℓ **:** the quaternionic form, $\ell = 2p+1 > 2$, $\mathfrak{g} = \mathfrak{sl}(p+1, \mathbb{H})$. We have:

$p = 1$ **:** the almost quaternionic geometries, dim $= 4$, two curvatures of homogeneity 2

$p > 1$ **:** the almost quaternionic geometries, dim $= 4p$, one torsion, one curvature of homogeneity 2

geometries modeled on quaternionic Grassmannians **:** two torsions

A_ℓ **:** $\ell = 2p - 1$ one type geometry for the algebra $\mathfrak{su}(p, p)$. We have:

$p = 2$ **:** two curvatures of homogeneity 2

$p > 2$ **:** two torsions

B_ℓ **:** the pseudo–conformal geometries in odd dimension ≥ 3. We have:

$\ell = 2$ **:** dim $= 3$, one curvature of homogeneity 3

$\ell > 2$ **:** dim $= 2\ell - 1$, one curvature of homogeneity 2

C_ℓ **:** the split form, $\ell > 2$, the almost Lagrangian geometries, one torsion

C_ℓ **:** one type of geometry corresponding to $\mathfrak{sp}(p, p)$, $2p = \ell$, one torsion

D_ℓ **:** the pseudo–conformal geometries in all even dimensions ≥ 4

$\ell = 3$ **:** dim $= 4$, two curvatures of homogeneity 2

$\ell > 3$ **:** dim ≥ 6, one curvature of homogeneity 2

D_ℓ **:** the real almost spinorial geometries $\mathfrak{g} = \mathfrak{so}(\ell, \ell)$

$\ell = 4$ **:** one curvature of homogeneity 2

$\ell \geq 5$ **:** one torsion

D_ℓ **:** the quaternionic almost spinorial geometries, $\mathfrak{g} = \mathfrak{u}^*(\ell, \mathbb{H})$, $\ell = 2m$, one torsion

E_6 **:** two exotic geometries with $\mathfrak{g}_0 = \mathfrak{so}(5, 5) \oplus \mathbb{R}$ and $\mathfrak{g}_0 = \mathfrak{so}(1, 9) \oplus \mathbb{R}$, one torsion

E_7 **:** two exotic geometries with $\mathfrak{g}_0 = EI \oplus \mathbb{R}$ and $\mathfrak{g}_0 = EIV \oplus \mathbb{R}$, one torsion

Let us remark, that in the low dimensional cases some of the algebras are isomorphic and the corresponding geometries are in fact equal. In particular, $\mathfrak{so}(3, 3) \simeq \mathfrak{sl}(4, \mathbb{R})$, $\mathfrak{so}(2, 4) \simeq \mathfrak{su}(2, 2)$, $\mathfrak{so}(1, 5) \simeq \mathfrak{sl}(2, \mathbb{H})$ and so all the four–dimensional conformal pseudo–Riemannian geometries are covered by the corresponding A_4–cases. Moreover, the spinorial geometries for D_4 are isomorphic to the conformal Riemannian geometries.

5. Weyl structures

We summarize here without proofs basic facts about this useful tool to study parabolic geometries. Later, we will work only with $|1|$–graded geometries and we give here most of the facts only for them. The complicated general formulas, which are very nice and clear in our special case, are the main reason for this simplification. General theory can be found in [**6, 19**].

Remind that there is the *underlying bundle* $\mathcal{G}_0 := \mathcal{G}/\exp \mathfrak{g}_1$, which is the principal bundle $p_0 : \mathcal{G}_0 \to M$ with structure group G_0. At the same time we get the principal bundle $\pi : \mathcal{G} \to \mathcal{G}_0$ with structure group $P_+ = \exp \mathfrak{g}_1$.

The *Weyl structure* σ for parabolic geometry $(p : \mathcal{G} \to M, \omega)$ is a global smooth G_0–equivariant section of the projection $\pi : \mathcal{G} \to \mathcal{G}_0$. We have the following situation:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\pi"'] & \mathcal{G}_0 \arrow[l, bend left, "\sigma"] \arrow[r, "p_0"'] & M \end{tikzcd}$$

There exists some Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ on an arbitrary parabolic geometry. For two Weyl structures σ and $\hat{\sigma}$ there exists exactly one G_0–equivariant mapping $\Upsilon : \mathcal{G}_0 \to \mathfrak{g}_1$ such that

$$\hat{\sigma}(u) = \sigma(u) \cdot \exp \Upsilon(u)$$

for all $u \in \mathcal{G}_0$.

Consequently, if we fix one Weyl structure σ, then all Weyl structures are exactly of the form $\sigma \cdot \exp \Upsilon$ for all possible G_0–equivariant functions Υ. Such a function Υ can be equivalently taken as a P–equivariant mapping $\mathcal{G} \to \mathfrak{g}_1$. At the points $\sigma(\mathcal{G}_0)$ it is given by the original G_0–equivariant function and the P–equivariancy gives the mapping elsewhere. This equivariant mapping is a frame form of some 1–form on M, i.e. a section of $\mathcal{A}^1 M \simeq \mathcal{A}_1 M$.

We get that the number of these possible functions is equal to the number of existing 1–forms $\Omega(M)$. The space of all Weyl structures is an affine space modeled over the vector space of all 1–forms and in this way we can write $\hat{\sigma} = \sigma + \Upsilon$. Using the Campbell–Baker–Haussdorf formula, see [**15**, p. 40], we compute

$$\begin{aligned} (\sigma + \Upsilon) + \Upsilon' = (\sigma \cdot \exp \Upsilon) \cdot \exp \Upsilon' = \sigma \cdot (\exp \Upsilon \cdot \exp \Upsilon') = \\ = \sigma \cdot \exp(\Upsilon + \Upsilon') = \sigma + (\Upsilon + \Upsilon'). \end{aligned}$$

The choice of the Weyl structure σ induces the decomposition of all tractor bundles into G_0–invariant pieces. In particular, the adjoint tractor bundle splits as

$$\mathcal{A}M = TM \oplus \mathrm{End}(TM) \oplus T^*M.$$

Thus the algebraic bracket of a vector field with a 1–form becomes an endomorphism on TM.

Similarly, the choice of some Weyl structure σ allows us to define the G_0–equivariant 1–form $\sigma^*\omega$ on the bundle $\mathcal{G}_0$. Because we have the G_0–invariant decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ we can decompose the form as

$$\sigma^*\omega = \sigma^*\omega_{-1} + \sigma^*\omega_0 + \sigma^*\omega_1.$$

The part $\sigma^*\omega_{-1}$ plays the role of the soldering form on the underlying bundle $\mathcal{G}_0$ and for this reason is called the *soldering form.* It does not change if we change the Weyl structure. The part

$$\sigma^*\omega_0 =: \gamma^\sigma \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$$

defines the principal connection on $p_0 : \mathcal{G}_0 \to M$. Every connection coming from some Weyl structure is called the *Weyl connection* (*associated to the Weyl structure* σ). For two Weyl structures σ and $\hat{\sigma} = \sigma \cdot \exp \Upsilon$ we have $(\gamma^{\hat{\sigma}} - \gamma^\sigma)(u)(\xi) = [\sigma^*\omega_{-1}(\xi), \Upsilon(u)]$ for $u \in \mathcal{G}_0$ and $\xi \in \mathfrak{X}(\mathcal{G}_0)$.

There is a similar description for all induced connections on associated bundles. Let V be a vector space with the left action λ of group G_0 and denote ∇^σ the connection induced from γ^σ on $\mathcal{G}_0 \times_{G_0} V$. These connections are called *Weyl connections*, too. If ∇^σ and $\nabla^{\hat\sigma}$ are Weyl connections corresponding to the structures σ and $\hat\sigma = \sigma \cdot \exp \Upsilon$, then we have

$$\nabla^{\hat\sigma}_\xi(s) = \nabla^\sigma_\xi(s) + \{\xi, \Upsilon\} \bullet s$$

for $\xi \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathcal{G}_0 \times_{G_0} V)$. Here $\bullet$ is the algebraic action derived from λ. In the case of the tangent bundle we get

$$\nabla^{\hat\sigma}_\xi(\eta) = \nabla^\sigma_\xi(\eta) + \{\{\xi, \Upsilon\}, \eta\}$$

for any $\xi, \eta \in \mathfrak{X}(M)$.

The positive component $\sigma^*\omega_1 =: \mathsf{P}^\sigma$ is called *Rho–tensor* (*associated to the Weyl structure* σ). This tensor is the analogy of the Rho–tensor from the conformal geometry and for $\xi \in \mathfrak{X}(M)$ and P^σ, $\mathsf{P}^{\hat\sigma}$ corresponding to σ and $\hat\sigma = \sigma \cdot \exp \Upsilon$ it transforms in the following way:

$$\mathsf{P}^{\hat\sigma}(\xi) = \mathsf{P}^\sigma(\xi) + \nabla^\sigma_\xi(\Upsilon) + \frac{1}{2}\{\Upsilon, \{\Upsilon, \xi\}\}.$$

Finally, we remark that the form $\sigma^*\omega_{-1} + \gamma^\sigma \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$ defines the Cartan connection on the G_0–bundle $\mathcal{G}_0$ for any σ and we get the reductive Cartan geometry $(\mathcal{G}_0 \to M, \sigma^*\omega_{-1} + \gamma^\sigma)$, which is of first order. Equivalently, we have a G_0–structure $\mathcal{G}_0 \to M$ with canonical form $\sigma^*\omega_{-1}$ and with the chosen principal connection γ^σ.

Action of automorphisms on Weyl structures

Let $\varphi : \mathcal{G} \to \mathcal{G}'$ be a morphism of parabolic geometries. Then there exists exactly one *underlying morphism* $\varphi_0 : \mathcal{G}_0 \to \mathcal{G}_0'$ such that $\varphi_0 \circ \pi = \pi' \circ \varphi$. If $\varphi : \mathcal{G} \to \mathcal{G}$ is an automorphism, then we have exactly one automorphism $\varphi_0 : \mathcal{G}_0 \to \mathcal{G}_0$ satisfying $\varphi_0 \circ \pi = \pi \circ \varphi$. If we fix a Weyl structure σ, the pullback

$$\hat\sigma = \varphi^{-1} \circ \sigma \circ \varphi_0 =: \varphi^*\sigma$$

clearly is again some Weyl structure, because it is a composition of G_0–equivariant mappings and then it is also G_0–equivariant. If the morphism φ is globally defined, then the resulting section is also globally defined. In fact, we are in the following situation.

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\pi"] \arrow[d, "\varphi"'] & \mathcal{G}_0 \arrow[l, bend left, "\hat\sigma"] \arrow[r, "p_0"] \arrow[d, "\varphi_0"] & M \arrow[d, equal] \\
\mathcal{G} \arrow[r, "\pi"] & \mathcal{G}_0 \arrow[l, bend left, "\sigma"] \arrow[r, "p_0"] & M
\end{tikzcd}$$

Here σ and $\hat\sigma$ are in general two different Weyl structures. There has to exist exactly one Υ such that $\varphi^*\sigma = \sigma + \Upsilon$.

Next, we shall check the compatibility of the affine structure on the Weyl structures with the pullback operation. Recall that for any 1–form Υ we have $\varphi^*\Upsilon = \Upsilon \circ \varphi_0$.

Proposition 5.1. *Let φ be an automorphism of $\mathcal{G}$ and let φ_0 be the induced automorphism of $\mathcal{G}_0$. Then*

$$\varphi^*(\sigma + \Upsilon) = \varphi^*\sigma + \varphi^*\Upsilon$$

holds for all Weyl structures σ and all 1–forms Υ, i.e. the pullback over φ respects the affine structure on the space of Weyl structures.

Proof. The definition of the affine structure is $(\sigma+\Upsilon)(u) = \sigma(u)\cdot \exp\Upsilon(u) = (\sigma\cdot\exp\Upsilon)(u)$ for all $u\in\mathcal{G}_0$ and we have the following computation:

$$\begin{aligned}\varphi^*(\sigma\cdot\exp\Upsilon)(u) &= \varphi^{-1}\circ(\sigma\cdot\exp\Upsilon)\circ\varphi_0(u) = \\ &= \varphi^{-1}\Big(\sigma(\varphi_0(u))\cdot\exp\Upsilon(\varphi_0(u))\Big) = \\ &= \varphi^{-1}\Big(\sigma(\varphi_0(u))\Big)\cdot\exp\Upsilon(\varphi_0(u)) = \\ &= \varphi^*\sigma(u)\cdot\exp\varphi^*\Upsilon(u)\end{aligned}$$

Again, the definition gives $\varphi^*\sigma(u)\cdot\exp\varphi^*\Upsilon(u) = (\varphi^*\sigma+\varphi^*\Upsilon)(u)$. □

If we take σ such that $\varphi^*\sigma = \sigma+\Upsilon$, we can write

$$\varphi^*(\sigma+\Upsilon') = \varphi^*\sigma+\varphi^*\Upsilon' = \sigma+\Upsilon+\varphi^*\Upsilon'.$$

Further, let φ_0 be an automorphism of $\mathcal{G}_0$ induced by φ and let σ be an arbitrary Weyl structure. It is an easy computation to show that its Weyl connection satisfies

$$\varphi_0^*\gamma^\sigma = \gamma^{\varphi^*\sigma} = \gamma^{\sigma+\Upsilon}.$$

The same idea works for induced connections coming from Weyl structures on associated bundles. In fact, the pullback over an arbitrary automorphism of parabolic geometry permutes all Weyl structures and also connections defined via Weyl structures.

If there is some Weyl structure σ such that $\varphi^*\sigma = \sigma$, then we have $\gamma^\sigma = \varphi_0^*\gamma^\sigma$ and we get a connection, which is invariant with respect to φ_0.

Curvatures of Weyl connections

Let $(\mathcal{G}\to M,\omega)$ be a $|1|$–graded geometry, K its curvature form and κ its curvature function. If we choose some Weyl structure σ, we can compute $\sigma^*\kappa = \kappa\circ\sigma : \mathcal{G}_0 \to \wedge^2\mathfrak{g}_{-1}^*\otimes\mathfrak{g}$. Thanks to the G_0–invariant decomposition of the tensor $\wedge^2\mathfrak{g}_{-1}^*\otimes\mathfrak{g}$, which depends only on the decomposition of the algebra $\mathfrak{g}$, we get

$$\sigma^*\kappa = \sigma^*\kappa_{-1}+\sigma^*\kappa_0+\sigma^*\kappa_1.$$

Similarly, we have the pullback σ^*K. We will not distinguish between the curvature K (resp. σ^*K) and its frame form κ (resp. $\sigma^*\kappa$).

Using the defining formula for the curvature of the Cartan connection we have for all $\xi,\eta\in\mathfrak{X}(\mathcal{G}_0)$

$$\begin{aligned}\sigma^*\kappa_{-1}(\xi,\eta) &= d\sigma^*\omega_{-1}(\xi,\eta)+[\sigma^*\omega_{-1}(\xi),\sigma^*\omega_0(\eta)]+[\sigma^*\omega_0(\xi),\sigma^*\omega_{-1}(\eta)],\\ \sigma^*\kappa_0(\xi,\eta) &= d\sigma^*\omega_0(\xi,\eta)+[\sigma^*\omega_0(\xi),\sigma^*\omega_0(\eta)]+\\ &\qquad [\sigma^*\omega_{-1}(\xi),\sigma^*\omega_1(\eta)]+[\sigma^*\omega_1(\xi),\sigma^*\omega_{-1}(\eta)],\\ \sigma^*\kappa_1(\xi,\eta) &= d\sigma^*\omega_1(\xi,\eta)+[\sigma^*\omega_1(\xi),\sigma^*\omega_0(\eta)]+[\sigma^*\omega_0(\xi),\sigma^*\omega_1(\eta)].\end{aligned}$$

Next, for the fixed Weyl structure σ we get the principal connection $\gamma^\sigma = \sigma^*\omega_0$ on the G_0–structure $(p_0:\mathcal{G}_0\to M,\theta)$, where $\theta=\sigma^*\omega_{-1}$. We also get associated connections ∇^σ. We can easily compute the curvature and the torsion of γ^σ. We have

$$\tau^\sigma(\xi,\eta) = d\theta(\xi,\eta) + [\theta(\xi),\gamma^\sigma(\eta)] + [\gamma^\sigma(\xi),\theta(\eta)],$$
$$\rho^\sigma(\xi,\eta) = d\gamma^\sigma(\xi,\eta) + [\gamma^\sigma(\xi),\gamma^\sigma(\eta)]$$

for all $\xi,\eta \in \mathfrak{X}(\mathcal{G}_0)$. Let us remark, that the form $\tau^\sigma + \rho^\sigma$ describes the curvature of first order Cartan geometry $(\mathcal{G}_0 \to M, \sigma^*\omega_{-1} + \gamma^\sigma)$.

The two 2–forms are horizontal and can be understand as forms on M. Clearly, they describe the curvature $R^\sigma \in \Gamma(\wedge^2 TM^* \otimes \mathrm{End}_0(TM))$ and the torsion $T^\sigma \in \Gamma(\wedge^2 TM^* \otimes TM)$ of associated connection on tangent bundle. In general, we have $TM \simeq \mathrm{gr}(TM)$ via the isomorphism given by the choice of σ.

We can easily reinterpret the curvature and the torsion of Weyl connection via the function $\sigma^*\kappa$. Clearly, the torsion corresponds to the part $\sigma^*\kappa_{-1}$. This part comes from the torsion of ω and does not depend on the choice of Weyl structure, because it is the lowest part of the decomposition. Then all Weyl connections share the same torsion and we will denote it by T.

The part $\sigma^*\kappa_0$ is the frame form of so called *Weyl curvature* W^σ of the connection ∇^σ. Comparing $\sigma^*\kappa_0$ with ρ^σ leads to the following relation between curvature of the Weyl connection and its Weyl curvature

$$W^\sigma(\xi,\eta) = R^\sigma(\xi,\eta) + \{\xi, \mathsf{P}^\sigma(\eta)\} + \{\mathsf{P}^\sigma(\xi),\eta\}$$

for all $\xi,\eta \in \mathfrak{X}(M)$. If we change Weyl structure, then the Weyl curvature transforms in the following way

$$W^{\hat\sigma}(\xi,\eta) = W^\sigma(\xi,\eta) + \{\Upsilon, T(\xi,\eta)\}$$

for all $\xi,\eta \in \mathfrak{X}(M)$ and $\hat\sigma = \sigma \cdot \exp\Upsilon$. Clearly, if the Weyl connection is torsion free (and then all Weyl connections), then the Weyl curvature does not depend on the choice of the Weyl structure.

We should remark that the part $\sigma^*\kappa_1$ corresponds to the so called *Cotton–York tensor*

$$Y^\sigma(\xi,\eta) = (\nabla^\sigma\mathsf{P}^\sigma)(\xi,\eta) - (\nabla^\sigma\mathsf{P}^\sigma)(\eta,\xi) + \mathsf{P}^\sigma(T(\xi,\eta))$$

of the connection ∇^σ for all $\xi,\eta \in \mathfrak{X}(M)$. If we change the Weyl structure, then we get the following formula for change of the Cotton–York tensor

$$Y^{\hat\sigma}(\xi,\eta) = Y^\sigma(\xi,\eta) + \{\Upsilon, W^\sigma(\xi,\eta)\} + \frac{1}{2}\{\Upsilon,\{\Upsilon,T(\xi,\eta)\}\}$$

for all $\xi,\eta \in \mathfrak{X}(M)$ and $\hat\sigma = \sigma\cdot\exp\Upsilon$.

We can shortly write that $\sigma^*\kappa$ corresponds to

$$T + W^\sigma + Y^\sigma = T + R^\sigma + \partial\mathsf{P}^\sigma + Y^\sigma,$$

where ∂ is the Lie algebra cohomology differential. It can be shown that if for one (and then for any) Weyl connection the torsion, the Weyl–curvature and the Cotton–York tensor vanish, then the manifold is as a parabolic geometry locally isomorphic to the homogeneous model.

Next, let φ be an automorphism of the parabolic geometry. Then its curvature is invariant with respect to φ. If we choose a Weyl structure σ such that $\varphi^*\sigma = \sigma$, then Weyl–curvature, Rho–tensor and other part of the curvature are invariant with respect to the underlying morphism φ_0.

Finally, we remark that all these objects are straight analogy of the well known objects from conformal geometry. More on this theory, which generalizes classical objects from the conformal geometry, can be found in [**6**] or in [**3, 4**].

Normal Weyl structures

Finally, we shall introduce special cases of Weyl structures which are closely related to normal coordinate systems for affine geometries.

For any $u \in \mathcal{G}$, there is a local diffeomorphism $\Phi_u : \mathfrak{g}_- \to M$ from the neighborhood of 0 to neighborhood of $p(u)$ defined by

$$\Phi_u(X) = p \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(u).$$

This mapping is called *normal coordinates* at u.

Each normal coordinates define a normal Weyl structure on any parabolic geometry as follows: Let Φ_u be normal coordinates defined on some neighborhood $U \subset \mathfrak{g}_-$, $0 \in U$. Over $\Phi_u(U)$, there is a unique G_0–equivariant section σ_u such that

$$\mathrm{Fl}_1^{\omega^{-1}(X)}(u) \subset \sigma_u(\mathcal{G}_0).$$

This section is called *normal Weyl structure at* $u \in \mathcal{G}$ and can be equivalently defined as the only G_0–equivariant section satisfying

$$\sigma_u \circ \pi \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(u) = \mathrm{Fl}_1^{\omega^{-1}(X)}(u).$$

Clearly, normal Weyl structures are in general defined only locally and can be used only for the study of local properties of the geometry. We prove:

Proposition 5.2. *Let φ be automorphism of $\mathcal{G}$ and σ be a normal Weyl structure. The pullback $\varphi^*\sigma$ is again some normal Weyl structure.*

Proof. We have the following computation:

$$\begin{aligned}
\varphi^*\sigma \circ \pi \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(u) &= \varphi^{-1} \circ \sigma \circ \varphi_0 \circ \pi \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(u) = \\
&= \varphi^{-1} \circ \sigma \circ \pi \circ \varphi \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(u) = \\
&= \varphi^{-1} \circ \sigma \circ \pi \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(\varphi(u)) = \\
&= \varphi^{-1} \circ \varphi \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(u) = \\
&= \mathrm{Fl}_1^{\omega^{-1}(X)}(u)
\end{aligned}$$

Clearly, the pullback $\varphi^*\sigma$ again satisfies the conditions on normal Weyl structures. $\square$

6. Symmetries on geometries

From the Section 3 we know, that it is quite difficult to find a universal definition of the symmetry for arbitrary Cartan geometry. In the case of parabolic geometries, the most reasonable possibility is to take the following definition:

Definition 6.1. Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry. The *symmetry at the point* x is a locally defined diffeomorphism s_x on M such that:

(i) $s_x(x) = x$

(ii) $T_x s_x|_{T_x^{-1}M} = -\operatorname{id}_{T^{-1}M}$

(iii) s_x is covered by an automorphism φ of the Cartan geometry, i.e. $s_x = \underline{\varphi}$ on some neighborhood of x.

The geometry is called (*locally*) *symmetric* if there is a symmetry at each point $x \in M$.

In other words, symmetries revert by the sign change only the smallest subspace in the filtration, while their actions on the rest are completely determined by the algebraic structure of the symbol algebra. It clearly resolves the problem with contact structures discussed in the end of first chapter.

In the case of $|1|$–graded geometries, however, the filtration is trivial and so we have $T^{-1}M = TM$. Thus the definition of the symmetries of $|1|$–graded geometries follow completely the classical intuitive idea.

Next, we will always work only with locally defined automorphisms. We will omit 'locally' and we will say only 'symmetric' geometry. We will denote φ an automorphism of Cartan geometry covering some symmetry and $\underline{\varphi}$ the corresponding symmetry. We have the following Lemma:

LEMMA 6.2. *If there is a symmetry in some x on a $|1|$–graded geometry of type (G, P), then there exists an element $g \in P$ such that*

$$\underline{\operatorname{Ad}}_g(X) = -X$$

for all $X \in \mathfrak{g}_{-1}$, where $\underline{\operatorname{Ad}}$ denotes the action on $\mathfrak{g}_{-1}$ induced from the adjoint action.

All such elements g are of the form $g = g_0 \exp Z$, where $g_0 \in G_0$ such that $\operatorname{Ad}_{g_0}(X) = -X$ for all $X \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_1$ is arbitrary.

PROOF. Let φ cover some symmetry in a point $x \in M$. The symmetry $\underline{\varphi}$ preserves the point x and hence the morphism φ preserves the fiber over x. Let u be an arbitrary fixed point in the fiber over x. There is an element $g \in P$ such that $\varphi(u) = u \cdot g$. We will study the action of this g on $\mathfrak{g}_{-1}$.

Let $\xi \in \mathfrak{X}(M)$ be a vector field on M. In the point x we have

$$T\underline{\varphi}.\xi(x) = -\operatorname{id}_{T_xM}(\xi(x)) = -\xi(x).$$

Using the identification $TM \simeq \mathcal{G} \times_P \mathfrak{g}_{-1}$ we have $\xi(x) = [\![u, X]\!]$ for some $X \in \mathfrak{g}_{-1}$. In the chosen frame u we then have

$$T\underline{\varphi}([\![u, X]\!]) = [\![u, -X]\!],$$

because the symmetry changes the sign of the coordinates X in the frame u.

The equivariancy gives in the fiber over x

$$T\underline{\varphi}([\![u, X]\!]) = [\![\varphi(u), X]\!] = [\![ug, X]\!] = [\![u, \underline{\operatorname{Ad}}_{g^{-1}}(X)]\!].$$

Comparing the coordinates in the frame u gives us the action of element $g \in P$ on $\mathfrak{g}_{-1}$. We have $\underline{\operatorname{Ad}}_g(-X) = X$ and the action of element g is the change of the sign for all elements from $\mathfrak{g}_{-1}$.

Next, because $g \in P$ we have $g = g_0 \exp Z$ for some $g_0 \in G_0$ and $Z \in \mathfrak{g}_1$, see Proposition 4.1. We have $\underline{\operatorname{Ad}}_{g_0 \exp Z}(X) = -X$ for all $X \in \mathfrak{g}_{-1}$. But the action of the component $\exp Z$ is trivial while the action of g_0 preserves the gradation, i.e. $\underline{\operatorname{Ad}}_{g_0} = \operatorname{Ad}_{g_0}$, and the element g_0 satisfies $\operatorname{Ad}_{g_0}(X) = -X$. $\square$

In fact, it is possible to formulate the latter Lemma also for other types of geometries, which are not necessarily $|1|$–graded parabolic. More precisely, similar fact can be shown for all Cartan geometries $(\mathcal{G} \to M, \omega)$ that carry an automorphism φ such that $\underline{\varphi}(x) = x$ and $T\underline{\varphi}|_{T_xM} = -\mathrm{id}_{T_xM}$ in some $x \in M$ (i.e. the symmetry $\underline{\varphi}$ reverts the whole T_xM). In all these cases we have:

Lemma 6.3. *Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P) such that there is a symmetry in x, which reverts the whole T_xM. Then there is an element $g \in P$ such that*

$$\underline{\mathrm{Ad}}_g(X + \mathfrak{p}) = -(X + \mathfrak{p})$$

for all $X + \mathfrak{p} \in \mathfrak{g}/\mathfrak{p}$, where $\underline{\mathrm{Ad}}$ is the action induced on the factor from the adjoint action.

Proof. In fact, the proof is the same as in the case of Lemma 6.2. For any $\xi \in TM \simeq \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ we can write $\xi(x) = [\![u, X + \mathfrak{p}]\!]$ for some $u \in p^{-1}(x)$ and $X + \mathfrak{p} \in \mathfrak{g}/\mathfrak{p}$. We then have

$$T\underline{\varphi}([\![u, X + \mathfrak{p}]\!]) = [\![\varphi(u), X + \mathfrak{p}]\!] = [\![ug, X + \mathfrak{p}]\!] = [\![u, \underline{\mathrm{Ad}}_{g^{-1}}(X + \mathfrak{p})]\!].$$

This is equal to $-\xi(x) = [\![u, -(X + \mathfrak{p})]\!]$ and it gives that g^{-1} is exactly the element we are looking for. Then also g acts by the change of the sign on $\mathfrak{g}/\mathfrak{p}$ and satisfies the equation. □

The Lemma 6.3 works for example for affine geometry. It says that there exists an element $A \in Gl(n, \mathbb{R})$ such that $\mathrm{Ad}_A(X) = AX = -X$ for all $X \in \mathbb{R}^n$ on the affine locally symmetric space, i.e. on symmetric Cartan geometry of type $(A(n, \mathbb{R}), Gl(n, \mathbb{R}))$, see Example 7.2. Conversely, it does not work for parabolic contact structures.

Remark 6.4. Suppose that there is an element g satisfying the corresponding condition from Lemmas on the geometry of given type. If some element differs from g by a conjugation by an element from P, then it has the same property. If g corresponds to the frame $u \in p^{-1}(x)$, then the element $h^{-1}gh$ for $h \in P$ corresponds to the frame $uh \in p^{-1}(x)$.

7. Homogeneous models

The previous Lemmas give us an efficient tool to study symmetries on homogeneous models. We have:

Proposition 7.1. *All symmetries of the homogeneous model $(G \to G/P, \omega_G)$ at the origin $o = eP$ are induced exactly by the left multiplications by elements $g \in P$ satisfying the conditions from the latter Lemmas for the geometry of appropriate type (G, P).*

Moreover, if there is a symmetry in the origin o, then the homogeneous model is symmetric.

Proof. All automorphisms of G/P are exactly left multiplications by elements from G, see Proposition 2.3. We look for elements from G, which give symmetries. Let us denote λ_g the left multiplication by element g and suppose, that λ_g covers some symmetry in the origin. Then the element g

must belong to P. We make similar computation as in the proofs of previous Lemmas.

Let $\xi \in \mathfrak{X}(G/P)$ be a vector field. In the origin of the homogeneous model we have $\xi(o) = [\![e, X]\!]$ for some $X \in \mathfrak{g}/\mathfrak{p} \simeq \mathfrak{g}_-$ and we get

$$T\underline{\lambda}_g([\![e, X]\!]) = [\![ge, X]\!] = [\![g, X]\!] = [\![e, \underline{\mathrm{Ad}}_{g^{-1}}(X)]\!].$$

It equals to $-\xi(o) = [\![e, -X]\!]$ and the element g must satisfy the conditions.

If there is a symmetry in the origin, we can use the conjugation to get symmetry in each point $hP \in G/P$. If λ_g covers a symmetry in the origin o for some $g \in P$, then $\lambda_{hgh^{-1}}$ covers some symmetry in the point hP. $\square$

Examples

Now we present some examples. We will simply look for elements from the latter Proposition to find possible symmetries.

We start with the affine geometry. This is not a parabolic geometry but our definition of the symmetry recovers the classical one and the latter Proposition holds. Then we give some $|1|$–graded examples.

Example 7.2. *Affine geometry.* We have $G = A(n, \mathbb{R})$, the affine group, and $P = Gl(n, \mathbb{R})$. Their algebras are $\mathfrak{g} = \mathfrak{a}(n, \mathbb{R})$ and $\mathfrak{p} = \mathfrak{gl}(n, \mathbb{R})$, see Example 2.1.

We look for elements $\left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right) \in A(n, \mathbb{R})$ satisfying $\left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 0 & 0 \\ -w & 0 \end{smallmatrix}\right)$ for all $\left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right) \in \mathfrak{g}_- \simeq \mathbb{R}^n$. Consequently, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ -w & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right)$ and thus $\left(\begin{smallmatrix} 0 & 0 \\ Aw & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ -w & 0 \end{smallmatrix}\right)$. We can see that there is only one element satisfying this equality and this is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -E \end{smallmatrix}\right)$, the conjugation by another element of P gives trivial change.

This matches the well known fact on the classical symmetric spaces. There can exist only one symmetry in each point on the affine (locally) symmetric space. The symmetry corresponds to the element we found above. The homogeneous model is the affine plane $\mathbb{R}^n \simeq A(n, \mathbb{R})/Gl(n, \mathbb{R})$. This clearly is a symmetric space and the symmetry in the origin is given by the left multiplication by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -E \end{smallmatrix}\right)$.

Example 7.3. *Projective structures.* We can make two reasonable choices of the Lie group G with the $|1|$–graded Lie algebra $\mathfrak{sl}(m+1, \mathbb{R})$. We can consider $G = Sl(m+1, \mathbb{R})$. Then the maximal P is the subgroup of all matrices of the form $\left(\begin{smallmatrix} d & W \\ 0 & D \end{smallmatrix}\right)$ such that $\frac{1}{d} = \det D$ for $D \in Gl(m, \mathbb{R})$ and $W \in \mathbb{R}^{m*}$, but we take only the connected component of the unit. It consists of all elements such that $\det D > 0$.

In this setting, the group G acts on rays in $\mathbb{R}^{m+1}$ and P is the stabilizer of the ray spanned by the first basis vector. Clearly, with this choice G/P is diffeomorphic to the m–dimensional sphere. The subgroup G_0 contains exactly elements of P such that $W = 0$, and this subgroup is isomorphic to $Gl^+(m, \mathbb{R})$.

The second reasonable choice is $G = PGl(m+1, \mathbb{R})$, the quotient of $Gl(m+1, \mathbb{R})$ by the subgroup of all multiples of the identity. This group acts on $\mathbb{R}P^m$ and as the subgroup P we take the stabilizer of the line generated by the first basis vector. Clearly G/P is diffeomorphic to $\mathbb{R}P^m$. The subgroup

G_0 is isomorphic to $Gl(m,\mathbb{R})$, because each class in G_0 has exactly one representant of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & D \end{smallmatrix}\right)$.

We can make the computation simultaneously and then discuss the cases separately. We have $\mathfrak{g} = \left\{ \left(\begin{smallmatrix} -tr(A) & Z \\ X & A \end{smallmatrix}\right) \;\middle|\; X, Z^T \in \mathbb{R}^m,\ A \in \mathfrak{gl}(m,\mathbb{R}) \right\}$ and elements from $\mathfrak{g}_{-1}$ look like $\left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right)$. The adjoint action of $a = \left(\begin{smallmatrix} b & 0 \\ 0 & B \end{smallmatrix}\right)$ on $V = \left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right)$ is $\mathrm{Ad}_a V = b^{-1}BX$. We look for elements $\left(\begin{smallmatrix} b & 0 \\ 0 & B \end{smallmatrix}\right)$ such that $BX = -bX$ for each X. It is easy to see that B is a diagonal matrix and that all elements on the diagonal are equal to $-b$. Thus we may represent the prospective solution as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -E \end{smallmatrix}\right)$.

Now, we discuss the choice $G = Sl(m+1,\mathbb{R})$ with $G/P \simeq S^m$. The element has the determinant ± 1 and the sign depends on the dimension of the geometry. If m is even, then the element gives a symmetry but if m is odd, then there is no symmetry on this model. The reason is obvious – our choice of the groups has lead to the oriented sphere with the canonical projective structure (represented e.g. by the metric connection of the round sphere metric) and the obvious symmetries are orientation preserving in the even dimensions only.

In the case of $G = PGl(m+1,\mathbb{R})$, the above element always represents the class in G_0 and thus yields the symmetry. In both cases, all elements giving symmetry are of the form $\left(\begin{smallmatrix} 1 & W \\ 0 & -E \end{smallmatrix}\right)$ for all $W \in \mathbb{R}^{m*}$.

These two choices of the group G correspond to projective structures on oriented and non–oriented manifolds. The non–oriented projective geometries can be symmetric, the homogeneous models are symmetric. The existence of a symmetry on the oriented projective geometry depends on its dimension. Only the even–dimensional geometries can be symmetric. Symmetric odd–dimensional oriented projective geometry does not exist.

Example 7.4. *Almost quaternionic structures.* Now we take almost quaternionic structures, we have $\mathfrak{g} = \mathfrak{sl}(m+1,\mathbb{H})$. There are again two interesting choices of the groups. We can choose $G = Sl(m+1,\mathbb{H})$ with the canonical action on $\mathbb{H}^{m+1}$. The parabolic subgroup P is the stabilizer of the quaternionic line spanned by the first basis vector in $\mathbb{H}^{m+1}$. Then $G_0 = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & A \end{smallmatrix}\right) \;\middle|\; |a|^4 \det_{\mathbb{R}} A = 1 \right\}$.

Next, we can take $G = PGl(m+1,\mathbb{H})$, the quotient of all invertible quaternionic linear endomorphisms by the subgroup of real multiples of identity. Let P be the (factor of the) stabilizer of the quaternionic line spanned by the first basis vector. The subgroup G_0 consists of classes in P of block diagonal matrices which are represented by matrices of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & A \end{smallmatrix}\right)$ such that $0 \neq a \in \mathbb{H}$ and $A \in Gl(m,\mathbb{H})$.

We have $\mathfrak{g}_{-1} = \left\{ \left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right) \;\middle|\; X \in \mathbb{H}^m \right\}$ and we look for elements $\left(\begin{smallmatrix} q & 0 \\ 0 & B \end{smallmatrix}\right)$ such that $BX = -Xq$ for each X. Again, such an element must be diagonal and the elements on the diagonal of B are equal to $-q$. Suppose that $q = a+bi+cj+dk$. If we choose $X = \left(\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right)$ we get $(-a-bi-cj-dk)i = -i(a+bi+cj+dk)$, thus $-ai+b+ck-dj = -ai+b-ck+dj$ and so $c = d = 0$. Then the choice $X = \left(\begin{smallmatrix} j \\ 0 \end{smallmatrix}\right)$ gives that q has to be real. We again get the element $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -E \end{smallmatrix}\right)$.

In the case of $PGl(m+1,\mathbb{H})$, this element clearly represents the class giving symmetry. In the case of $Sl(m+1,\mathbb{H})$ it should again depend on the dimension of the manifold. But the real dimension equals to $4m$ and also in this case, the symmetry is well defined. All elements giving symmetries look like $\left(\begin{smallmatrix} 1 & W \\ 0 & -E \end{smallmatrix}\right)$ for all $W \in \mathbb{H}^{m*}$.

EXAMPLE 7.5. *Conformal Riemannian structures.* There is a lot of possible choices of the corresponding group and we show only the most common one. The usual choice is $G = O(p+1, q+1)$ and P the Poincaré conformal group, see Example 4.2.

The adjoint action of some $b = \left(\begin{smallmatrix} \lambda & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & \lambda^{-1} \end{smallmatrix}\right) \in G_0$ on arbitrary $V = \left(\begin{smallmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^T J & 0 \end{smallmatrix}\right) \in \mathfrak{g}_{-1}$ is $\mathrm{Ad}_b V = \lambda^{-1} C X$ and we require $\lambda^{-1} C X = -X$. Thus we look for $\lambda \in \mathbb{R} \setminus \{0\}$ and $C \in O(p,q)$ such that $CX = -\lambda X$ for each $X \in \mathbb{R}^{p+q}$. Clearly, C has to be diagonal and all elements on the diagonal have to be equal to 1 or -1 because $\det C$ is equal to 1 eventually -1. We get two elements satisfying all conditions: $\left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -E & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

Both elements belong to $O(p+1, q+1)$ and their left multiplications give symmetries on homogeneous model. This two elements differ in the multiplication by element $-E$. The choice $G = O(p+1, q+1)$ gives not effective geometry and the kernel of this geometry is $\{\pm \mathrm{id}\}$. Thus our elements differ in multiplication by some element from the kernel. Left multiplication by elements from the kernel induce identity on the base manifold and then this two elements give the same symmetry. All elements inducing some symmetry are of the form $\left(\begin{smallmatrix} -1 & Z & 0 \\ 0 & E & -JZ^T \\ 0 & 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & Z & 0 \\ 0 & -E & -JZ^T \\ 0 & 0 & 1 \end{smallmatrix}\right)$ for all $Z \in \mathbb{R}^{p+q*}$.

It is possible to take an effective model, e.g. to start with $PO(p+1, q+1)$, the factor of orthogonal group by its center. With this choice we clearly get exactly one element from G_0 satisfying the condition, which is the class represented by the element $\left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$. It is the analogy to the choice $G = PGl(m+1)$ in the projective case.

We could take only effective geometries but we will rather work with all infinitesimally effective geometries to include standard choices of groups for all $|1|$–graded parabolic geometries. It brings no complications. We return to the discussion of the efficiency of geometries in more general setting in Section 9.

EXAMPLE 7.6. *Almost Grassmannian structures.* We have $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{R})$ and we first take $G = Sl(p+q, \mathbb{R})$. This group acts on $\mathbb{R}^{p+q}$ and the subgroup P is the stabilizer of $\mathbb{R}^q$. Elements of $G_0 = S(Gl(p, \mathbb{R}) \times Gl(q, \mathbb{R}))$ look like $\left(\begin{smallmatrix} C & 0 \\ 0 & D \end{smallmatrix}\right)$ where $\det C \cdot \det D = 1$. The homogeneous model is the Grassmannian of p–dimensional subspaces of $\mathbb{R}^{p+q}$.

The algebra $\mathfrak{g}$ consists of block elements $\left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right)$ with block size p and q where $tr(X) + tr(W) = 0$ and elements of $\mathfrak{g}_{-1}$ are those with X, Y, W vanishing.

The adjoint action of some $\left(\begin{smallmatrix} S & 0 \\ 0 & T \end{smallmatrix}\right) \in G_0$ on $\left(\begin{smallmatrix} 0 & 0 \\ V & 0 \end{smallmatrix}\right) \in \mathfrak{g}_{-1}$ is TVS^{-1} and we look for S and T such that $TV = -VS$ for all $V \in L(\mathbb{R}^p, \mathbb{R}^q)$. The properties of matrix multiplication give that T and S are diagonal, elements on the diagonal of T are equal, elements on the diagonal of S are equal and elements on the diagonal of T are equal to minus elements from S. The condition on determinant gives that only elements $\left(\begin{smallmatrix} E & 0 \\ 0 & -E \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -E & 0 \\ 0 & E \end{smallmatrix}\right)$ satisfy all latter restrictions.

We have to discuss the dependence on p and q to resolve whether some of these two elements belong to $Sl(p+q,\mathbb{R})$ and give a symmetry. We get some symmetry if at least one of p and q is even. If only p is even, then all symmetries are given by elements $\left(\begin{smallmatrix} -E & X \\ 0 & E \end{smallmatrix}\right)$ for all $X \in L(\mathbb{R}^q, \mathbb{R}^p)$. If only q is even, then all symmetries are given by elements $\left(\begin{smallmatrix} E & X \\ 0 & -E \end{smallmatrix}\right)$ for all $X \in L(\mathbb{R}^q, \mathbb{R}^p)$. If both sizes are even, then all latter elements give symmetries. If p and q are odd, then there is no symmetry.

The situation where p and q are both even looks similar to the situation in some latter examples of non–effective models. In this special case, the geometry has nontrivial kernel, too. This is of the form $\{\left(\begin{smallmatrix} E & 0 \\ 0 & E \end{smallmatrix}\right), \left(\begin{smallmatrix} -E & 0 \\ 0 & -E \end{smallmatrix}\right)\}$. Clearly, the second element belongs to the group $Sl(p+q,\mathbb{R})$ if and only if p and q are both even (or odd, but this case is not interesting). In this case, there are two different elements giving the same symmetry and these two elements differ by the multiplication by $-E$. Again, we can take $G = PSl(p+q,\mathbb{R})$ instead of $Sl(p+q,\mathbb{R})$ to get effective geometry. With this choice, each symmetry is given by exactly one class represented by some latter element.

Let us point out some observations coming from these examples. The existence of symmetry on the homogeneous model depends on the choice of the groups. We know, that if there is no element satisfying the condition from Lemma 6.2, then the homogeneous model of the corresponding type is not symmetric. In addition, none of the Cartan geometry of the same type is symmetric. Let us mention the oriented projective structures in odd dimension. These Cartan geometries cannot be symmetric. If we forget the orientation (i.e. if we consider different groups) then we get geometries which can be symmetric.

Remark on generalized geodesics

There is an equivalent definition of affine (locally) symmetric space. The symmetry in x on the manifold M with linear connection ∇ can be defined as an automorphism of some neighborhood $U \subset M$, $x \in U$, which turns the parametrized geodesics of ∇ going through x around this point. The manifold with affine connection is symmetric if there is a symmetry in each point $x \in M$.

The affine geodesic is uniquely given by the 1–jet, i.e. the point and the direction. Also in every point, there is only one geodesic in each direction and the reverting is given uniquely. The image of the geodesic going in some direction is the geodesic going through the same point in the reversed direction. The question is following: Is this mapping the affine transformation? There can exist at most one symmetry, exactly the latter mapping. It corresponds to the fact, that affine structures are of first order and each (auto)morphism is uniquely given by its 1–jet in one point. If we prescribe the symmetry on this $Gl(n,\mathbb{R})$–structure by its 1–jet, then we look for a connection, which is invariant with respect to our symmetry. This approach leads to classical results, see [**13**].

Analogies of the affine geodesics for parabolic geometries $(p : \mathcal{G} \to M, \omega)$ and more generally for all split Cartan geometries are generalized geodesics. They are defined as follows:

Definition 7.7. The *generalized geodesic* on a split Cartan geometry $(p : \mathcal{G} \to M, \omega)$ is a parametrized curve in M given as the projection of the flow line of a horizontal vector field, i.e. $c^{u,X} := p \circ \mathrm{Fl}_t^{\omega^{-1}(X)}(u)$ with $u \in \mathcal{G}$ and $X \in \mathfrak{g}_-$.

In the affine case we get exactly affine geodesics of corresponding linear connection. More detailed discussion on generalized geodesics can be found in [**8, 23**].

Clearly, our definition of symmetry gives that symmetries are exactly those automorphisms, which revert the 'classes' of generalized geodesics through the point in one direction. More precisely, the symmetry maps some generalized geodesic in some direction to some generalized geodesic in the reversed direction.

Generalized geodesics on parabolic geometries are given by its higher jets. In particular, in the $|1|$–graded case, all geodesics are given by its 2–jet in one point. The condition of symmetry prescribes only the point and the direction and there can exist several different geodesics. There is a certain freedom in the last coordinate of their assigned 2–jets. In this case, it is not reasonable to define the symmetry by reverting of this curves. The symmetries do not only revert each geodesic in the fixed direction, but also mix all of them together. It corresponds to the fact that parabolic geometries are second order structures and their morphisms are given by 2–jets in one point.

We can illustrate this situation on the Riemannian and the conformal sphere. The Riemannian sphere is the unit sphere with the metric induced from the Euclidean space. It can be realized as a homogeneous model of first order Cartan geometry, see [**18**]. Generalized geodesics, which are exactly geodesics of the Levi-Civita connection, are all great circles. Each symmetry in some point has to revert all great circles going through the point. It maps the circle onto the same circle with the reversed orientation.

The situation is very different in the case of conformal sphere. The conformal sphere is the homogeneous model of conformal geometry of positive definite signature, see Example 4.2. On this $|1|$–graded parabolic geometry, generalized geodesics going through one point are all (parametrized) circles going through this point. The symmetry in the point maps each circle going through the point in some direction onto another circle going in the opposite direction. There are many different possibilities.

Remark 7.8. There is the well known fact, that all morphisms of the conformal sphere are exactly all mappings, which maps circles onto circles. In our special case we can simply say, that all symmetries in one point are exactly the morphisms, which in addition map each circle going through the point onto some circle going in the opposite direction.

CHAPTER 3

Symmetric $|1|$–graded geometries

In this chapter, we formulate main results on symmetries on $|1|$–graded geometries. In Section 8 we summarize, which types of geometries have to be locally flat if they are symmetric and which types can carry some symmetry in the point and still allow some nonzero curvature at this point. In the Section 9 we return to Weyl structures. We discuss Weyl structures fixed by the symmetries, which play an important role later. We also discuss symmetries on effective geometries. In the Section 10 we show some corollaries of previous results for projective and conformal geometries. The motivation for the Section 10 and partly for Section 9 is the article [**16**]. The author treats the projective case in more classical way there.

8. Torsion restrictions

We would like to find some restriction on the curvature of $|1|$–graded geometry coming from the existence of the symmetry. The following Proposition is fundamental.

Proposition 8.1. *Symmetric $|1|$–graded parabolic geometries are torsion free.*

Proof. Let us choose an arbitrary $x \in M$ on a $|1|$–graded geometry of type (G, P) and let φ cover some symmetry in x. The symmetry $\underline{\varphi}$ fixes x and thus φ preserves the fiber over x. The curvature function satisfies $\kappa = \kappa \circ \varphi$ and we have

$$\kappa(u) = \kappa(\varphi(u)) = \kappa(u \cdot g) = g^{-1} \cdot \kappa(u)$$

for appropriate $g \in P$. The torsion is identified with the part κ_{-1} and we have the same equation for this correctly defined component (we have just to keep in mind the proper action of P on the quotient space $\mathfrak{g}_{-1} \simeq \mathfrak{g}/\mathfrak{p}$). We compare κ_{-1} in the frames u and $\varphi(u)$ from $p^{-1}(x)$. We arrive at

$$\begin{aligned}\kappa_{-1}(\varphi(u))(X, Y) = \kappa_{-1}(u \cdot g)(X, Y) &= g^{-1} \cdot \kappa_{-1}(u)(X, Y) = \\ &= \underline{\mathrm{Ad}}_{g^{-1}}(\kappa_{-1}(u)(\underline{\mathrm{Ad}}_g X, \underline{\mathrm{Ad}}_g Y)) = \\ &= -\kappa_{-1}(u)(-X, -Y) = -\kappa_{-1}(u)(X, Y),\end{aligned}$$

because g is exactly the element from Lemma 6.2 which acts as $-\mathrm{id}$ on $\mathfrak{g}_{-1}$. This is equal to $\kappa_{-1}(u)(X, Y)$ and we obtain $\kappa_{-1}(u)(X, Y) = -\kappa_{-1}(u)(X, Y)$ for all $X, Y \in \mathfrak{g}_{-1}$. Thus $\kappa_{-1}(u)$ vanishes and this holds for all frames $u \in \mathcal{G},\ p(u) = x$. The torsion then vanishes in x.

If the geometry is symmetric, then there is some symmetry in all $x \in M$. Then κ_{-1} vanishes in all $x \in M$ and the geometry is torsion free. $\square$

The latter Proposition is the analogy of the classical result for the affine (locally) symmetric space. Moreover, in the case of normal $|1|$–graded geometries, we get as a corollary of the general theory on harmonic curvature the following Theorem.

THEOREM 8.2. *Let $(\mathcal{G} \to M, \omega)$ be a normal $|1|$–graded parabolic geometry such that its homogeneous components of the harmonic curvature are only of degree* 1. *If there is a symmetry at a point $x \in M$, then the whole curvature vanishes in this point.*

In particular, if the geometry is symmetric than it is locally isomorphic with the homogeneous model.

PROOF. The existence of a symmetry forces κ_{-1} to vanish, see Proposition 8.1. If all harmonic curvature is concentrated to this homogeneity, then the whole curvature κ has to vanish, see Theorem 4.3. Then the geometry is locally flat, see Theorem 2.2. □

Using the overview of all $|1|$–graded geometries (see the page 15) we can easily list all types of geometries satisfying the condition in the previous Theorem.

COROLLARY 8.3. *The following symmetric normal $|1|$–graded geometries always have to be locally flat:*

- *almost Grassmannian geometries such that $p > 2$ and $q > 2$*
- *geometries modeled on quaternionic Grassmannians (but not the almost quaternionic ones)*
- *geometries for the algebra $\mathfrak{sp}(p,p)$ where $p > 2$*
- *all geometries coming from the algebras of types C_ℓ*
- *spinorial geometries in the D_ℓ types with $\ell > 4$*
- *all exotic geometries.*

The crucial point in the above considerations was the odd homogeneity degree of the components in harmonic curvatures. We can use similar argument for all geometries where the only available homogeneity is three (see the overview on the page 15):

PROPOSITION 8.4. *The following symmetric normal geometries are locally flat:*

- *conformal geometries in all signatures of dimension* 3
- *projective geometries of dimension* 2.

PROOF. First we prove that κ_1 is zero. Suppose that φ covers a symmetry in $x \in M$. In arbitrary frame u over x we have

$$\begin{aligned}\kappa_1(\varphi(u))(X,Y) = \kappa_1(u \cdot g)(X,Y) = g^{-1} \cdot \kappa_1(u)(X,Y) = \\ = \underline{\mathrm{Ad}}_{g^{-1}}(\kappa_1(u)(\underline{\mathrm{Ad}}_g X, \underline{\mathrm{Ad}}_g Y)) = \\ = \underline{\mathrm{Ad}}_{g^{-1}}(\kappa_1(u)(-X,-Y)) = \underline{\mathrm{Ad}}_{g^{-1}}(\kappa_1(u)(X,Y))\end{aligned}$$

and this is equal to $\kappa_1(u)(X,Y)$.

We know that the action of g on $\mathfrak{g}_{-1}$ is $-\mathrm{id}$. We have that $\mathfrak{g}_1$ is dual to $\mathfrak{g}_{-1}$ with respect to the Killing form and the adjoint action of P on $\mathfrak{g}_1$ is then the dual action of the adjoint action on $\mathfrak{g}_{-1}$. The action of the element g on $\mathfrak{g}_1$ is then the dual action of $-\mathrm{id}$ and it is again $-\mathrm{id}$. We get

$\kappa_1(u)(X,Y) = -\kappa_1(u)(X,Y)$ for all u over x and therefore κ_1 vanishes in x. If we have a symmetry in each point, then κ_1 vanishes.

By Theorem 4.3 and our list of the features of all $|1|$–graded geometries, the geometries in question have no homogeneous parts of curvature of degree 1 and 2. They have only one homogeneous harmonic part of degree 3. This component belongs to κ_1 and therefore has to vanish. But then the harmonic part of curvature κ_H vanishes and so the curvature κ vanishes and the geometries are locally flat. □

The curvature of a symmetric $|1|$–graded geometry looks like $\kappa = \kappa^0$ and its lowest part $\kappa_0 : \mathcal{G} \to \mathfrak{g}^* \wedge \mathfrak{g}^* \otimes \mathfrak{g}_0$ is a correctly defined (quotient) object. Unfortunately, comparing $\kappa_0(u)$ with $\kappa_0(\varphi(u))$ does not give us any new information. Indeed,

$$\begin{aligned}\kappa_0(\varphi(u))(X,Y) &= \kappa_0(u\cdot g)(X,Y) = g^{-1}\cdot\kappa_0(u)(X,Y) = \\ &= \underline{\mathrm{Ad}}_{g^{-1}}(\kappa_0(u)(-X,-Y)) = \underline{\mathrm{Ad}}_{g^{-1}}(\kappa_0(u)(X,Y))\end{aligned}$$

is equal to $\kappa_0(u)(X,Y)$ for all $X,Y \in \mathfrak{g}_{-1}$. Since $\mathfrak{g}_0 \subseteq \mathfrak{gl}(\mathfrak{g}_{-1}) \simeq \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$, and the action on $\mathfrak{g}_{-1}$ and $\mathfrak{g}^*_{-1}$ is $-\mathrm{id}$, the action of g on the tensor is obviously trivial.

Again, we can easily find all remaining $|1|$–graded normal parabolic geometries allowing some component of harmonic curvature of homogeneous degree 2 (see the overview on the page 15). These are just four lines of examples:

- projective geometries of dim > 2
- conformal Riemannian geometries in all signatures of dim > 3
- almost quaternionic geometries
- almost Grassmannian structures such that $p = 2$ or $q = 2$.

We should remind that all the other geometries allowing curvature in our list above are in fact isomorphic with some of these four types.

9. Weyl structures and symmetries

We discuss here the behavior of Weyl structures with respect to the action of some symmetry in x on a $|1|$–graded geometry. We start with the Proposition which is essential for us.

PROPOSITION 9.1. *Let $(\mathcal{G} \to M, \omega)$ be a $|1|$–graded geometry. Suppose that there is a symmetry $\underline{\varphi}$ in $x \in M$ covered by the automorphism φ. There is a Weyl structure σ such that*

$$\varphi^*\sigma|_{p_0^{-1}(x)} = \sigma|_{p_0^{-1}(x)},$$

i.e. the pullback of σ along φ equals to the same structure σ over x.

PROOF. Let us choose arbitrary Weyl structure $\hat\sigma : \mathcal{G}_0 \longrightarrow \mathcal{G}$ and compute the pullback of this structure along φ. We get another Weyl structure $\varphi^*\hat\sigma = \hat\sigma + \Upsilon$. The Weyl structure $\hat\sigma + \frac{1}{2}\Upsilon$ then satisfies (see Section 5 for notation)

$$\varphi^*(\hat\sigma + \frac{1}{2}\Upsilon) = \varphi^*\hat\sigma + \varphi^*\frac{1}{2}\Upsilon = \hat\sigma + \Upsilon + \frac{1}{2}\varphi^*\Upsilon.$$

Now we show that

$$\varphi^*\Upsilon(u) = (\Upsilon \circ \varphi_0)(u) = -\Upsilon(u)$$

holds for all $u \in p_0^{-1}(x) \subset \mathcal{G}_0$. We have that $\varphi_0 : \mathcal{G}_0 \to \mathcal{G}_0$ preserves $p_0^{-1}(x)$ and in fact, it is equal to the right action of some suitable element from G_0. It is exactly the element g_0 from Lemma 6.2 corresponding to the frame u. Thanks to the equivariancy and the fact that the values of Υ are in $\mathfrak{g}_1$, the dual of $\mathfrak{g}_{-1}$, the action of the suitable element changes the sign and we get $(\Upsilon \circ \varphi_0)(u) = \Upsilon(ug_0) = -\Upsilon(u)$.

The latter fact gives

$$\hat{\sigma} + \Upsilon + \frac{1}{2}\varphi^*\Upsilon = \hat{\sigma} + \Upsilon - \frac{1}{2}\Upsilon = \hat{\sigma} + \frac{1}{2}\Upsilon$$

and if we put all together we get $\varphi^*(\hat{\sigma} + \frac{1}{2}\Upsilon) = \hat{\sigma} + \frac{1}{2}\Upsilon$. The morphism φ then preserves the Weyl structure $\sigma := \hat{\sigma} + \frac{1}{2}\Upsilon$ in the fiber over $x \in M$. □

Remark 9.2. We know that in $p^{-1}(x) \subset \mathcal{G}$, the morphism φ is equal to the right multiplication by some suitable element from P. Clearly, the points from the fiber that are simultaneously in $\sigma(\mathcal{G}_0)$ are exactly points for which φ is equal to the right multiplication by an element from G_0.

In the sequel, we call any such Weyl structure *fixed Weyl structure* (*for φ in x*). In fact, the next Lemma shows that in the fiber $p_0^{-1}(x)$, the fixed Weyl structure in x is unique.

Lemma 9.3. *Let φ be an automorphism covering some symmetry in x on a $|1|$–graded geometry and let σ, $\bar{\sigma}$ be two different fixed Weyl structures for φ in x. Then σ and $\bar{\sigma}$ are equal in the fiber over x.*

Proof. Let $\sigma, \bar{\sigma}$ be different fixed Weyl structures for φ in x. We know that $\bar{\sigma} = \sigma + \Upsilon$ for some nonzero $\Upsilon : \mathcal{G}_0 \to \mathfrak{g}_1$. In the point x we get

$$\bar{\sigma} = \varphi^*\bar{\sigma} = \varphi^*(\sigma + \Upsilon) = \varphi^*\sigma + \varphi^*\Upsilon = \sigma + \varphi^*\Upsilon.$$

The relation $\sigma + \Upsilon = \sigma + \varphi^*\Upsilon$ implies $\Upsilon = \varphi^*\Upsilon$ in x. Because $\varphi^*\Upsilon = -\Upsilon$ holds in the fiber over x we have that Υ vanishes in x. □

In general, we know nothing about (the difference between) the fixed Weyl structure σ and $\varphi^*\sigma$ over the neighborhood of x. Nevertheless, the latter facts allow us to prove the following Theorem.

Theorem 9.4. *Suppose that φ covers some symmetry in x on a $|1|$–graded geometry. There is exactly one normal Weyl structure σ such that*

$$\varphi^*\sigma = \sigma$$

over some neighborhood of x.

Proof. Let σ be an arbitrary fixed Weyl structure in x, i.e. let $\varphi^*\sigma = \sigma$ in the fiber over x. We take the normal Weyl structure at $\sigma(v) = u$ defined on a suitable neighborhood $U \subset M$ of x. Here $v \in p_0^{-1}(x) \subset \mathcal{G}_0$. We then have the Weyl structure σ_u such that

$$\begin{aligned} \sigma_u(v) &= \sigma(v) && \text{for } p_0(v) = x, \\ \sigma_u(p \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(\sigma(v))) &= \mathrm{Fl}_1^{\omega^{-1}(X)}(\sigma(v)) && \text{otherwise.} \end{aligned}$$

Pullback of this Weyl structure is again some normal Weyl structure. But we know that σ and σ_u are equal in x and we have $\varphi^*\sigma_u = \sigma_u$ in x. Then $\varphi^*\sigma_u$ has to be the original normal Weyl structure σ_u and we get $\varphi^*\sigma_u = \sigma_u$ over U.

Finally, the resulting normal structure does not depend on the choice of the fixed Weyl structure σ, because all these structures are equal in x, see Lemma 9.3. $\square$

Actually, we have some analogy of the fact from affine locally symmetric spaces. On the first order structures, there is the classical concept of normal coordinates and the symmetry clearly respects it. In these coordinates the symmetry only reverts the straight lines going through the point.

On parabolic geometries, there can exist many different normal coordinate systems. They are given by the choice of the (second order) frame in the fiber. We showed that on $|1|$–graded geometry carrying some symmetry in x, there is exactly one coordinate system in x such that the (covering of the) symmetry only reverts the straight lines going through the point in these coordinates.

REMARK 9.5. Simultaneously, we get that on some neighborhood of x, the pullback with respect to the covering φ of the symmetry $\underline{\varphi}$ in x permutes together all fixed Weyl structures in x. (They are all equal only in the fiber over x.) Clearly, $\varphi^*\sigma$ is again fixed Weyl structure in x because $\varphi^*\sigma = \sigma$ implies $\varphi^*(\varphi^*\sigma) = \varphi^*\sigma$ in x. In addition, there is at least one Weyl structure such that the pullback over φ preserves this structure on its place in the neighborhood of x (we constructed the normal one above).

If we start with the fixed normal Weyl structure σ_u, then for any $\sigma_u + \Upsilon$ we have

$$\varphi^*(\sigma_u + \Upsilon) = \varphi^*\sigma_u + \varphi^*\Upsilon = \sigma_u + \varphi^*\Upsilon$$

over some neighborhood of x. Consequently, all Weyl structures σ such that $\varphi^*\sigma = \sigma$ on some neighborhood of x differ from σ_u by some Υ satisfying $\Upsilon = \varphi^*\Upsilon$ on the neighborhood.

SYMMETRIES ON EFFECTIVE GEOMETRIES

We know, that each covering of some symmetry in x has some fixed Weyl structure in x. Let us take two different automorphisms that cover two different symmetries in x. We ask, whether the corresponding fixed Weyl structures in x can be equal. We start with one useful Lemma.

LEMMA 9.6. *Let ϕ be an automorphism of a $|1|$–graded geometry $(\mathcal{G} \to M, \omega)$ such that its base morphism $\underline{\phi}$ preserves some $x \in M$. If there is some $u \in p^{-1}(x)$ such that $\phi(u) = u \cdot h$ for some $h \in G_0$ and $\mathrm{Ad}_h(X) = X$ for all $X \in \mathfrak{g}_-$, then $\underline{\phi} = \mathrm{id}_M$ on some neighborhood of x.*

PROOF. We have $\underline{\phi}(x) = x$ and we use the normal coordinates at $u \in p^{-1}(x)$ to describe the neighborhood of x. Any point from the suitable neighborhood of x can be written as $p \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(u)$ for suitable $X \in \mathfrak{g}_{-1}$ and we have

$$\underline{\phi} \circ p \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(u) = p \circ \phi \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(u) = p \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(\phi(u)) = \\ = p \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(uh).$$

The equivariancy of ω and the fact that $\underline{\mathrm{Ad}}_h = \mathrm{Ad}_h$ for $h \in G_0$ gives that the curve $p \circ \mathrm{Fl}_t^{\omega^{-1}(X)}(uh)$ coincides with the curve $p \circ \mathrm{Fl}_t^{\omega^{-1}(\mathrm{Ad}_{h^{-1}} X)}(u)$. (See [**8, 23**] for details on generalized geodesics.)

The action of h is trivial and then the action of h^{-1} is trivial too. We have

$$p \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(uh) = p \circ \mathrm{Fl}_1^{\omega^{-1}(\mathrm{Ad}_{h^{-1}} X)}(u) = p \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(u).$$

This holds for all $X \in \mathfrak{g}_-$ and $\underline{\phi}$ is the identity on M locally. □

Let us remark that if some $h \in G_0$ acts trivially on $\mathfrak{g}_{-1}$, then it acts trivially on the whole $\mathfrak{g}$ because the action of elements from G_0 on $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ respects the gradation, while the action on $\mathfrak{g}_0 \simeq \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1$ and $\mathfrak{g}_1 \simeq \mathfrak{g}_{-1}^*$ is induced from trivial action on $\mathfrak{g}_{-1}$ and thus trivial. So the elements h we use here are exactly elements from the kernel of the adjoint action on $\mathfrak{g}$.

In addition, we defined parabolic geometries as infinitesimally effective geometries and in this case the kernel K of the geometry is discrete. Then thanks to the smoothness of the multiplication we have

$$\exp(-tX)k\exp(tX) = k$$

for all $k \in K$ and $X \in \mathfrak{g}$ and the differentiating in $t = 0$ gives $T\rho^k.X - T\lambda_k.X = 0$ and thus $\mathrm{Ad}_k(X) = X$. Then K lies in the kernel of the adjoint action and among others it then lies in G_0. Thus K equals to the kernel of the adjoint action because the latter subgroup is normal and so contained in K. The elements $h \in G_0$ from the latter Lemma are exactly elements from the kernel of the geometry.

Proposition 9.7. *Let φ and ψ be two coverings of two (in general different) symmetries in x on a $|1|$–graded geometry sharing the same fixed Weyl structure σ in x, i.e. $\varphi^*\sigma|_{p_0^{-1}(x)} = \sigma|_{p_0^{-1}(x)} = \psi^*\sigma|_{p_0^{-1}(x)}$. Then*

$$\psi(u) = \varphi(u) \cdot h$$

for all u over some suitable neighborhood of x and for some h from the kernel of the geometry.

Proof. Suppose that $\varphi^*\sigma = \sigma$ and $\psi^*\sigma = \sigma$ in x. Then we have $\varphi^{-1} \circ \sigma \circ \varphi_0 = \psi^{-1} \circ \sigma \circ \psi_0$ and this is equivalent with the fact that

$$(\psi \circ \varphi^{-1}) \circ \sigma(v) = \sigma \circ (\psi_0 \circ \varphi_0^{-1})(v)$$

holds for each $v \in p_0^{-1}(x)$. The morphism φ_0 preserves the fiber over x and in fixed v is equal to the right multiplication by some $k \in G_0$ such that $\mathrm{Ad}_k(X) = -X$ for all $X \in \mathfrak{g}_{-1}$. The morphism ψ_0 also coincides in v with the action of some $g \in G_0$ satisfying $\mathrm{Ad}_g(X) = -X$ for all $X \in \mathfrak{g}_{-1}$. We can accordingly write

$$(\psi \circ \varphi^{-1}) \circ \sigma(v) = \sigma(v)h,$$

where $h = k^{-1}g$ acts trivially on $\mathfrak{g}_{-1}$, because we have

$$\mathrm{Ad}_h X = \mathrm{Ad}_{k^{-1}} \mathrm{Ad}_g X = \mathrm{Ad}_{k^{-1}}(-X) = X.$$

Thus element $u = \sigma(v) \in p^{-1}(x)$ is exactly element from the fiber with the property, that $(\psi \circ \varphi^{-1})(u) = u \cdot h$ where $h \in G_0$ acts trivially on $\mathfrak{g}_{-1}$ and Lemma 9.6 gives that

$$\underline{\psi \circ \varphi}^{-1} = \mathrm{id}_M.$$

The Theorem 2.4 gives, that $\psi \circ \varphi^{-1}(u) = \mathrm{id}_{\mathcal{G}} \cdot f(u)$ holds for some function f over some neighborhood of x. According to our definition of parabolic geometries, they are always infinitesimally effective and the function f has to be constant. The value of f has to be the latter element $h \in G_0$, which clearly belongs to the kernel. We have $\psi \circ \varphi^{-1}(u) = uh$ on some neighborhood of x. If we apply the automorphism φ first, we get $\psi \circ \varphi^{-1}(\varphi(u)) = \varphi(u)h$ over some suitable neighborhood of x. This implies $\psi(u) = \varphi(u)h$. □

As an easy consequence of the latter facts we get the following Proposition on symmetries.

Proposition 9.8. *Let φ and ψ be two coverings of two symmetries in $x \in M$ on a $|1|$–graded geometry which share the same fixed Weyl structure in x. Then $\underline{\varphi} = \underline{\psi}$ on some neighborhood of x, i.e. they cover the same symmetry on the base manifold M.*

Proof. One can use the proof of Proposition 9.7. We have $\underline{\psi \circ \varphi}^{-1} = \mathrm{id}_M$ on some neighborhood of the point x and then clearly $\underline{\varphi} = \underline{\psi}$ on the neighborhood of x. □

In particular, if φ covers some symmetry, then clearly φ^{-1} covers some symmetry, too. Moreover, φ and φ^{-1} share the same fixed Weyl structure in x because if we have $\varphi^* \sigma = \sigma$, then we also have

$$(\varphi^{-1})^* \varphi^* \sigma = (\varphi^{-1})^* \sigma$$

and simultaneously we get

$$(\varphi^{-1})^* \varphi^* \sigma = (\varphi \circ \varphi^{-1})^* \sigma = \mathrm{id}^* \sigma = \sigma.$$

We then get $(\varphi^{-1})^* \sigma = \sigma$ in x. Thus φ and φ^{-1} cover the same symmetry.

Corollary 9.9. *Each symmetry $\underline{\varphi}$ in $x \in M$ on a $|1|$–graded geometry is involutive.*

Remark 9.10. There is an equivalent definition of the symmetry on the manifold, see [**16**] or [**13**]. One can define symmetry in x as a (locally defined) involutive automorphism such that the point x is the isolated fixed point of this automorphism. Symmetries defined in this manner clearly satisfy the condition on the differential and they are symmetries in our sense. Conversely, the Corollary 9.9 says that our symmetries correspond to the latter definition.

If we start with an effective geometry, we in addition have following consequences of Proposition 9.7:

Corollary 9.11. *Let φ and ψ cover two symmetries in x on some effective $|1|$–graded geometry and suppose that they share the same fixed Weyl structure in x. Then $\varphi = \psi$ over the neighborhood of x. In addition, each covering of some symmetry is involutive.*

Proof. From the proof of Proposition 9.7 we get that $(\psi \circ \varphi^{-1}) \circ \sigma(v) = \sigma(v)$ holds in the case of effective geometries. The rest follows immediately from the formula. $\square$

Clearly, there can exist more then one covering of one symmetry in x in general. If we suppose that two automorphisms cover the same symmetry, then the Theorem 2.4 directly says, that they differ by a multiplication by some element from the kernel. We can show the statement opposite to the Proposition 9.7.

Lemma 9.12. *Suppose that φ and ψ cover the same symmetry in x on a $|1|$–graded geometry. Then they share the same fixed Weyl structure in x.*

Proof. We have $\psi(u) = \varphi(u) \cdot h$ over some suitable neighborhood of x, where h belongs to the kernel. Then clearly their underlying automorphisms on $\mathcal{G}_0$ satisfy $\psi_0(v) = \phi_0(v) \cdot h$ for $v \in \mathcal{G}_0$. Suppose that $\varphi^*\sigma(v) = \varphi^{-1} \circ \sigma \circ \varphi_0(v) = \sigma(v)$ for all $v \in p_0^{-1}(x)$. We have

$$\begin{aligned}\psi^*\sigma(v) &= (\psi^{-1} \circ \sigma \circ \psi_0)(v) = \varphi^{-1}(\sigma \circ \psi_0(v)) \cdot h^{-1} = \\ &= (\varphi^{-1} \circ \sigma \circ \varphi_0)(v) \cdot h \cdot h^{-1} = \varphi^*\sigma(v) = \sigma(v)\end{aligned}$$

for all $v \in p_0^{-1}(x)$ and they share the same fixed Weyl structure σ in x. $\square$

Among others, this two coverings then share the same fixed normal Weyl structure on the neighborhood of x. If we put all together, we get:

Corollary 9.13. *Each symmetry $\underline{\varphi}$ in x allows exactly one fixed Weyl structure in x. It does not depend on choice of the covering of the symmetry $\underline{\varphi}$. The symmetry can have several coverings, but all of them differ by the multiplication of some element from the kernel and all of them share in the fiber over x the same fixed Weyl structure in x.*

Since the fixed Weyl structure in x does not depend on the choice of the covering of the symmetry, we can speak about fixed Weyl structure in x corresponding to the symmetry $\underline{\varphi}$.

Remark 9.14. The effectiveness of the geometry depends on the particular choice of the groups G and P. In concrete examples, this amounts to matrix computation of all elements in G_0 which act trivially on $\mathfrak{g}_{-1}$ (and thus on $\mathfrak{g}$). Although we could restrict ourselves to effective geometries, there are standard choices, which give well known geometries and which are not effective in general. We show some examples:

Example 9.15. *Projective geometries.* We take both possible choices of groups together and finally we discuss each of them, see Example 7.3. We look for all $\begin{pmatrix} d & 0 \\ 0 & D \end{pmatrix}$ such that $\begin{pmatrix} d & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & D \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}$ for all $X \in \mathbb{R}^m$. We then solve $\begin{pmatrix} d & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & D \end{pmatrix}$ and this is equivalent to $DX = dX$. There are two possible solutions: $\begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -E \end{pmatrix}$.

In the case $G = PGl(m+1,\mathbb{R})$ these two elements represent the same class, the unit element. In the case $G = Sl(m+1,\mathbb{R})$ the first element is the unit and we should discuss the second element. The elements from G_0 look like $\left(\begin{smallmatrix} d & 0 \\ 0 & D \end{smallmatrix}\right)$ where $\det D = \frac{1}{d}$ is positive. For the element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -E \end{smallmatrix}\right)$ we have that if m is odd, then $D = -E$ has the negative determinant and the whole element is not contained in G_0. If m is even, then the whole element is not contained in G because its determinant is negative. In both cases of projective geometries, only the unit acts trivially. We have effective geometries and each possible symmetry has exactly one covering.

Example 9.16. *Conformal geometries.* We take the standard choice of group, see Example 4.2. We look for elements $\left(\begin{smallmatrix} \lambda & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & \lambda^{-1} \end{smallmatrix}\right) \in G_0$ such that

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^TJ & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^TJ & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}$$

holds for all $X \in \mathbb{R}^{p+q}$. We have the equation $CX = \lambda X$ and there are two possible solutions: $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -E & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$. The first solution is the unit. The second element also belongs to the group $G_0 \subset O(p+1,q+1)$. They clearly form the kernel of the geometry. Each possible symmetry has exactly two possible coverings. See Example 7.5 to compare this with the homogeneous model.

10. Further curvature restrictions

We know that if there exists a symmetry in one point on the homogeneous model, then there are many symmetries in any point of the base manifold. We would like to study the following question: How many (different) symmetries can exist in one point on a geometry, which is not locally flat?

In the Section 8 we showed that in many cases, if there exists some symmetry in x, than the geometry has zero curvature in x. But there are some exceptions. All geometries which have some homogeneous component of harmonic curvature of degree 2 may carry symmetries and need not to be locally flat.

The curvature of these geometries looks like $\kappa = \kappa^0 : \mathcal{G} \to \wedge^2\mathfrak{g}_{-1}^* \otimes \mathfrak{g}^0$. If we choose some Weyl structure σ, we can take the decomposition $\sigma^*\kappa^0 = \sigma^*\kappa_0 + \sigma^*\kappa_1$. The Weyl curvature

$$\sigma^*\kappa_0 : \mathcal{G}_0 \to \wedge^2\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$$

does not change, if we change the Weyl structure, because it is the lowest part of decomposition. Next we will use W for it (instead of W^σ). This part corresponds to the interesting part of the whole curvature and we should concentrate on it.

Suppose, that φ covers some symmetry in x. Let us remind that for any Weyl structure σ we have $\varphi^*\sigma = \sigma + \Upsilon$ and similar fact holds for all connections coming from Weyl structures. See Section 5 for details. We should also remind, that all ideas in this section do not depend on the choice of the covering of the symmetry, see Section 9.

We will study algebraic actions and covariant derivatives of the Weyl curvature W with respect to Weyl connections.

LEMMA 10.1. *Let σ be an arbitrary Weyl structure on a $|1|$–graded geometry and let $\underline{\varphi}$ be a symmetry in x covered by some φ. Then*

$$\{\xi, \Upsilon\} \bullet W + 2\nabla^\sigma_\xi W = 0$$

holds in x for all $\xi \in \mathfrak{X}(M)$, where Υ is defined by $\varphi^*\sigma = \sigma + \Upsilon$.

PROOF. We take $\nabla^\sigma_\xi W(\eta, \mu)$ for each $\xi, \eta, \mu \in \mathfrak{X}(M)$. We compute the pullback of the connection with respect to the symmetry $\underline{\varphi}$ in the point x. In the point x we have

$$(\underline{\varphi}^*\nabla^\sigma)_\xi W(\eta, \mu) = (T\underline{\varphi} \otimes T\underline{\varphi}^{-1}).\nabla^\sigma_{T\underline{\varphi}.\xi} W(T\underline{\varphi}.\eta, T\underline{\varphi}.\mu) = -\nabla^\sigma_\xi W(\eta, \mu)$$

for each $\xi, \eta, \mu \in \mathfrak{X}(M)$.

Next, we have $\varphi^*\sigma = \sigma + \Upsilon$ for some Υ and it gives

$$(\underline{\varphi}^*\nabla^\sigma)_\xi W(\eta, \mu) = \nabla^{\sigma+\Upsilon}_\xi W(\eta, \mu)$$

for each $\xi, \eta, \mu \in \mathfrak{X}(M)$.

If we put it together, we get that the following identity

$$-\nabla^\sigma_\xi W(\eta, \mu) = \nabla^{\sigma+\Upsilon}_\xi W(\eta, \mu)$$

holds in x. Using the formula for change of Weyl connection we can rewrite it as

$$-\nabla^\sigma_\xi W(\eta, \mu) = \nabla^\sigma_\xi W(\eta, \mu) + (\{\xi, \Upsilon\} \bullet W)(\eta, \mu).$$

This holds in the point x for each $\xi, \eta, \mu \in \mathfrak{X}(M)$ and therefore it gives exactly the equation. $\square$

As an easy consequence we get the following Proposition.

PROPOSITION 10.2. *Let $\underline{\varphi}$ be some symmetry in x on a $|1|$–graded geometry. There exists a Weyl connection ∇^σ such that $\nabla^\sigma W = 0$ in x. The connection corresponds to the fixed Weyl structure in x.*

PROOF. Let σ be the fixed Weyl structure in x. Then in the point x, we have $\varphi^*\sigma = \sigma$ for a covering φ of the symmetry $\underline{\varphi}$ and the connection ∇^σ is invariant with respect to the symmetry $\underline{\varphi}$ in x.

We use the Lemma 10.1. In this case, we have $\Upsilon = 0$ in x thanks to the invariance and the algebraic bracket from the expression in the latter Lemma has to vanish for all $\xi \in \mathfrak{X}(M)$. But then $2\nabla^\sigma_\xi W = 0$ holds for all ξ and then $\nabla^\sigma W = 0$. $\square$

In fact, this is an analogy of the result from affine symmetric spaces. There is only one connection given on the affine geometry and we know that its curvature is covariantly constant with respect to it.

In our case, there is a class of interesting connections and we showed that there is at least one connection such that the Weyl curvature, which is equal for all Weyl structures in this case, is covariantly constant with respect to it.

Using all latter facts we show some algebraic restriction on the Weyl curvature of symmetric $|1|$–graded geometries.

Proposition 10.3. *Assume there are two different symmetries in x on a $|1|$–graded geometry. Then*

$$\{\xi, \Upsilon\} \bullet W = 0 \tag{1}$$

holds in x for any $\xi \in \mathfrak{X}(M)$ and one fixed 1–form Υ given by the fixed Weyl structures of the mentioned symmetries.

Proof. Let us take two different symmetries $\underline{\varphi}$ and $\underline{\psi}$ with coverings φ and ψ. In the point x, the symmetries have different fixed Weyl structures, see Section 9. Let σ be fixed Weyl structure in x for $\underline{\varphi}$, i.e. $\varphi^*\sigma = \sigma$ in x. Then σ cannot be fixed for symmetry ψ and we have $\underline{\psi}^*\sigma = \sigma + \Upsilon$, where Υ is nonzero in x.

The Lemma 10.1 and Proposition 10.2 give that

$$\nabla^\sigma_\xi W = 0,$$
$$\{\xi, \Upsilon\} \bullet W + 2\nabla^\sigma_\xi W = 0$$

hold for all $\xi \in \mathfrak{X}(M)$. If we put it together, we get exactly the required expression. □

The expression (1) can be equivalently written in the following way.

Corollary 10.4. *If there are two different symmetries in x on a $|1|$–graded geometry, then*

$$\begin{aligned} &\{\{\xi, \Upsilon\}, W(\eta, \mu)(\nu)\} - W(\{\{\xi, \Upsilon\}, \eta\}, \mu)(\nu) \; - \\ &W(\eta, \{\{\xi, \Upsilon\}, \mu\})(\nu) - W(\eta, \mu)(\{\{\xi, \Upsilon\}, \nu\}) = 0 \end{aligned} \tag{2}$$

holds in x for any $\xi, \eta, \mu, \nu \in \mathfrak{X}(M)$ and one fixed 1–form Υ, which is given by the fixed Weyl structures of mentioned symmetries.

Proof. The expression $\{\xi, \Upsilon\} \bullet W$ is of the type $\wedge^2 T^*M \otimes T^*M \otimes TM$ for any field ξ and we evaluate it on vector fields η, μ and ν. We get:

$$\begin{aligned} (\{\xi, \Upsilon\} \bullet W)(\eta, \mu)(\nu) = \{\{\xi, \Upsilon\}, W(\eta, \mu)(\nu)\} - W(\{\{\xi, \eta\}, \eta\}, \mu)(\nu) \; - \\ W(\eta, \{\{\xi, \Upsilon\}, \mu\})(\nu) - W(\eta, \mu)(\{\{\xi, \Upsilon\}, \nu\}). \end{aligned}$$

It gives exactly the required formula. □

Suppose that there are several symmetries in x. If we choose one, then we get the restriction (1) (resp. (2)) for any Υ, which is given by the fixed Weyl structure of the chosen symmetry and the fixed Weyl structure of any other symmetry. Clearly, for higher number of symmetries we get more conditions and there is higher chance that we get some stronger restriction on the Weyl curvature.

Extremal cases are symmetric homogeneous models. All symmetries (in o) are all left multiplications by $g_0 \exp Z$ for suitable $g_0 \in G_0$ and arbitrary $Z \in \mathfrak{g}_1$. Then we get the previous restriction for all possible Υ. We know that the curvature of homogeneous model vanishes and $W = 0$ is also the only possible solution, that satisfies $\{\xi, \Upsilon\} \bullet W = 0$ for all ξ and Υ.

Now, we can ask how many symmetries we need to get some better restriction on the Weyl curvature and then on the whole curvature of some geometry, which is not locally flat. The exact form of the algebraic bracket, which is in the restrictive condition, is different in each geometry and we take some concrete geometries and study the latter question for them.

PROJECTIVE GEOMETRIES

We shall use the conventions introduced in Example 7.3. We start with a small Lemma which works for all projective structures in general.

LEMMA 10.5. *In the projective geometry*

$$\{\{\xi, \Upsilon\}, \eta\} = \Upsilon(\xi) \cdot \eta + \Upsilon(\eta) \cdot \xi$$

holds for any vector fields ξ, η *and any* 1*–form* Υ.

PROOF. For $\left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ Y & 0 \end{smallmatrix}\right)$ in $\mathfrak{g}_{-1}$ and $\left(\begin{smallmatrix} 0 & Z \\ 0 & 0 \end{smallmatrix}\right)$ in $\mathfrak{g}_1$ we have

$$\left[\left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & Z \\ 0 & 0 \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} -ZX & 0 \\ 0 & XZ \end{smallmatrix}\right),$$

$$\left[\left[\left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & Z \\ 0 & 0 \end{smallmatrix}\right)\right], \left(\begin{smallmatrix} 0 & 0 \\ Y & 0 \end{smallmatrix}\right)\right] = \left[\left(\begin{smallmatrix} -ZX & 0 \\ 0 & XZ \end{smallmatrix}\right), \left(\begin{smallmatrix} Y & 0 \\ 0 & 0 \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} 0 & 0 \\ X \cdot ZY + Y \cdot ZX & 0 \end{smallmatrix}\right).$$

Here ZY and ZX correspond to the evaluation of a 1–form on a vector field and we get directly the bracket. □

REMARK 10.6. Let us remind that from this we can explicitly write the formula for the change of Weyl connections in the projective geometry, see Section 5. For two Weyl structures σ and $\hat{\sigma} = \sigma + \Upsilon$ we have

$$\nabla^{\hat{\sigma}}_{\xi}(\eta) = \nabla^{\sigma}_{\xi}(\eta) + \Upsilon(\xi) \cdot \eta + \Upsilon(\eta) \cdot \xi$$

for all $\xi, \eta \in \mathfrak{X}(M)$. This is exactly the formula for two projectively equivalent affine connections.

Now, we can start the discussion on symmetries on projective geometries. At first, we rewrite the expression (2). Applying the formula for the bracket we have:

$$\begin{aligned} &\Upsilon(\xi) \cdot W(\eta, \mu)(\nu) + \Upsilon(W(\eta, \mu)(\nu)) \cdot \xi - \\ &\Upsilon(\xi) \cdot W(\eta, \mu)(\nu) - \Upsilon(\eta) \cdot W(\xi, \mu)(\nu) - \\ &\Upsilon(\xi) \cdot W(\eta, \mu)(\nu) - \Upsilon(\mu) \cdot W(\eta, \xi)(\nu) - \\ &\Upsilon(\xi) \cdot W(\eta, \mu)(\nu) - \Upsilon(\nu) \cdot W(\eta, \mu)(\xi) = \ 0 \end{aligned}$$

After some arrangements we get:

$$\begin{aligned} \Upsilon(W(\eta, \mu)(\nu)) \cdot \xi = 2\Upsilon(\xi) \cdot W(\eta, \mu)(\nu) + \Upsilon(\eta) \cdot W(\xi, \mu)(\nu) + \\ \Upsilon(\mu) \cdot W(\eta, \xi)(\nu) + \Upsilon(\nu) \cdot W(\eta, \mu)(\xi) \end{aligned} \tag{3}$$

LEMMA 10.7. *Suppose that there exist two different symmetries in* x *on a projective geometry and let* Υ *be given as in the proof of Proposition 10.3. Then in the point* x *we have*

$$\Upsilon(W(\eta, \mu)(\nu)) = 0$$

for any $\eta, \mu, \nu \in \mathfrak{X}(M)$*, i.e. the values of* W *are in kernel of* Υ.

PROOF. We start with the expression (3) which holds thanks to the assumption and Proposition 10.3. The 1–form Υ is nonzero in x because it is given by two fixed Weyl structures corresponding to different symmetries in x. We evaluate the 1–form Υ on the expression. We have

$$\begin{aligned} \Upsilon(W(\eta, \mu)(\nu)) \cdot \Upsilon(\xi) = 2\Upsilon(\xi) \cdot \Upsilon(W(\eta, \mu)(\nu)) + \Upsilon(\eta) \cdot \Upsilon(W(\xi, \mu)(\nu)) + \\ \Upsilon(\mu) \cdot \Upsilon(W(\eta, \xi)(\nu)) + \Upsilon(\nu) \cdot \Upsilon(W(\eta, \mu)(\xi)) \end{aligned}$$

and we get

$$
-\Upsilon(W(\eta,\mu)(\nu)) \cdot \Upsilon(\xi) = \Upsilon(\eta) \cdot \Upsilon(W(\xi,\mu)(\nu)) + \\
\Upsilon(\mu) \cdot \Upsilon(W(\eta,\xi)(\nu)) + \\
\Upsilon(\nu) \cdot \Upsilon(W(\eta,\mu)(\xi)). \tag{4}
$$

It works in x for any ξ and we choose such that $\Upsilon(\xi) \neq 0$. Any vector field can be then written as the sum of some multiple of ξ and some field from $\ker \Upsilon$. We now divide the computations into the cases when any of η, μ, ν is in $\ker \Upsilon$ or when it is not in $\ker \Upsilon$ and we can take it as equal to ξ. We have to discuss all possibilities:

If $\eta, \mu, \nu \in \ker \Upsilon$, then we get from (4) that $-\Upsilon(W(\eta,\mu)(\nu)) \cdot \Upsilon(\xi) = 0$. Because $\Upsilon(\xi)$ is nonzero, we get $\Upsilon(W(\eta,\mu)(\nu)) = 0$.

In the case $\mu, \nu \in \ker \Upsilon$ and $\eta = \xi$ we have

$$
-\Upsilon(W(\xi,\mu)(\nu)) \cdot \Upsilon(\xi) = \Upsilon(\xi) \cdot \Upsilon(W(\xi,\mu)(\nu))
$$

and again, because $\Upsilon(\xi)$ is nonzero we get $\Upsilon(W(\xi,\mu)(\nu)) = 0$. The other cases when two of the fields are in the kernel of Υ and one is not there work similarly. We always get $2\Upsilon(\xi) \cdot \Upsilon(W(\eta,\mu)(\nu)) = 0$ for corresponding choice of fields.

Further cases are analogical. If one of the fields is in $\ker \Upsilon$ and the other two are equal to ξ, we get from (4) that

$$
3\Upsilon(\xi) \cdot \Upsilon(W(\eta,\mu)(\nu)) = 0
$$

for corresponding choice of fields. The case when all fields are equal to ξ is trivial. □

Consequently, we have the following equation:

$$
0 = 2\Upsilon(\xi) \cdot W(\eta,\mu)(\nu) + \Upsilon(\eta) \cdot W(\xi,\mu)(\nu) + \\
\Upsilon(\mu) \cdot W(\eta,\xi)(\nu) + \Upsilon(\nu) \cdot W(\eta,\mu)(\xi). \tag{5}
$$

PROPOSITION 10.8. *Suppose that there exist two different symmetries in x on the projective geometry. Then W vanishes in x.*

PROOF. We use the equation (5). Here Υ is again given by the two fixed Weyl structures of corresponding symmetries and it has to be nonzero in x. We choose some ξ such that $\Upsilon(\xi) \neq 0$ and we divide the proof into the cases when any of the field η, μ and ν is in $\ker \Upsilon$ and when we can take it as equal to the field ξ.

If $\eta, \mu, \nu \in \ker \Upsilon$, then (5) gives $2\Upsilon(\xi) \cdot W(\eta,\mu)(\nu) = 0$. If $\eta, \mu \in \ker \Upsilon$ and $\nu = \xi$, then we get

$$
0 = 2\Upsilon(\xi) \cdot W(\eta,\mu)(\xi) + \Upsilon(\xi) \cdot W(\eta,\mu)(\xi) = 3\Upsilon(\xi) \cdot W(\eta,\mu)(\xi).
$$

In both cases, we get $W(\eta,\mu)(\nu) = 0$ for the corresponding choices of fields because $\Upsilon(\xi)$ is nonzero. If $\eta, \nu \in \ker \Upsilon$ and $\mu = \xi$, then we get

$$
3\Upsilon(\xi) \cdot W(\eta,\xi)(\nu) = 0
$$

and analogical result we get for the case $\mu, \nu \in \ker \Upsilon$ and $\eta = \xi$.

If $\eta \in \ker \Upsilon$ and $\mu = \xi = \nu$, then we get

$$
4\Upsilon(\xi) \cdot W(\eta,\xi)(\xi) = 0.
$$

The case $\mu \in \ker \Upsilon$ and $\eta = \xi = \nu$ is analogical. The case $\eta = \mu = \xi$ is trivial. We again get $W(\eta,\mu)(\nu) = 0$ for corresponding choices of fields.

This works for all $\xi \notin \ker \Upsilon$ and for arbitrary η, μ, ν. But then W vanishes on some open subset and then it has to vanish everywhere. □

Now we can show:

THEOREM 10.9. *Let $(\mathcal{G} \to M, \omega)$ be a projective geometry and suppose that there exist two different symmetries in x. Then the curvature vanishes in x.*

If there are two different symmetries in each point, then the geometry is locally flat.

PROOF. We know that if there is some symmetry in x, then the torsion vanishes in x, see Proposition 8.1. Existence of two different symmetries in x kills the Weyl curvature in x and then $\kappa = \kappa_1$. But κ_1 has to vanish too, we use the same argument as in the proof of Theorem 8.4. Then the whole curvature vanishes in x.

If there are two different symmetries in each point, then the curvature vanishes in all points and the geometry is locally isomorphic with homogeneous model. □

We can nicely reformulate the latter Theorem in following way.

COROLLARY 10.10. *For projective geometries, there can exist at most one symmetry in each point where the curvature does not vanish.*

CONFORMAL GEOMETRIES

Let us first remark that the conformal structure of signature (p, q) on the manifold M can be viewed as an equivalence class of pseudo Riemannian metrics such that two metrics are equivalent if and only if they differ by a multiplication by a smooth positive function. It is convenient to write $\hat{g} = \Omega^2 g$ for two metrics from the class.

The values of the metrics in the point $x \in M$ form a ray in $S^2T^*_xM$ and we get a bundle such that its sections are exactly the metrics from the conformal class. It is a ray subbundle of S^2T^*M, i.e. a principal subbundle with structure group $\mathbb{R}_+$. The conformal metric is thus a section of the bundle of rays in $S^2_0T^*M \subset S^2T^*M$, the subspace of regular symmetric forms. It is usually denoted as $\boldsymbol{g}$, but we shall use the usual letter g instead in the rest of this section. The conformal metric is a weighted tensor, thus we may use g in order to raise or lower indices on the cost of adding the appropriate weights. In particular, the same applies if the metric is understood as isomorphism between the tangent and cotangent bundles, and no weight is added if both the conformal metric and its inverse are used together. For example, for a given vector ξ, the evaluation $g(\xi, \xi)$ provides rather a density of weight two than a number (but its (non)vanishing is well defined anyhow).

The computations will be performed in the setting of Example 4.2. We again start with a small Lemma on the bracket which holds for conformal geometry in general.

LEMMA 10.11. *In conformal geometry*

$$\{\{\xi, \Upsilon\}, \eta\} = \Upsilon(\xi) \cdot \eta + \Upsilon(\eta) \cdot \xi - g(\xi, \eta) \cdot g^{-1}(\Upsilon)$$

holds for any vector fields ξ, η *and any* 1*–form* Υ.

PROOF. For $\begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^T J & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ Y & 0 & 0 \\ 0 & -Y^T J & 0 \end{pmatrix}$ in $\mathfrak{g}_{-1}$ and $\begin{pmatrix} 0 & Z & 0 \\ 0 & 0 & -JZ^T \\ 0 & 0 & 0 \end{pmatrix}$ in $\mathfrak{g}_1$ we have

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^T J & 0 \end{pmatrix}, \begin{pmatrix} 0 & Z & 0 \\ 0 & 0 & -JZ^T \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} -ZX & 0 & 0 \\ 0 & XZ - JZ^T X^T J & 0 \\ 0 & 0 & ZX \end{pmatrix},$$

$$\left[\begin{pmatrix} -ZX & 0 & 0 \\ 0 & XZ - JZ^T X^T J & 0 \\ 0 & 0 & ZX \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ Y & 0 & 0 \\ 0 & -Y^T J & 0 \end{pmatrix}\right] =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ ZY \cdot X + ZX \cdot Y - JZ^T \cdot (X^T JY) & 0 & 0 \\ 0 & -ZY \cdot X^T J - Z(X) \cdot Y^T J + Y^T JX \cdot Z & 0 \end{pmatrix}.$$

Here ZX and ZY correspond to the evaluation of a 1–form on a vector field, $(X^T JY)$ is the evaluation of the standard scalar product on X and Y and JZ^T is dual to Z with respect to the standard product. □

REMARK 10.12. Again, from this we can explicitly write the formula for the change of Weyl connection in conformal geometry. For two Weyl structures σ and $\hat{\sigma} = \sigma + \Upsilon$ and for all $\xi, \eta \in \mathfrak{X}(M)$ we have

$$\nabla^{\hat{\sigma}}_\xi(\eta) = \nabla^\sigma_\xi(\eta) + \Upsilon(\xi) \cdot \eta + \Upsilon(\eta) \cdot \xi - g(\xi, \eta) \cdot g^{-1}(\Upsilon).$$

This is exactly the classical formula for the change of the connection under a conformal rescaling of a metric.

Now, we start the discussion of symmetries. In the conformal geometry, the expression (1) can be rewritten in the following way: If we apply the formula for the bracket, we get

$$\begin{aligned}
&\Upsilon(\xi) \cdot W(\eta, \mu)(\nu) + \Upsilon(W(\eta, \mu)(\nu)) \cdot \xi - g(\xi, W(\eta, \mu)(\nu)) \cdot g^{-1}(\Upsilon) - \\
&\Upsilon(\xi) \cdot W(\eta, \mu)(\nu) - \Upsilon(\eta) \cdot W(\xi, \mu)(\nu) + g(\xi, \eta) \cdot W(g^{-1}(\Upsilon), \mu)(\nu) - \\
&\Upsilon(\xi) \cdot W(\eta, \mu)(\nu) - \Upsilon(\mu) \cdot W(\eta, \xi)(\nu) + g(\xi, \mu) \cdot W(\eta, g^{-1}(\Upsilon))(\nu) - \\
&\Upsilon(\xi) \cdot W(\eta, \mu)(\nu) - \Upsilon(\nu) \cdot W(\eta, \mu)(\xi) + g(\xi, \nu) \cdot W(\eta, \mu)(g^{-1}(\Upsilon)) = 0.
\end{aligned}$$

Some rearrangements give

$$\begin{aligned}
&\Upsilon(W(\eta, \mu)(\nu)) \cdot \xi - g(\xi, (W(\eta, \mu)(\nu)) \cdot g^{-1}(\Upsilon) = \\
&= 2\Upsilon(\xi) \cdot W(\eta, \mu)(\nu) + \Upsilon(\eta) \cdot W(\xi, \mu)(\nu) + \\
&\Upsilon(\mu) \cdot W(\eta, \xi)(\nu) + \Upsilon(\nu) \cdot W(\eta, \mu)(\xi) - \\
&g(\xi, \eta) \cdot W(g^{-1}(\Upsilon), \mu)(\nu) - \\
&g(\xi, \mu) \cdot W(\eta, g^{-1}(\Upsilon))(\nu) - \\
&g(\xi, \nu) \cdot W(\eta, \mu)(g^{-1}(\Upsilon)).
\end{aligned} \tag{6}$$

PROPOSITION 10.13. *Suppose that there exist two different symmetries in* x *on a conformal geometry. Let* Υ *be given by the corresponding fixed Weyl structures as in the proof of Proposition 10.3 and suppose that* $|\Upsilon|_g$ *is nonzero. Then* W *vanishes in* x.

PROOF. We can use the equation (6) thanks to the assumptions and Proposition 10.3. Let us choose ξ such that $\Upsilon = g(\xi, -)$, i.e. $\xi = g^{-1}(\Upsilon)$. If we put it into the expression (6) we get

$$\begin{aligned}
&\Upsilon(W(\eta,\mu)(\nu)) \cdot g^{-1}(\Upsilon) - g(g^{-1}(\Upsilon), (W(\eta,\mu)(\nu)) \cdot g^{-1}(\Upsilon) = \\
&= 2\Upsilon(g^{-1}(\Upsilon)) \cdot W(\eta,\mu)(\nu) + \Upsilon(\eta) \cdot W(g^{-1}(\Upsilon),\mu)(\nu) + \\
&\Upsilon(\mu) \cdot W(\eta, g^{-1}(\Upsilon))(\nu) + \Upsilon(\nu) \cdot W(\eta,\mu)(g^{-1}(\Upsilon)) - \\
&g(g^{-1}(\Upsilon),\eta) \cdot W(g^{-1}(\Upsilon),\mu)(\nu) - \\
&g(g^{-1}(\Upsilon),\mu) \cdot W(\eta, g^{-1}(\Upsilon))(\nu) - \\
&g(g^{-1}(\Upsilon),\nu) \cdot W(\eta,\mu)(g^{-1}(\Upsilon)).
\end{aligned}$$

Simultaneously, we have that the form $g(g^{-1}(\Upsilon), -) = g(\xi, -)$ is just Υ and the expression reduces heavily. We get

$$2\Upsilon(g^{-1}(\Upsilon)) \cdot W(\eta,\mu)(\nu) = 0.$$

The assumption is that $\Upsilon(g^{-1}(\Upsilon)) = g(g^{-1}(\Upsilon), g^{-1}(\Upsilon)) = |\Upsilon|_g$ is nonzero. But then we get $W(\eta,\mu)(\nu) = 0$ for all $\eta, \mu, \nu \in \mathfrak{X}(M)$ in x and it gives that $W = 0$ in x. □

THEOREM 10.14. *Let $(\mathcal{G} \to M, \omega)$ be a conformal geometry of arbitrary signature and suppose that there exist two different symmetries in x. Let Υ be given by the corresponding fixed Weyl structures and suppose that $|\Upsilon|_g$ is nonzero. Then the curvature vanishes in x.*

If there are two such different symmetries in each point, then the geometry is locally flat.

PROOF. We use exactly the same ideas as in the projective case, see Theorem 10.9. If there is some symmetry in x, then the torsion vanishes in x, see Proposition 8.1. Existence of two different symmetries satisfying the assumption on corresponding Υ kills the Weyl curvature in x and then $\kappa = \kappa_1$. But κ_1 has to vanish too, we use the same argument as in the proof of Theorem 8.4, and the whole curvature vanishes in x.

If there are two such different symmetries in each point, then the curvature vanishes in all points and the geometry is locally isomorphic with homogeneous model. □

As an easy consequence of the latter Theorem we get:

THEOREM 10.15. *Suppose that there exist two different symmetries in x on a conformal geometry of positive definite signature or negative definite signature. Then the curvature vanishes in x.*

If there are two different symmetries in each point, then the geometry is locally flat.

PROOF. If the geometry is of positive definite signature, then the corresponding Υ have nonzero length (if Υ itself is nonzero). The same property holds for the negative definite signature. The rest follows from the latter facts. □

Further geometries

The list of interesting $|1|$–graded geometries, see page 31, includes two more items – the almost Grasmannian and the almost quaternionic geometries. We shall study these examples in detail elsewhere. Let us conclude with low dimensional cases, where they coincide with the already discussed ones.

THEOREM 10.16. *Suppose that there exist two different symmetries in x on an almost quaternionic geometry of (real) dimension 4. Then W vanishes in x and thus the whole curvature vanishes in x.*

If there are two different symmetries in each point, then the geometry is locally flat.

PROOF. In fact, the geometry in the question is isomorphic to the conformal geometry of dimension 4 of positive definite signature (see p. 15). The rest follows from the Theorem 10.15. □

THEOREM 10.17. *Suppose that there exist two different symmetries in x on an almost Grassmannian structure of a dimension 4 (such that $p = q = 2$). Let Υ be given by the corresponding fixed Weyl structures and suppose that Υ has maximal rank. Then W vanishes in x and thus the whole curvature vanishes in x.*

If there are two such different symmetries in each point, then the geometry is locally flat.

PROOF. In fact, the geometry in the question is isomorphic to the conformal geometry of dimension 4 of indefinite signature $(2, 2)$ (see p. 15). The condition on rank corresponds to the condition on the length of Υ from the conformal case. The rest follows from the Theorem 10.14. □

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