MASARYK UNIVERSITY
Faculty of Science
Department of Mathematics and Statistics

# DISSERTATION THESIS

Brno 2011 JOSEF REBENDA

# MASARYK UNIVERSITY
## Faculty of Science
### Department of Mathematics and Statistics



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# Asymptotic behaviour of solutions of a real two-dimensional differential system with a finite number of nonconstant delays

Ph. D. Thesis


**Supervisor: Doc. RNDr. Josef Kalas, CSc.** **Brno 2011**

## Bibliographic entry

| | |
|---|---|
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| Title | Asymptotic behaviour of solutions of a real two-dimensional differential system with a finite number of nonconstant delays |
| Study programme | Mathematics |
| Study field | Mathematical analysis |
| Supervisor | Doc. RNDr. Josef Kalas, CSc. |
| Year | 2011 |
| Keywords | delayed differential equations, asymptotic behaviour, nonconstant delay, stability, asymptotic stability, instability, boundedness of solutions, Lyapunov method, Ważewski topological principle |

# Acknowledgment

I would like to sincerely thank my supervisor doc. RNDr. Josef Kalas, CSc. for his helpful advice, willingness, lots of worthy suggestions and great patience.

I also thank my wife for immense support and motivation to finish the thesis.

# Abstract

In the thesis, we study asymptotic behaviour of solutions of a real two-dimensional differential system with a finite number of nonconstant delays.

We consider the system

$$x'(t) = \mathbf{A}(t)x(t) + \sum_{k=1}^{m} \mathbf{B}_k(t)x(\theta_k(t)) + \mathbf{h}(t, x(t), x(\theta_1(t)), \ldots, x(\theta_m(t)))$$

with unbounded nonconstant delays $t - \theta_k(t) \geq 0$ satisfying $\lim_{t\to\infty} \theta_k(t) = \infty$ for $k \in \{1, \ldots, m\}$. Here $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{h}$ are supposed to be matrix functions and a vector function, respectively.

The conditions for the stable and unstable properties of solutions together with the conditions for the existence of bounded solutions are given. The methods are based on the transformation of the considered real system to one equation with complex-valued coefficients. Asymptotic properties are studied by means of the Lyapunov-Krasovskii functional and a suitable version of Ważewski topological principle.



MSC 2010: 34K20, 34K12, 34K25

# Contents

# Chapter 1

# Introduction

This thesis deals with the study of asymptotic behaviour of solutions of a real two-dimensional differential system with one or more constant or nonconstant delays. More precisely, we study the system

$$x'(t) = \mathbf{A}(t)x(t) + \sum_{k=1}^{m} \mathbf{B}_k(t)x(\theta_k(t)) + \mathbf{h}(t, x(t), x(\theta_1(t)), \dots, x(\theta_m(t))) \tag{1.1}$$

with generally unbounded delays $t - \theta_k(t) \geq 0$ satisfying $\lim_{t\to\infty} \theta_k(t) = \infty$. Here $\mathbf{A}(t) = \big(a_{ij}(t)\big)$, $\mathbf{B}_k(t) = \big(b_{ijk}(t)\big)$ $(i, j = 1, 2)$ for $k \in \{1, \dots, m\}$ are real square matrices, $\mathbf{h}(t, x, y_1, \dots, y_m) = \big(h_1(t, x, y_1, \dots, y_m), h_2(t, x, y_1, \dots, y_m)\big)$ is a real vector function and $x = (x_1, x_2)$, $y_k = (y_{k1}, y_{k2})$ for $k \in \{1, \dots, m\}$ are real two-dimensional vectors. We suppose that the functions $\theta_k$, $a_{ij}$ are locally absolutely continuous on $[t_0, \infty)$, $b_{ijk}$ are locally Lebesgue integrable on $[t_0, \infty)$ and the function $\mathbf{h}$ satisfies Carathéodory conditions on $[t_0, \infty) \times \mathbb{R}^{2(m+1)}$.

There are a lot of papers dealing with the stability and asymptotic behaviour of $n$-dimensional real vector equations with delay. Since the plane has special topological properties different from those of $n$-dimensional space, where $n \geq 3$ or $n = 1$, it is interesting to study asymptotic behaviour of two-dimensional systems by using tools which are typical and effective for two-dimensional systems. The convenient tool is the combination of the method of complexification and the method of Lyapunov-Krasovskii functional. For the case of instability, it is useful to add to this combination the version of Ważewski topological principle formulated by Rybakowski in the papers [44], [45]. Using these techniques we obtain new and easy applicable results on stability, asymptotic stability, instability or boundedness of solutions of the system (1.1).

Differential systems with delay were studied by many mathematicians since the second half of 20th century. The asymptotic theory and stability of solutions were studied, among others we mention the papers [2], [3], and the monographs [4], [16], [28], [30], [34]. The asymptotic theory is still developed, we should mention the recent results of Bainov, Markova and Simeonov [1], Čermák [6], Čermák and Dvořáková [7], Diblík and Khusainov [8], Diblík and Svoboda [9], [10], [11], Diblík, Svoboda and Šmarda [12], Džurina and Kotorová [13], Graef, Qian and Zhang [14], Györi and Pituk [15], Koplatadze and Kvinikadze [29], Kusano and Marušiak [31], Marušiak and Janík [32], Matucci [33], Philos and Purnaras [35], Pituk [36], Staněk [47], Špániková and Šamajová [48]. In the paper of Campos and Mawhin [5], the method of complexification was used for investigating other properties of solutions.

The main idea of the investigation, the combination of the method of complexification and the method of Lyapunov-Krasovskii functional, was introduced for ordinary differential equations in the paper by Ráb and Kalas [39] in 1990. The principle was transferred to differential equations with delay by Kalas and Baráková [23] in 2002. The results in the case of instability were obtained for ODE's by Kalas and Osička [24] in 1994 and for delayed differential equations by Kalas [19] in 2005.

The thesis is organized as follows. In Chapter 2, we recall methods and results for ordinary differential systems. We start with transformation of the system

$$x' = \mathbf{A}_0(t)x + \mathbf{h}_0(t,x) \tag{1.2}$$

into one equation with complex-valued coefficients. Then we introduce all possible cases which can be studied. The cases originate from the classification of singular point 0 for the autonomous homogeneous system

$$x' = \mathbf{A}x, \tag{1.3}$$

where $\mathbf{A}$ is supposed to be regular. Next, for each case, we formulate assumptions under which the results are valid for the system (1.2) and we conclude Chapter 2 with the results for the system (1.2).

Chapter 3 is devoted to the study of solutions of the system (1.1) in the situation corresponding to the case when the singular point of the system (1.3) is stable. First we introduce the assumptions, then we continue with the main results on the stability and asymptotic stability of the solutions and the proofs (the results are being prepared for publication). We finish the Chapter with several consequences and examples.

Chapter 4 has similar content as Chapter 3 except that we develop the theory for solutions of (1.1) in the situation corresponding to the case when the singular point of the system (1.3) is unstable. Hence, instead of stability and asymptotic stability, the conditions for instability and the existence of bounded solutions are given. The first part of the results was published in [26], the second part is in preparation for publication.

Chapter 5 is Conclusion, where we recall the significance of the results and we suggest some directions for further investigation.

The last chapter of this thesis is Appendix. It contains several well known theorems which were used in the proofs throughout the thesis.

At the end of this introduction we append a brief overview of notation used in the thesis.

| | |
|---|---|
| $\mathbb{R}$ | the set of all real numbers, |
| $\mathbb{R}_+$ | the set of all positive real numbers, |

| | |
|---|---|
| $\mathbb{R}_+^0$ | the set of all non-negative real numbers, |
| $\mathbb{R}_-$ | the set of all negative real numbers, |
| $\mathbb{R}_-^0$ | the set of all non-positive real numbers, |
| $\mathbb{C}$ | the set of all complex numbers, |
| $\mathcal{C}$ | the class of all continuous functions $[-r, 0] \to \mathbb{C}$, |
| $AC_{\mathrm{loc}}(I, M)$ | the class of all locally absolutely continuous functions $I \to M$, |
| $L_{\mathrm{loc}}(I, M)$ | the class of all locally Lebesgue integrable functions $I \to M$, |
| $Lip_{\mathrm{loc}}(I \times \Omega, M)$ | the class of all functions $I \times \Omega \to M$ locally Lipschitzian with respect to the second variable, |
| $K(I \times \Omega, M)$ | the class of all functions $I \times \Omega \to M$ satisfying Carathéodory conditions on $I \times \Omega$, |
| $\operatorname{Re} z$ | the real part of $z$, |
| $\operatorname{Im} z$ | the imaginary part of $z$, |
| $\bar{z}$ | the complex conjugate of $z$, |
| $\overline{\Omega}$ | the closure of the set $\Omega$, |
| $\partial\Omega$ | the boundary of the set $\Omega$, |
| $\mathrm{int}\Omega$ | the interior of the set $\Omega$. |

# Chapter 2

# Results for ordinary differential systems

The results on ordinary differential systems are recalled in this chapter. We consider the real two-dimensional differential system

$$x' = \mathbf{A}_0(t)x + \mathbf{h}_0(t, x), \tag{2.1}$$

where $\mathbf{A}_0(t)$ is a real two-dimensional square matrix and $\mathbf{h}_0(t, x) = (h_{0_1}(t, x), h_{0_2}(t, x))$ is a real vector function.

For investigating this system, we make use of the properties of linear systems. The system (2.1) is regarded as a perturbation of linear system $x'(t) = \mathbf{A}_0(t)x(t)$.

## 2.1 Transformation of the system into one equation

We use the method of complexification to transform the system (2.1) into one equation with complex-valued coefficients. The method is described in papers [37], [38], [18] or [39], we refer to these papers for further details.

The transformation goes as follows. The real plane is converted into the complex plane by assigning the complex number $z = x_1 + ix_2$ to the point $[x_1, x_2]$. We obtain

$$z' = a(t)z + b(t)\bar{z} + g(t, z, \bar{z}), \tag{2.2}$$

where

$$\begin{aligned}
a(t) &= \frac{1}{2}(a_{11}(t) + a_{22}(t)) + \frac{i}{2}(a_{21}(t) - a_{12}(t)), \\
b(t) &= \frac{1}{2}(a_{11}(t) - a_{22}(t)) + \frac{i}{2}(a_{21}(t) + a_{12}(t)), \\
g(t, z, \bar{z}) &= h_1(t, \frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z})) + +ih_2(t, \frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z})).
\end{aligned}$$

The inverse transformation of the equation (2.2) into the real system (2.1) is expressed by the relations

$$\begin{aligned}
a_{11}(t) &= \operatorname{Re}[a(t)+b(t)], \qquad a_{12}(t) = \operatorname{Im}[b(t)-a(t)],\\
a_{21}(t) &= \operatorname{Im}[a(t)+b(t)], \qquad a_{22}(t) = \operatorname{Re}[a(t)-b(t)],\\
h_1(t,x_1,x_2) &= \operatorname{Re} g(t, x_1+ix_2, x_1-ix_2),\\
h_2(t,x_1,x_2) &= \operatorname{Im} g(t, x_1+ix_2, x_1-ix_2).
\end{aligned}$$

## 2.2 Results

The recapitulation of the most important results for ordinary differential systems is made in this section. We introduce three cases which arise from the study of singular points of the autonomous system (1.3).

We start from the study of the equation

$$z' = az + b\bar{z}, \quad a, b \in \mathbb{C} \tag{2.3}$$

which corresponds to the system (1.3). To distinguish the cases, we put $q = |a|^2 - |b|^2$, $-p = 2\operatorname{Re} a$, $\delta = p^2 - 4q = 4(|b|^2 - (\operatorname{Im} a)^2)$.

### 2.2.1 Stable case

Stable case corresponds to the situation when the singular point 0 of the system (1.3) is stable and it is a focus, a centre or a node. This holds true for the equation (2.3) when the conditions $q > 0$, $p \geq 0$ are met. The results were published in [39].

For the purpose of this subsection, denote

$$\begin{aligned}
\gamma(t) &= |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2},\\
c(t) &= \frac{\bar{a}(t)b(t)}{|a(t)|},\\
\alpha(t) &= 1 + \left|\frac{b(t)}{a(t)}\right| \operatorname{sgn} \operatorname{Re} a(t),\\
\vartheta(t) &= \frac{\operatorname{Re}(\gamma(t)\gamma'(t) - \bar{c}(t)c'(t)) + |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2},\\
\Theta(t) &= \alpha(t)\operatorname{Re} a(t) + \vartheta(t) + \varkappa(t).
\end{aligned}$$

We make the following assumptions about the equation (2.2):

$$(\mathrm{A}) \begin{cases} (\mathrm{A}_1) & \text{The functions } a, b\colon [t_0,\infty) \to \mathbb{C} \text{ have continuous first derivatives}\\ (\mathrm{A}_2) & \text{The function } g\colon [t_0,\infty) \times \{z \in \mathbb{C}, |z| < r \leq \infty\}^2 \to \mathbb{C} \text{ is continuous}\\ & \text{and any initial value problem to (2.2) has a unique solution.} \end{cases}$$

(B) $\liminf_{t\to\infty}(|a(t)| - |b(t)|) > 0$.

(C) There exist continuous functions $\varkappa, \varrho\colon [\tau,\infty) \to \mathbb{R}$, $\tau \geq t_0$, such that

$$|\gamma(t)g(t,z,\bar{z}) + c(t)\bar{g}(t,z,\bar{z})| \leq \varkappa(t)|\gamma(t)z + c(t)\bar{z}| + \varrho(t),\ t \geq \tau,\ |z| < r.$$

**Theorem 2.1.** *Let the assumptions* (A), (B), *and* (C) *with* $\varrho(t) \equiv 0$ *be fulfilled.*

*a) If*

$$\limsup_{t\to\infty} \int^{t} \Theta(s)ds < \infty, \tag{2.4}$$

*then the trivial solution of* (2.2) *is stable;*

*b) if*

$$\lim_{t\to\infty} \int^{t} \Theta(s)ds = -\infty, \tag{2.5}$$

*then the trivial solution of* (2.2) *is asymptotically stable.*

**Theorem 2.2.** *Let the assumptions* (A), (B), *and* (C) *be fulfilled. Let* $z = z(t)$ *be any solution of* (2.2) *defined for* $t \to \infty$ *and* $V(t) = |\gamma(t)z(t) + c(t)\bar{z}(t)|$. *Then there exists* $\mu > 0$ *such that*

$$\mu|z(t)| \leq V(s)\exp\left(\int_s^t \Theta(\xi)d\xi\right) + \int_s^t \varrho(\xi)\exp\left(\int_\xi^t \Theta(\sigma)d\sigma\right)d\xi \tag{2.6}$$

*for* $t \geq s > \tau$.

**Theorem 2.3.** *Let* (A), (B), *and* (C) *be fulfilled. Let* $\Theta(t) \leq 0$ *for* $t \geq \tau \geq t_0$,

$$\lim_{t\to\infty} \int^{t} \Theta(s)ds = -\infty, \quad \textit{and} \quad \varrho(t) = o(\Theta(t)). \tag{2.7}$$

*Then any solution* $z(t)$ *of the equation* (2.2) *existing for* $t \to \infty$ *satisfies*

$$\lim_{t\to\infty} z(t) = 0.$$

Some improved results can be obtained in special case when $\delta < 0$. The condition (B) is replaced by $\liminf_{t\to\infty}(|\operatorname{Im} a(t)| - |b(t)|) > 0$ and the functions $\tilde{\gamma}(t) = \operatorname{Im} a(t) + \sqrt{(\operatorname{Im} a(t))^2 - |b(t)|^2}\operatorname{sgn}(\operatorname{Im} a(t))$, $\tilde{c}(t) = -ib(t)$ are involved instead of the functions $\gamma(t)$, $c(t)$. For details see [39].

### 2.2.2 Unstable case

Unstable case corresponds to the situation when the singular point 0 of the system (1.3) is unstable and it is a focus, a centre or a node. This situation occurs for the equation (2.3) when $q > 0$, $p < 0$. The results can be found in [24].

For the purpose od this subsection, denote

$$
\begin{aligned}
\gamma(t) &= |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2},\\
c(t) &= \frac{\bar{a}(t)b(t)}{|a(t)|},\\
\alpha(t) &= 1 - \left|\frac{b(t)}{a(t)}\right| \operatorname{sgn} \operatorname{Re} a(t),\\
\vartheta(t) &= \frac{\operatorname{Re}(\gamma(t)\gamma'(t) - \bar{c}(t)c'(t)) - |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2},\\
\Theta_n(t) &= \alpha(t) \operatorname{Re} a(t) + \vartheta(t) - \varkappa_n(t).
\end{aligned}
$$

The results for the equation (2.2) were obtained subject to the following assumptions:

(A) The functions $a, b\colon [t_0, \infty) \to \mathbb{C}$ have continuous first derivatives, the function $g\colon [t_0, \infty) \times \mathbb{C}^2 \to \mathbb{C}$ is continuous and any initial value problem to (2.2) has a unique solution.

(B) $|a(t)| > |b(t)|$ for $t \geq t_0$.

($\mathrm{C_n}$) There exist a $\varrho_n$ and a continuous function $\varkappa_n\colon [t_0, \infty) \to \mathbb{R}$ such that

$$|\gamma(t)g(t, z, \bar{z}) + c(t)\bar{g}(t, z, \bar{z})| \leq \varkappa_n(t)|\gamma(t)z + c(t)\bar{z}| \text{ for } t \geq t_n\ (\geq t_0),\ |z| > \varrho_n.$$

**Theorem 2.4.** *Let the assumptions* (A), (B) *and* ($\mathrm{C_0}$) *be fulfilled. Suppose that there exists a $t_1 \geq t_0$ such that*

$$\inf_{t \geq t_1} \left[ \int_{t_1}^{t} \Theta_0(s)ds + \ln \frac{\gamma(t_1) - |c(t_1)|}{\gamma(t) + |c(t)|} \right] = \mu > -\infty. \tag{2.8}$$

*Let $z = z(t)$ be any solution of* (2.2) *satisfying*

$$|z(t_1)| > \varrho_0 \cdot \mathrm{e}^{-\mu}. \tag{2.9}$$

*Then*

$$|z(t)| \geq \frac{\gamma(t_1) - |c(t_1)|}{\gamma(t) + |c(t)|} |z(t_1)| \exp\left\{ \int_{t_1}^{t} \Theta_0(s)ds \right\} \tag{2.10}$$

*for all $t \geq t_1$ for which $z(t)$ is defined.*

**Theorem 2.5.** *Let the conditions* (A), (B), ($\mathrm{C}_0$) *be satisfied. Suppose that* $\xi\colon [t_0,\infty)\to\mathbb{R}$ *is a continuous function such that*

$$\xi(t) < \Theta_0(t), \quad t\in[t_0,\infty), \tag{2.11}$$

*and*

$$\liminf_{t\to\infty}\left[\int\limits_{t_0}^{t}\xi(s)ds - \ln(\gamma(t)+|c(t)|)\right] = \mu_1 > -\infty. \tag{2.12}$$

*Then for any* $C > \varrho_0\cdot \mathrm{e}^{-\mu_1}$ *there is a* $t_1 > t_0$ *and a solution* $z_0(t)$ *of* (2.2) *satisfying*

$$|z_0(t)| \le \frac{C}{\gamma(t)-|c(t)|}\exp\left\{\int\limits_{t_0}^{t}\xi(s)ds\right\} \tag{2.13}$$

*for* $t\ge t_1$.

**Theorem 2.6.** *Let the conditions* (A), (B) *be satisfied and let* ($\mathrm{C_n}$) *hold for* $n\in\mathbb{N}$, *where* $\inf\limits_{n\in\mathbb{N}}\varrho_n =: \varrho_0$. *Assume that* $\xi_n\colon[t_0,\infty)\to\mathbb{R}$, *(*$n=1,2,\ldots$*), are continuous functions such that*

$$\xi_n(t) < \Theta_n(t) \tag{2.14}$$

*for* $t\ge t_n$,

$$\liminf_{t\to\infty}\left[\int\limits_{t_0}^{t}\xi_n(s)ds - \ln(\gamma(t)+|c(t)|)\right] > -\infty, \tag{2.15}$$

$$\limsup_{t\to\infty}\left[\int\limits_{t_0}^{t}(\Theta_n(s)-\xi_n(s))ds + \ln\frac{\gamma(t)-|c(t)|}{\gamma(t)+|c(t)|}\right] = \infty \tag{2.16}$$

*and*

$$\inf_{t_n\le s\le t<\infty}\left[\int\limits_{s}^{t}(\Theta_n(\sigma)d\sigma + \ln\frac{\gamma(s)-|c(s)|}{\gamma(t)+|c(t)|}\right] \ge \mu > -\infty \tag{2.17}$$

*for* $n\in\mathbb{N}$.

*Then there exists a solution* $z_0(t)$ *of* (2.2) *such that*

$$\limsup_{t\to\infty}|z_0(t)| \le \varrho_0\cdot\mathrm{e}^{-\mu}. \tag{2.18}$$

### 2.2.3 Semistable case

This case corresponds to the situation when the singular point of the system (1.3) is a saddle point. This happens to the equation (2.3) if $q < 0$. The results were introduced in [25].

For the purpose od this subsection, denote

$$
\begin{aligned}
c(t) &= i \operatorname{Im} a(t), & d(t) &= \sqrt{|b(t)|^2 - (\operatorname{Im} a(t))^2} + b(t), \\
f(t,z,\bar z) &= c(t) g(t,z,\bar z) + d(t)\bar g(t,z,\bar z), & W(t,z,\bar z) &= c(t) z + d(t)\bar z, \\
\alpha(t) &= \operatorname{Re} a, & \beta(t) &= \sqrt{|b(t)|^2 - (\operatorname{Im} a(t))^2}, \\
p(t) &= \frac{\bar c(t) c'(t) - \bar d(t) d'(t)}{|c(t)|^2 - |d(t)|^2}, & q(t) &= \frac{c(t) d'(t) - c'(t) d(t)}{|c(t)|^2 - |d(t)|^2}.
\end{aligned}
$$

The assumptions under which the equation (2.2) was studied are:

(A) The functions $a, b\colon [t_0,\infty) \to \mathbb{C}$ have continuous first derivatives, the function $g\colon [t_0,\infty) \times \mathbb{C}^2 \to \mathbb{C}$ is continuous and any initial value problem to (2.2) has a unique solution.

(B) $|b(t)| > |\operatorname{Im} a(t)|$, $\sqrt{|b(t)|^2 - (\operatorname{Im} a(t))^2} + \operatorname{Re} b(t) \neq 0$ for $t \in [t_0,\infty)$.

(C) There exist continuous functions $\kappa_1, \kappa_2, \lambda_1, \lambda_2\colon [t_0,\infty) \to \mathbb{R}$ such that

$$\operatorname{Re} f(t,z,\bar z) \operatorname{sgn} \operatorname{Re} W(t,z,\bar z) \geq \kappa_1(t) |\operatorname{Re} W(t,z,\bar z)| + \lambda_1(t) |\operatorname{Im} W(t,z,\bar z)|, \tag{2.19}$$
$$\operatorname{Im} f(t,z,\bar z) \operatorname{sgn} \operatorname{Im} W(t,z,\bar z) \leq \lambda_2(t) |\operatorname{Re} W(t,z,\bar z)| + \kappa_2(t) |\operatorname{Im} W(t,z,\bar z)| \tag{2.20}$$

for $t \in [t_0,\infty)$, $z \in \mathbb{C}$.

(D) There exist continuous functions $\kappa_3, \kappa_4, \lambda_3, \lambda_4\colon [t_0,\infty) \to \mathbb{R}$ such that

$$\operatorname{Re}[(1+i) f(t,z,\bar z)] \geq \kappa_3(t) \operatorname{Re}[(1+i) W(t,z,\bar z)] + \lambda_3(t) \operatorname{Im}[(1+i) W(t,z,\bar z)], \tag{2.21}$$
$$\operatorname{Im}[(1+i) f(t,z,\bar z) \geq \lambda_4(t) \operatorname{Re}[(1+i) W(t,z,\bar z)] + \kappa_4(t) \operatorname{Im}[(1+i) W(t,z,\bar z)] \tag{2.22}$$

for $t \in [t_0,\infty)$, $z \in \mathbb{C}$.

In the following let $p_1 = \operatorname{Re} p$, $p_2 = \operatorname{Im} p$, $q_1 = \operatorname{Re} q$, $q_2 = \operatorname{Im} q$, so that $p_1, p_2, q_1, q_2\colon [t_0,\infty) \to \mathbb{R}$ are continuous functions for all $t$ for which $|c(t)| \neq |d(t)|$.

**Theorem 2.7.** *Let the hypotheses* (A), (B), (C) *and*

$$\lambda_1 \leq |p_2 - q_2|, \quad -\lambda_2 \leq |p_2 + q_2|, \tag{2.23}$$
$$-\kappa_1 + \kappa_2 - \lambda_1 + \lambda_2 - 2q_1 + 2\max(|p_2|, |q_2|) < 2\beta \tag{2.24}$$

*be satisfied. Suppose that $\theta\colon [t_0,\infty) \to \mathbb{R}$ is a continuous function such that*

$$\theta > \alpha - \beta + \kappa_2 + \lambda_2 + p_1 - q_1 + |p_2 + q_2| \quad \text{on } [t_0,\infty). \tag{2.25}$$

*Then for any* $\nu \in \mathbb{R}$, $\nu > 0$ *there exists at least one-parameter family of solutions* $z(t)$ *of* (2.2) *satisfying*

$$|z(t)| < \frac{\nu}{|c(t)| - |d(t)|} \exp\left\{ \int_{t_0}^{t} \theta(s) ds \right\} \tag{2.26}$$

*for* $t \geq t_0$.

**Theorem 2.8.** *Let the assumptions* (A), (B) *and* (D) *be fulfilled. Let*

$$p_2 - q_1 - \lambda_3 < \beta, \quad -p_2 - q_1 - \lambda_4 < \beta \quad \text{for } t \geq t_0. \tag{2.27}$$

*If* $\nu \in \mathbb{R}$, $\nu > 0$ *and* $\psi\colon [t_0, \infty) \to \mathbb{R}$ *is any continuous function satisfying*

$$\psi(t) < \alpha + \beta + p_1 + q_1 + \min(\kappa_3 + \lambda_3 - p_2 - q_2, \kappa_4 + \lambda_4 + p_2 + q_2) \tag{2.28}$$

*then there exists two-parameter family of solutions* $z(t)$ *of* (2.2) *satisfying*

$$|z(t)| > \frac{\nu}{|c(t)| + |d(t)|} \exp\left\{ \int_{t_0}^{t} \psi(s) ds \right\} \tag{2.29}$$

*for* $t \in [t_0, \omega)$, *where* $[t_0, \omega)$ *is the right maximal interval of existence of* $z$.

# Chapter 3

# Differential systems with delay: The stable case

In this chapter we consider conditions analogous to the case when the singular point 0 of the autonomous system (1.3) is stable.

## 3.1 Transformation to one equation with complex-valued coefficients

This section contains the first of the main ideas - the method of complexification. We use a transformation that reduces real two-dimensional system to one equation with complex-valued coefficients. The transformation for the considered system with a finite number of nonconstant delays can be found in [26].

We study the system

$$x'(t) = \mathbf{A}(t)x(t) + \sum_{k=1}^{m} \mathbf{B}_k(t)x(\theta_k(t)) + \mathbf{h}(t, x(t), x(\theta_1(t)), \dots, x(\theta_m(t))) \tag{3.1}$$

with generally unbounded delays $t - \theta_k(t) \geq 0$ satisfying $\lim\limits_{t\to\infty} \theta_k(t) = \infty$. Here $\mathbf{A}(t) = \big(a_{ij}(t)\big)$, $\mathbf{B}_k(t) = \big(b_{ijk}(t)\big)$ $(i, j = 1, 2)$ for $k \in \{1, \dots, m\}$ are real square matrices, $\mathbf{h}(t, x, y_1, \dots, y_m) = \big(h_1(t, x, y_1, \dots, y_m), h_2(t, x, y_1, \dots, y_m)\big)$ is a real vector function and $x = (x_1, x_2)$, $y_k = (y_{k1}, y_{k2})$ for $k \in \{1, \dots, m\}$ are real two-dimensional vectors. We suppose that the functions $\theta_k$, $a_{ij}$ are locally absolutely continuous on $[t_0, \infty)$, $b_{ijk}$ are locally Lebesgue integrable on $[t_0, \infty)$ and the function $\mathbf{h}$ satisfies Carathéodory conditions on $[t_0, \infty) \times \mathbb{R}^{2(m+1)}$.

Introducing complex variables $z = x_1 + ix_2$, $w_1 = y_{11} + iy_{12}$, …, $w_m = y_{m1} + iy_{m2}$, we can rewrite the system (3.1) into an equivalent equation with complex-valued coefficients

$$
\begin{aligned}
z'(t) = & a(t)z(t) + b(t)\bar{z}(t) + \sum_{k=1}^{m} \big[A_k(t)z(\theta_k(t)) + B_k(t)\bar{z}(\theta_k(t))\big] \\
& + g(t, z(t), z(\theta_1(t)), \dots, z(\theta_m(t))),
\end{aligned} \tag{3.2}
$$

where $\theta_k \in AC_{\mathrm{loc}}(J,\mathbb{R})$ for $k = 1,\dots,m$, $A_k, B_k \in L_{\mathrm{loc}}(J,\mathbb{C})$, $a, b \in AC_{\mathrm{loc}}(J,\mathbb{C})$, $g \in K(J \times \mathbb{C}^{m+1},\mathbb{C})$, $J = [t_0,\infty)$.

The relations between the functions are following:

$$
\begin{aligned}
a(t) &= \frac{1}{2}(a_{11}(t) + a_{22}(t)) + \frac{i}{2}(a_{21}(t) - a_{12}(t)), \\
b(t) &= \frac{1}{2}(a_{11}(t) - a_{22}(t)) + \frac{i}{2}(a_{21}(t) + a_{12}(t)), \\
A_k(t) &= \frac{1}{2}(b_{11k}(t) + b_{22k}(t)) + \frac{i}{2}(b_{21k}(t) - b_{12k}(t)), \\
B_k(t) &= \frac{1}{2}(b_{11k}(t) - b_{22k}(t)) + \frac{i}{2}(b_{21k}(t) + b_{12k}(t)), \\
g(t, z, w_1, \dots, w_m) &= h_1(t, \frac{1}{2}(z+\bar{z}), \frac{1}{2i}(z-\bar{z}), \frac{1}{2}(w_1+\bar{w}_1), \frac{1}{2i}(w_1-\bar{w}_1), \dots, \frac{1}{2i}(w_m-\bar{w}_m)) + \\
&+ ih_2(t, \frac{1}{2}(z+\bar{z}), \frac{1}{2i}(z-\bar{z}), \frac{1}{2}(w_1+\bar{w}_1), \frac{1}{2i}(w_1-\bar{w}_1), \dots, \frac{1}{2}(w_m+\bar{w}_m), \frac{1}{2i}(w_m-\bar{w}_m)).
\end{aligned}
$$

Conversely, putting

$$
\begin{aligned}
a_{11}(t) &= \mathrm{Re}[a(t) + b(t)], & a_{12}(t) &= \mathrm{Im}[b(t) - a(t)], \\
a_{21}(t) &= \mathrm{Im}[a(t) + b(t)], & a_{22}(t) &= \mathrm{Re}[a(t) - b(t)], \\
b_{11k}(t) &= \mathrm{Re}[A_k(t) + B_k(t)], & b_{12k}(t) &= \mathrm{Im}[B_k(t) - A_k(t)], \\
b_{21k}(t) &= \mathrm{Im}[A_k(t) + B_k(t)], & b_{22k}(t) &= \mathrm{Re}[A_k(t) - B_k(t)],
\end{aligned}
$$

$$
\begin{aligned}
h_1(t, x, y_1, \dots, y_m) &= \mathrm{Re}\, g(t, x_1 + ix_2, y_{11} + iy_{12}, \dots, y_{m1} + iy_{m2}), \\
h_2(t, x, y_1, \dots, y_m) &= \mathrm{Im}\, g(t, x_1 + ix_2, y_{11} + iy_{12}, \dots, y_{m1} + iy_{m2}),
\end{aligned}
$$

the equation (3.2) can be written in the real form (3.1) as well.

## 3.2 The case $\liminf\limits_{t\to\infty}(|a(t)| - |b(t)|) > 0$

We consider the equation (3.2) in the case when

$$
\liminf_{t\to\infty}\big(|a(t)| - |b(t)|\big) > 0 \tag{3.3}
$$

and study the behavior of solutions of (3.2) under this assumption. This situation corresponds to the case when the the autonomous system (1.3) has the singular point 0 of type centre, focus or node.

### 3.2.1 Assumptions

Regarding (3.3) and since the delay functions $\theta_k$ satisfy $\lim_{t\to\infty} \theta_k(t) = \infty$, there are numbers $T_1 \ge t_0$, $T \ge T_1$ and $\mu > 0$ such that

$$|a(t)| > |b(t)| + \mu \text{ for } t \ge T_1, \quad t \ge \theta_k(t) \ge T_1 \text{ for } t \ge T \ (k = 1, \dots, m). \tag{3.4}$$

Put

$$\gamma(t) = |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2}, \quad c(t) = \frac{\bar{a}(t)b(t)}{|a(t)|}. \tag{3.5}$$

Since $\gamma(t) > |a(t)|$ and $|c(t)| = |b(t)|$, the inequality

$$\gamma(t) > |c(t)| + \mu \tag{3.6}$$

holds for all $t \ge T_1$. It is easy to verify that $\gamma, c \in AC_{\text{loc}}([T_1, \infty), \mathbb{C})$.

For the rest of this section denote

$$\alpha(t) = 1 + \left|\frac{b(t)}{a(t)}\right| \operatorname{sgn} \operatorname{Re} a(t), \tag{3.7}$$

$$\vartheta(t) = \frac{\operatorname{Re}(\gamma(t)\gamma'(t) - \bar{c}(t)c'(t)) + |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2}. \tag{3.8}$$

The stability mentioned in the title of the chapter is studied under the following assumptions:

(i) The numbers $T_1 \ge t_0$, $T \ge T_1$ and $\mu > 0$ are such that (3.4) holds.

(ii) There exist functions $\varkappa, \kappa_k, \varrho\colon [T, \infty) \to \mathbb{R}$ such that

$$|\gamma(t)g(t, z, w_1, \dots, w_m) + c(t)\bar{g}(t, z, w_1, \dots, w_m)| \le \varkappa(t)|\gamma(t)z + c(t)\bar{z}| + \sum_{k=1}^{m} \kappa_k(t)|\gamma(\theta_k(t))w_k + c(\theta_k(t))\bar{w}_k| + \varrho(t)$$

for $t \ge T$, $z, w_k \in \mathbb{C}$ $(k = 1, \dots, m)$, where $\varkappa$, $\varrho \in L_{\text{loc}}([T, \infty), \mathbb{R})$.

(iii) $\beta \in AC_{\text{loc}}([T, \infty), \mathbb{R}_+^0)$ is a function satisfying

$$\theta_k'(t)\beta(t) \ge \lambda_k(t) \quad \text{a. e. on } [T, \infty), \tag{3.9}$$

where $\lambda_k$ is defined for $t \ge T$ by

$$\lambda_k(t) = \kappa_k(t) + (|A_k(t)| + |B_k(t)|)\frac{\gamma(t) + |c(t)|}{\gamma(\theta_k(t)) - |c(\theta_k(t))|}. \tag{3.10}$$

(iv) There exists a function $\Lambda \in L_{\mathrm{loc}}([T,\infty),\mathbb{R})$ which satisfies the inequalities $\beta'(t) \le \Lambda(t)\beta(t)$, $\Theta(t) \le \Lambda(t)$ for almost all $t \in [T,\infty)$, where the function $\Theta$ is defined by

$$\Theta(t) = \alpha(t) \operatorname{Re} a(t) + \vartheta(t) + \varkappa(t) + m\beta(t). \tag{3.11}$$

If $A_k$, $B_k$, $\kappa_k$, $\theta_k'$ are locally absolutely continuous on $[T,\infty)$ and $\lambda_k(t) \ge 0$, $\theta_k'(t) > 0$ on $[T,\infty)$, the choice $\beta(t) = \max\limits_{k=1,\dots,m} [\lambda_k(t)(\theta_k'(t))^{-1}]$ is admissible in (iii).

Under the assumption (i), we can estimate

$$\begin{aligned} |\vartheta| &\le \frac{|\operatorname{Re}(\gamma\gamma' - \bar{c}c')| + |\gamma c' - \gamma' c|}{\gamma^2 - |c|^2} \le \frac{(|\gamma'| + |c'|)(\gamma + |c|)}{\gamma^2 - |c|^2} = \\ &= \frac{|\gamma'| + |c'|}{\gamma - |c|} \le \frac{1}{\mu}(|\gamma'| + |c'|), \end{aligned}$$

hence the function $\vartheta$ is locally Lebesgue integrable on $[T,\infty)$. Moreover, if $\beta \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_+)$ and $\varkappa \in L_{\mathrm{loc}}([T,\infty)$, then we can choose

$$\Lambda(t) = \max\Big(\Theta(t), \frac{\beta'(t)}{\beta(t)}\Big) \tag{3.12}$$

in (iv).

Finally, if $\varrho(t) \equiv 0$ in (ii), then the equation (3.2) has the trivial solution $z(t) \equiv 0$. Notice that in this case the condition (ii) implies that the functions $\varkappa(t)$, $\kappa_k(t)$ are nonnegative on $[T,\infty)$ for $k = 1,\dots,m$, and due to this, $\lambda_k(t) \ge 0$ on $[T,\infty)$. The case $\varrho(t) < 0$ is omitted since it can be replaced by $\varrho(t) \equiv 0$.

### 3.2.2 Main results

The aim is to generalize the results for ordinary differential equations recalled in Chapter 2 as well as the results contained in [23] (one constant delay), [40] (a finite number of constant delays) and [22] (one nonconstant delay). In the proof of the crucial theorem we use the following auxiliary result.

**Lemma 3.1.** *Let $a_1$, $a_2$, $b_1$, $b_2 \in \mathbb{C}$ and $|a_2| > |b_2|$. Then*

$$\operatorname{Re} \frac{a_1 z + b_1 \bar{z}}{a_2 z + b_2 \bar{z}} \le \frac{\operatorname{Re}(a_1 \bar{a}_2 - b_1 \bar{b}_2) + |a_1 b_2 - a_2 b_1|}{|a_2|^2 - |b_2|^2}$$

*for $z \in \mathbb{C}$, $z \ne 0$.*

The proof of Lemma 3.1 can be found for example in [39], [40].

The following theorem is the main result of this section.

**Theorem 3.1.** *Let the conditions* (i), (ii), (iii) *and* (iv) *hold and $\varrho(t) \equiv 0$.*

*a) If*

$$\limsup_{t\to\infty} \int^{t} \Lambda(s)ds < \infty, \tag{3.13}$$

*then the trivial solution of* (3.2) *is stable on* $[T,\infty)$*;*

*b) if*

$$\lim_{t\to\infty} \int^{t} \Lambda(s)ds = -\infty, \tag{3.14}$$

*then the trivial solution of* (3.2) *is asymptotically stable on* $[T,\infty)$*.*

*Proof.* In fact, we obtain the proof as the combination of the proof of Theorem 1 in [40] and the proof of Theorem 2.2 in [22].

Choose arbitrary $t_1 \geq T$. Let $z(t)$ be any solution of (3.2) satisfying the condition $z(t) = z_0(t)$ for $t \in [T_1, t_1]$, where $z_0(t)$ is a continuous complex-valued initial function defined on $t \in [T_1, t_1]$. Consider the Lyapunov functional

$$V(t) = U(t) + \beta(t)\sum_{k=1}^{m} \int_{\theta_k(t)}^{t} U(s)ds, \tag{3.15}$$

where

$$U(t) = |\gamma(t)z(t) + c(t)\bar{z}(t)|.$$

To simplify the computations, denote $w_k(t) = z(\theta_k(t))$ and write the functions of variable $t$ without brackets, for example, $z$ instead of $z(t)$.

From (3.15) we get

$$V' = U' + \beta'\sum_{k=1}^{m} \int_{\theta_k(t)}^{t} U(s)ds + m\beta|\gamma z + c\bar{z}| - \sum_{k=1}^{m} \theta_k'\beta|\gamma(\theta_k)w_k + c(\theta_k)\bar{w}_k| \tag{3.16}$$

for almost all $t \geq t_1$ for which $z(t)$ is defined and $U'(t)$ exists.

Denote $\mathcal{K} = \{t \geq t_1 \colon z(t) \text{ exists}, U(t) \neq 0\}$ and $\mathcal{M} = \{t \geq t_1 \colon z(t) \text{ exists}, U(t) = 0\}$. It is clear that the derivative $U'(t)$ exists for almost all $t \in \mathcal{K}$, hence we focus on the set $\mathcal{M}$.

In view of (3.6) we have $z(t) = 0$ for $t \in \mathcal{M}$. For almost all $t \in \mathcal{M}$ we compute

$$\begin{aligned} U'_{\pm}(t) &= \lim_{\tau\to t\pm} \frac{U(\tau) - U(t)}{\tau - t} = \lim_{\tau\to t\pm} \frac{U(\tau)}{\tau - t} = \lim_{\tau\to t\pm} \frac{|\gamma(\tau)[z(\tau) - z(t)] - c(\tau)[\bar{z}(\tau) - \bar{z}(t)]|}{\tau - t} \\ &= \pm|\gamma(t)z'(t) + c(t)\bar{z}'(t)| = \pm|\gamma(t)g^*(t) + c(t)\bar{g}^*(t)|, \end{aligned}$$

where

$$g^*(t)=\sum_{k=1}^{m}\big(A_k(t)w_k(t)+B_k(t)\bar{w}_k(t)\big)+g(t,0,w_1(t),\dots,w_m(t)).$$

Hence $U$ has one-sided derivatives almost everywhere in $\mathcal{M}$. According to [46], Chapter IX., Theorem (1.1), or [17], the set of all $t$ such that $U'_+(t)\neq U'_-(t)$ can be at most countable, thus the derivative $U'$ exists for almost all $t\in\mathcal{M}$, and for these $t$, $U'(t)=0$.

In particular, the derivative $U'$ exists for almost all $t\geq t_1$ for which $z(t)$ is defined, thus (3.16) holds for almost all $t\geq t_1$ for which $z(t)$ is defined.

Now return the attention to the set $\mathcal{K}$. Since

$$az+b\bar{z}=\frac{a}{2|a|}(\gamma z+c\bar{z})+\frac{b}{2\gamma}(\gamma\bar{z}+\bar{c}z),$$

the equation (3.2) can be written in the form

$$z'=\frac{a}{2|a|}(\gamma z+c\bar{z})+\frac{b}{2\gamma}(\gamma\bar{z}+\bar{c}z)+\sum_{k=1}^{m}\big(A_kw_k+B_k\bar{w}_k\big)+g(t,z,w_1,\dots,w_m).\qquad(3.17)$$

Short computation leads to

$$\operatorname{Re}\left[\frac{\gamma a}{2|a|}+\frac{c\bar{b}}{2\gamma}\right]=\operatorname{Re}a,\qquad \frac{b}{2}+\frac{c\bar{a}}{2|a|}=b\frac{\operatorname{Re}a}{a}.$$

In view of this and (3.17) we have

$$\begin{aligned}
&UU'=U\big(\sqrt{(\gamma z+c\bar{z})(\bar{\gamma}\bar{z}+\bar{c}z)}\big)'=\operatorname{Re}\big[(\gamma\bar{z}+\bar{c}z)(\gamma'z+\gamma z'+c'\bar{z}+c\bar{z}')\big]=\\
&=\operatorname{Re}\Big\{(\gamma\bar{z}+\bar{c}z)\Big[\gamma'z+c'\bar{z}+\gamma\Big(\frac{a}{2|a|}(\gamma z+c\bar{z})+\frac{b}{2\gamma}(\gamma\bar{z}+\bar{c}z)+\sum_{k=1}^{m}\big(A_kw_k+B_k\bar{w}_k\big)+g\Big)+\\
&+c\Big(\frac{\bar{a}}{2|a|}(\gamma\bar{z}+\bar{c}z)+\frac{\bar{b}}{2\gamma}(\gamma z+c\bar{z})+\sum_{k=1}^{m}\big(\bar{A}_k\bar{w}_k+\bar{B}_kw_k\big)+\bar{g}\Big)\Big]\Big\}\leq\\
&\leq|\gamma z+c\bar{z}|^2\Big(\operatorname{Re}a+|b|\frac{|\operatorname{Re}a|}{|a|}\Big)+\operatorname{Re}\Big\{(\gamma\bar{z}+\bar{c}z)\Big[\gamma'z+c'\bar{z}+\gamma\Big(\sum_{k=1}^{m}\big(A_kw_k+B_k\bar{w}_k\big)+g\Big)+\\
&+c\Big(\sum_{k=1}^{m}\big(\bar{A}_k\bar{w}_k+\bar{B}_kw_k\big)+\bar{g}\Big)\Big]\Big\}
\end{aligned}$$

for almost all $t\in\mathcal{K}$.

If we recall the definition of $\alpha(t)$ in (3.7), then

$$
\begin{aligned}
UU' \le U^2\alpha\,\mathrm{Re}\,a + \mathrm{Re}\Big\{(\gamma\bar z+\bar c z)\big[\gamma\sum_{k=1}^{m}\big(A_k w_k + B_k\bar w_k\big) + c\sum_{k=1}^{m}\big(\bar A_k\bar w_k+\bar B_k w_k\big)\big]\Big\}+\\
+\mathrm{Re}[(\gamma\bar z+\bar c z)(\gamma g+c\bar g)] + \mathrm{Re}[(\gamma\bar z+\bar c z)(\gamma' z+c'\bar z)] \le\\
\le U^2\alpha\,\mathrm{Re}\,a + U(\gamma+|c|)\Big(\sum_{k=1}^{m}|A_k w_k+B_k\bar w_k|\Big) + U|\gamma g+c\bar g| + U^2\,\mathrm{Re}\,\frac{\gamma' z+c'\bar z}{\gamma z+c\bar z}.
\end{aligned}
$$

Applying Lemma 3.1 to the last term, we obtain

$$
\mathrm{Re}\,\frac{\gamma' z+c'\bar z}{\gamma z+c\bar z} \le \vartheta.
$$

Using this inequality together with (3.10) and the assumption (ii) we get

$$
\begin{aligned}
UU' \le U^2(\alpha\,\mathrm{Re}\,a+\vartheta+\varkappa) + U\sum_{k=1}^{m}\big(\kappa_k|\gamma(\theta_k)w_k+c(\theta_k)\bar w_k|\big)+\\
+U(\gamma+|c|)\Big(\sum_{k=1}^{m}\frac{|A_k||w_k|+|B_k||\bar w_k|}{\gamma(\theta_k)-|c(\theta_k)|}(\gamma(\theta_k)-|c(\theta_k)|)\Big) \le\\
\le U^2(\alpha\,\mathrm{Re}\,a+\vartheta+\varkappa)+\\
+U\Big\{\sum_{k=1}^{m}\Big[\kappa_k+(|A_k|+|B_k|)\frac{\gamma+|c|}{\gamma(\theta_k)-|c(\theta_k)|}\Big]|\gamma(\theta_k)w_k+c(\theta_k)\bar w_k|\Big\} \le\\
\le U^2(\alpha\,\mathrm{Re}\,a+\vartheta+\varkappa) + U\sum_{k=1}^{m}\lambda_k|\gamma(\theta_k)w_k+c(\theta_k)\bar w_k|
\end{aligned}
$$

for almost all $t\in\mathcal{K}$.

Consequently,

$$
U' \le U(\alpha\,\mathrm{Re}\,a+\vartheta+\varkappa) + \sum_{k=1}^{m}\lambda_k|\gamma(\theta_k)w_k+c(\theta_k)\bar w_k| \tag{3.18}
$$

for almost all $t\in\mathcal{K}$.

Recalling that $U'(t)=0$ for almost all $t\in\mathcal{M}$, we can see that the inequality (3.18) is valid for almost all $t\ge t_1$ for which $z(t)$ is defined.

From (3.16) and (3.18) we have

$$
\begin{aligned}
V' \le U(\alpha\,\mathrm{Re}\,a+\vartheta+\varkappa+m\beta) + \sum_{k=1}^{m}(\lambda_k-\theta_k'\beta)|\gamma(\theta_k)w_k+c(\theta_k)\bar w_k|+\\
+\beta'\sum_{k=1}^{m}\int_{\theta_k(t)}^{t}|\gamma(s)z(s)+c(s)\bar z(s)|ds.
\end{aligned}
$$

As $\beta(t)$ fulfills the condition (3.9), we obtain

$$V'(t) \le U(t)\Theta(t) + \beta'(t) \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} |\gamma(s)z(s) + c(s)\bar{z}(s)| ds,$$

hence

$$V'(t) - \Lambda(t)V(t) \le 0 \tag{3.19}$$

for almost all $t \ge t_1$ for which the solution $z(t)$ exists.

Notice that, with respect to (3.6),

$$V(t) \ge (\gamma(t) - |c(t)|)|z(t)| \ge \mu|z(t)| \tag{3.20}$$

for all $t \ge t_1$ for which $z(t)$ is defined.

Suppose that the condition (3.13) holds, and choose arbitrary $\varepsilon > 0$. Put

$$\Delta = \max_{s\in[T_1,t_1]} (\gamma(s) + |c(s)|), \qquad L = \sup_{T\le t<\infty} \int_{T}^{t} \Lambda(s)ds$$

and

$$\delta = \mu\varepsilon\Delta^{-1}\big(1 + m\beta(t_1)(t_1 - T_1)\big)^{-1} \exp\Big\{\int_{T}^{t_1} \Lambda(s)ds - L\Big\},$$

where $\mu > 0$, $T_1 \ge t_0$ and $T \ge T_1$ are the numbers from the condition (i).

If the initial function $z_0(t)$ of the solution $z(t)$ satisfies $\max\limits_{s\in[T_1,t_1]} |z_0(s)| < \delta$, then the multiplication of (3.19) by $\exp\{-\int_{t_1}^{t} \Lambda(s)ds\}$ and the integration over $[t_1, t]$ yield

$$V(t)\exp\Big\{-\int_{t_1}^{t} \Lambda(s)ds\Big\} - V(t_1) \le 0 \tag{3.21}$$

for all $t \ge t_1$ for which $z(t)$ is defined. From (3.20) and (3.21) we gain

$$\mu|z(t)| \le V(t) \le V(t_1)\exp\Big\{\int\limits_{t_1}^{t}\Lambda(s)ds\Big\} \le \Big[(\gamma(t_1)+|c(t_1)|)|z(t_1)|+$$

$$+\beta(t_1)\max_{s\in[T_1,t_1]}|z(s)|\Big(\sum_{k=1}^{m}\int\limits_{\theta_k(t_1)}^{t_1}(\gamma(s)+|c(s)|)ds\Big)\Big]\exp\Big\{\int\limits_{t_1}^{t}\Lambda(s)ds\Big\} \le$$

$$\le \Big[\Delta\max_{s\in[T_1,t_1]}|z_0(s)|+\beta(t_1)\max_{s\in[T_1,t_1]}|z_0(s)|\Delta\sum_{k=1}^{m}(t_1-\theta_k(t_1))\Big]\exp\Big\{\int\limits_{t_1}^{t}\Lambda(s)ds\Big\},$$

i. e.

$$\mu|z(t)| \le \Delta\max_{s\in[T_1,t_1]}|z_0(s)|\big(1+m\beta(t_1)(t_1-T_1)\big)\exp\Big\{L-\int\limits_{T}^{t_1}\Lambda(s)ds\Big\} < \mu\varepsilon.$$

Thus we have $|z(t)| < \varepsilon$ for all $t \ge t_1$ and we conclude that the trivial solution of the equation (3.2) is stable.

Now suppose that the condition (3.14) is valid. Then, in view of the first part of Theorem 3.1, for $K > 0$ there is a $\rho > 0$ such that $\max\limits_{s\in[T_1,t_1]}|z_0(s)| < \rho$ implies that the solution $z(t)$ of (3.2) exists for all $t \ge t_1$ and satisfies $|z(t)| < K$, where $K$ is arbitrary real constant. Hence, from this and (3.20), we have

$$|z(t)| \le \mu^{-1}V(t) \le \mu^{-1}V(t_1)\exp\Big\{\int\limits_{t_1}^{t}\Lambda(s)ds\Big\}$$

for all $t \ge t_1$. This inequality with the condition (3.14) give

$$\lim_{t\to\infty} z(t) = 0,$$

which completes the proof. □

*Remark* 3.1. Theorem 3.1 represents a generalization of previous results.

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2,\dots,m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2,\dots,m$, $\theta_1(t) = t - r$, where $r > 0$, we get Theorem 1 from [23].

If we take $\theta_k(t) = t - r_k$, where $r_k > 0$, $k = 1,\dots,m$, we obtain Theorem 1 from [40].

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2,\dots,m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2,\dots,m$, $\theta_1(t) = \theta(t)$, we get Theorem 2.2 from [22].

The next theorem involves the function $\varrho$ in (ii), thus it is more general than Theorem 3.1. A part of the proof of Theorem 3.1 is utilized in the proof of Theorem 3.2.

**Theorem 3.2.** *Let the assumptions* (i), (ii), (iii) *and* (iv) *hold and*

$$V(t) = |\gamma(t)z(t) + c(t)\bar{z}(t)| + \beta(t) \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} |\gamma(s)z(s) + c(s)\bar{z}(s)|ds, \tag{3.22}$$

*where* $z(t)$ *is any solution of* (3.2) *defined on* $[t_1, \infty)$. *Then*

$$\mu|z(t)| \leq V(s) \exp\left(\int_s^t \Lambda(\xi)d\xi\right) + \int_s^t \varrho(\xi) \exp\left(\int_\xi^t \Lambda(\sigma)d\sigma\right)d\xi \tag{3.23}$$

*for* $t \geq s \geq t_1$ *where* $t_1 \geq T$.

*Proof.* Following the proof of Theorem 3.1, we have

$$V'(t) \leq |\gamma(t)z(t)+c(t)\bar{z}(t)|\Theta(t)+\beta'(t) \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} |\gamma(s)z(s)+c(s)\bar{z}(s)|ds+\varrho(t) \leq \Lambda(t)V(t)+\varrho(t)$$

a. e. on $[t_1, \infty)$. Using this inequality, we get

$$V'(t) - \Lambda(t)V(t) \leq \varrho(t) \tag{3.24}$$

a. e. on $[t_1, \infty)$. Multiplying (3.24) by $\exp\bigl(-\int_s^t \Lambda(\xi)d\xi\bigr)$ gives

$$\left[V(t) \exp\left(-\int_s^t \Lambda(\xi)d\xi\right)\right]' \leq \varrho(t) \exp\left(-\int_s^t \Lambda(\xi)d\xi\right)$$

a. e. on $[t_1, \infty)$. Integration over $[s, t]$ yields

$$V(t) \exp\left(-\int_s^t \Lambda(\xi)d\xi\right) - V(s) \leq \int_s^t \varrho(\xi) \exp\left(-\int_s^\xi \Lambda(\sigma)d\sigma\right)d\xi, \tag{3.25}$$

and multiplying (3.25) by $\exp\bigl(\int_s^t \Lambda(\xi)d\xi\bigr)$, we obtain

$$V(t) \leq V(s) \exp\left(\int_s^t \Lambda(\xi)d\xi\right) + \int_s^t \varrho(\xi) \exp\left(\int_\xi^t \Lambda(\sigma)d\sigma\right)d\xi.$$

The statement now follows from (3.20). $\square$

*Remark* 3.2. Theorem 3.2 generalizes theorems contained in previous papers.

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = t - r$, where $r > 0$, we get Theorem 2 from [23].

If we take $\theta_k(t) = t - r_k$, where $r_k > 0$, $k = 1, \ldots, m$, we obtain Theorem 2 from [40].

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = \theta(t)$, we get Theorem 2.7 from [22].

The last of the main propositions gives the conditions under which all solutions of (3.2) are tending to zero.

**Theorem 3.3.** *Let the assumptions* (i), (ii), (iii) *and* (iv) *be satisfied. Let* $\Lambda(t) \le 0$ *a. e. on* $[T^*, \infty)$, *where* $T^* \in [T, \infty)$. *If*

$$\lim_{t\to\infty} \int^{t} \Lambda(s)ds = -\infty \quad \textit{and} \quad \varrho(t) = o(\Lambda(t)), \tag{3.26}$$

*then any solution* $z(t)$ *of the equation* (3.2) *existing for* $t \to \infty$ *satisfies*

$$\lim_{t\to\infty} z(t) = 0.$$

*Proof.* Choose arbitrary $\varepsilon > 0$. According to (3.26), there is $s \ge T^*$ such that $\varrho(t) \le \frac{\mu\varepsilon}{2}|\Lambda(t)|$ for $t \ge s$ and

$$\begin{aligned}\int_s^t \varrho(\tau)\exp\left(\int_\tau^t \Lambda(\sigma)d\sigma\right)d\tau &\le \frac{\mu\varepsilon}{2}\int_s^t [-\Lambda(\tau)]\exp\left(\int_\tau^t \Lambda(\sigma)d\sigma\right)d\tau = \\ &= \frac{\mu\varepsilon}{2}\int_s^t \left(\frac{d}{d\tau}\left[\exp\left(\int_\tau^t \Lambda(\sigma)d\sigma\right)\right]\right)d\tau = \frac{\mu\varepsilon}{2}\left[\exp\left(\int_\tau^t \Lambda(\sigma)d\sigma\right)\right]_s^t = \\ &= \frac{\mu\varepsilon}{2}\left[1 - \exp\left(\int_s^t \Lambda(\tau)d\tau\right)\right] < \frac{\mu\varepsilon}{2}\end{aligned}$$

for $t \ge s$. From (3.26) we have $\exp\big(\int_s^t \Lambda(\tau)d\tau\big) \to 0$ as $t \to \infty$, hence there is $S \ge s$ such that $\exp\big(\int_s^t \Lambda(\tau)d\tau\big) < \frac{\mu\varepsilon}{2V(s)}$ for $t \ge S$. Considering this fact and (3.23), we get

$$\mu|z(t)| < V(s)\frac{\mu\varepsilon}{2V(s)} + \frac{\mu\varepsilon}{2} = \mu\varepsilon$$

for $t \ge S$. This completes the proof. $\square$

*Remark* 3.3. Theorem 3.3 is a generalization of results published in the papers [23], [40] and [22].

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \dots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \dots, m$, $\theta_1(t) = t - r$, where $r > 0$, we get Theorem 3 from [23].

If we take $\theta_k(t) = t - r_k$, where $r_k > 0$, $k = 1, \dots, m$, we obtain Theorem 3 from [40].

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \dots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \dots, m$, $\theta_1(t) = \theta(t)$, we get Theorem 2.14 from [22].

### 3.2.3 Corollaries and examples

From Theorem 3.1 we easily obtain the following corollaries.

**Corollary 3.1.** *Let $a(t) \equiv a \in \mathbb{C}$, $b(t) \equiv b \in \mathbb{C}$, $|a| > |b|$. Suppose that $\lim_{t\to\infty} \theta_k(t) = \infty$, $\theta_k(t) \le t$ for $t \ge T_1$, where $T_1 \ge t_0$. Let $\rho_0, \rho_1, \dots, \rho_m \colon [T_1, \infty) \to \mathbb{R}$ be such that*

$$|g(t, z, w_1, \dots, w_m)| \le \rho_0(t)|z| + \sum_{k=1}^{m} \rho_k(t)|w_k| \tag{3.27}$$

*for $t \ge T_1$, $|z| < R$, $|w_k| < R$, $k = 1, \dots, m$, $R > 0$ and $\rho_0 \in L_{\mathrm{loc}}([T_1, \infty), \mathbb{R})$.*

*Let $\beta \in AC_{\mathrm{loc}}([T_1, \infty), \mathbb{R}_+)$ satisfy*

$$\theta_k'(t)\beta(t) \ge \left(\frac{|a| + |b|}{|a| - |b|}\right)^{\frac{1}{2}} \big(\rho_k(t) + |A_k(t)| + |B_k(t)|\big) \quad \text{a. e. on } [T_1, \infty) \text{ for } k = 1, \dots, m.$$

*If*

$$\limsup_{t\to\infty} \int^{t} \max\left(\frac{|a| - |b|}{|a|} \operatorname{Re} a + \left(\frac{|a| + |b|}{|a| - |b|}\right)^{\frac{1}{2}} \rho_0(s) + m\beta(s), \frac{\beta'(s)}{\beta(s)}\right) ds < \infty, \tag{3.28}$$

*then the trivial solution of equation (3.2) is stable. If*

$$\lim_{t\to\infty} \int^{t} \max\left(\frac{|a| - |b|}{|a|} \operatorname{Re} a + \left(\frac{|a| + |b|}{|a| - |b|}\right)^{\frac{1}{2}} \rho_0(s) + m\beta(s), \frac{\beta'(s)}{\beta(s)}\right) ds = -\infty, \tag{3.29}$$

*then the trivial solution of (3.2) is asymptotically stable.*

*Proof.* Choose $T \ge T_1$ such that $\theta_k(t) \ge T_1$ for $t \ge T$, $k = 1, \dots, m$. Denote $z = z(t)$ and $w_k = z(\theta_k(t))$ again. Since $a, b \in \mathbb{C}$ are constants, then also $\gamma$ and $c$ are constants and we have $\vartheta(t) \equiv 0$. Using the condition (3.27) we get

$$
\begin{aligned}
|\gamma g(t,z,w_1,\dots,w_m)+c\bar g(t,z,w_1,\dots,w_m)| &\le (\gamma+|c|)\bigl(\rho_0(t)|z|+\sum_{k=1}^m \rho_k(t)|w_k|\bigr)= \\
&= \frac{\gamma+|c|}{\gamma-|c|}(\gamma-|c|)\bigl(\rho_0(t)|z|+\sum_{k=1}^m \rho_k(t)|w_k|\bigr)\le \\
&\le \frac{\gamma+|c|}{\gamma-|c|}\bigl(\rho_0(t)|\gamma z+c\bar z|+\sum_{k=1}^m \rho_k(t)|\gamma w_k+c\bar w_k|\bigr)
\end{aligned}
$$

and it follows that the condition (ii) holds with

$$
\varkappa(t)=\frac{\gamma+|c|}{\gamma-|c|}\rho_0(t), \qquad \kappa_k(t)=\frac{\gamma+|c|}{\gamma-|c|}\rho_k(t)
$$

and $\varrho(t)\equiv 0$.

The condition (3.28) implies that $\operatorname{Re} a\le 0$. Since

$$
\alpha = 1+\frac{|b|}{|a|}\operatorname{sgn}\operatorname{Re} a=\frac{|a|+|b|\operatorname{sgn}\operatorname{Re} a}{|a|}\ge\frac{|a|-|b|}{|a|}
$$

and

$$
\frac{\gamma+|c|}{\gamma-|c|}=\frac{|a|+\sqrt{|a|^2-|b|^2}+|b|}{|a|+\sqrt{|a|^2-|b|^2}-|b|}=\left(\frac{|a|+|b|}{|a|-|b|}\right)^{\frac12},
$$

in view of (3.11) we obtain

$$
\begin{aligned}
\lambda_k(t) &= \left(\frac{|a|+|b|}{|a|-|b|}\right)^{\frac12}\bigl\{\rho_k(t)+|A_k(t)|+|B_k(t)|\bigr\},\\
\Theta(t) &= \alpha\operatorname{Re} a+\frac{\gamma+|c|}{\gamma-|c|}\rho_0(t)+m\beta(t)\le\frac{|a|-|b|}{|a|}\operatorname{Re} a+\left(\frac{|a|+|b|}{|a|-|b|}\right)^{\frac12}\rho_0(t)+m\beta(t),
\end{aligned}
$$

and the assertion follows from (3.12) and Theorem 3.1. □

We can formulate a corollary that is simpler for application than the previous one. However, it is concerned only in stability.

**Corollary 3.2.** *Assume that the conditions* (i), (ii) *and* (iii) *are valid with* $\varrho(t)\equiv 0$. *If* $\beta(t)$ *is monotone and bounded on* $[T,\infty)$ *and if*

$$
\limsup_{t\to\infty}\int^{t}[\Theta(s)]_+ds<\infty,
$$

*where* $[\Theta(t)]_+=\max\{\Theta(t),0\}$, *then the trivial solution of* (3.2) *is stable.*

*Proof.* Suppose firstly that $\beta$ is non-increasing on $[T,\infty)$. Then $\beta' \le 0$ a. e. on $[T,\infty)$.

If $\beta(T_2)=0$ for some $T_2 \ge T$, then $\beta(t)\equiv 0$ on $[T_2,\infty)$. Consequently, $\Lambda$ has to satisfy only the inequality $\Theta(t)\le\Lambda(t)$ a. e. on $[T_2,\infty)$, so we may choose $\Lambda(t)=\Theta(t)$ on $[T_2,\infty)$. It follows that $\Lambda(t)=\Theta(t)\le\max\{\Theta(t),0\}=[\Theta(t)]_+$.

On the other way, if $\beta(t)>0$ on $[T,\infty)$, we may put $\Lambda(t)=\max\big\{\Theta(t),\frac{\beta'(t)}{\beta(t)}\big\}$. Then

$$\Lambda(t)=\max\big\{\Theta(t),\frac{\beta'(t)}{\beta(t)}\big\}\le\max\{\Theta(t),0\}=[\Theta(t)]_+.$$

In both cases, $\Lambda$ satisfies the condition (iv) and the inequality $\Lambda(t)\le[\Theta(t)]_+$ on $[T_2,\infty)$, hence

$$\limsup_{t\to\infty}\int^t\Lambda(s)ds\le\limsup_{t\to\infty}\int^t[\Theta(s)]_+ds<\infty.$$

Now assume that $\beta$ is non-decreasing on $[T,\infty)$. Then $\beta'\ge 0$ a. e. on $[T,\infty)$.

If $\beta(t)\equiv 0$ on $[T,\infty)$, we may treat it as above.

Otherwise, there is some $T_3\ge T$ such that $\beta(t)>0$ on $[T_3,\infty)$ and we may choose $\Lambda(t)=\max\big\{\Theta(t),\frac{\beta'(t)}{\beta(t)}\big\}$ on $[T_3,\infty)$. Clearly $\Lambda$ satisfies the condition (iv) on $[T_3,\infty)$. Since $\beta'\ge 0$ a. e. on $[T,\infty)$, it follows that $\frac{\beta'}{\beta}\ge 0$ a. e. on $[T_3,\infty)$. Hence

$$\Lambda(t)=\max\big\{\Theta(t),\frac{\beta'(t)}{\beta(t)}\big\}\le\max\big\{[\Theta(t)]_+,\frac{\beta'(t)}{\beta(t)}\big\}\le[\Theta(t)]_++\frac{\beta'(t)}{\beta(t)}$$

and then

$$\begin{aligned}\limsup_{t\to\infty}\int^t\Lambda(s)ds&\le\limsup_{t\to\infty}\int^t[\Theta(s)]_+ds+\limsup_{t\to\infty}\int^t\frac{\beta'(t)}{\beta(t)}ds\le\\&\le\limsup_{t\to\infty}\int^t[\Theta(s)]_+ds+\limsup_{t\to\infty}\big(\ln(\beta(t))\big)-\ln(\beta(T_3))<\infty\end{aligned}$$

since $\beta$ is bounded on $[T,\infty)$.

The statement follows from Theorem 3.1. □

*Remark* 3.4. If the function $\beta$ does not satisfy the assumptions of Corollary 3.2, we can try to find a function $\beta^*(t)\ge\beta(t)$ which is monotone and bounded on $[T,\infty)$ such that all conditions stated in Corollary 3.2 become true.

However, in some cases it is not possible to find such function $\beta^*$ while the trivial solution of the equation (3.2) is stable, as we can see in an example.

**Example 3.1.** Let us study the equation (3.2) where $a(t)\equiv-12-5i$, $b(t)\equiv 1$, $A_k(t)\equiv 0$, $B_k(t)\equiv 0$,

$$g(t,z,w_1,\dots,w_m)=\frac{\sqrt6}{\sqrt7}\Big(\frac{105}{13}-\sin t\Big)\,\mathrm{e}^{it}\,z+\sum_{k=1}^{m}\frac{\sqrt6}{\sqrt7}\frac{k}{mt}(2+\sin t)w_k.$$

Assume that $t_0=m$, $R=\infty$ and $\theta_k(t)=k\ln t$, $k=1,\dots,m$. Put $T_1=t_0$, $T=\mathrm{e}^{T_1}=\mathrm{e}^{m}$. As $g$ satisfies the inequality (3.27) from Corollary 3.1 with $\rho_0(t)=\frac{\sqrt6}{\sqrt7}\big(\frac{105}{13}-\sin t\big)$ and $\rho_k(t)=\frac{\sqrt6}{\sqrt7}\frac{k}{mt}(2+\sin t)$, the condition (ii) is valid for $\varkappa(t)=\big(\frac{105}{13}-\sin t\big)$, $\kappa_k(t)=\frac{k}{mt}(2+\sin t)$ and $\varrho(t)\equiv0$. Moreover, $\vartheta(t)\equiv0$ and $\alpha(t)\equiv\frac{12}{13}$.

Since $\kappa_k(t)\in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_+)$ and $\theta_k'(t)>0$ on $[T,\infty)$, we may choose $\beta(t)=\max\limits_{k=1,\dots,m}(\theta_k'(t))^{-1}\kappa_k(t)=\frac{1}{m}(2+\sin t)$, hence the minimal function $\beta^*$ which satisfies the conditions of Corollary 3.2 is the function $\beta^*\equiv\frac{3}{m}$. For this $\beta^*$ we have $\Theta^*(t)=\alpha(t)\operatorname{Re}a+\vartheta(t)+\varkappa(t)+m\beta^*(t)=-\frac{144}{13}+0+\big(\frac{105}{13}-\sin t\big)+m\frac{3}{m}=-\sin t$. It is not difficult to compute that $\limsup\limits_{t\to\infty}\int\limits_T^t[\Theta^*(s)]_+ds=\infty$, hence Corollary 3.2 is not suitable to prove stability of the trivial solution of the equation (3.2).

Nevertheless, for $\beta$ we get $\Theta(t)=\alpha(t)\operatorname{Re}a+\vartheta(t)+\varkappa(t)+m\beta(t)=-\frac{144}{13}+0+\big(\frac{105}{13}-\sin t\big)+m\frac{1}{m}(2+\sin t)=-1$ and since $\beta>0$ on $[T,\infty)$, we may choose $\Lambda(t)=\max\big\{\Theta(t),\frac{\beta'(t)}{\beta(t)}\big\}=\frac{\cos t}{2+\sin t}$. Then $\limsup\limits_{t\to\infty}\int\limits_T^t\Lambda(s)ds\le\ln3<\infty$, hence, in view of Theorem 3.1, the trivial solution of the equation (3.2) is stable.

We can derive several consequences from Theorem 3.2.

**Corollary 3.3.** *Let the conditions* (i), (ii), (iii) *and* (iv) *be fulfilled and*

$$\limsup_{t\to\infty}\int\limits_s^t\varrho(\xi)\exp\left(-\int\limits_s^\xi\Lambda(\sigma)d\sigma\right)d\xi<\infty$$

*for some $s\ge T$.*

*If $z(t)$ is any solution of* (3.2) *existing for $t\to\infty$, then*

$$z(t)=O\left[\exp\left(\int\limits_s^t\Lambda(\xi)d\xi\right)\right].$$

*Proof.* From the assumptions and (3.25) we can see that there are $K>0$ and $S\ge s$ such that for $t\ge S$ we have

$$V(t)\exp\left(-\int\limits_s^t\Lambda(\xi)d\xi\right)-V(s)\le\int\limits_s^t\varrho(\xi)\exp\left(-\int\limits_s^\xi\Lambda(\sigma)d\sigma\right)d\xi\le K<\infty.$$

Then

$$\mu|z(t)| \le V(t) \le (K + V(s)) \exp\left(\int_s^t \Lambda(\xi) d\xi\right).$$

□

**Corollary 3.4.** *Let the assumptions* (i), (ii), (iii) *and* (iv) *hold and let*

$$\limsup_{t\to\infty} \Lambda(t) < \infty \quad \text{and} \quad \varrho(t) = O(\mathrm{e}^{\eta t}), \tag{3.30}$$

*where* $\eta > \limsup_{t\to\infty} \Lambda(t)$. *If* $z(t)$ *is any solution of* (3.2) *existing for* $t \to \infty$, *then* $z(t) = O(\mathrm{e}^{\eta t})$.

*Proof.* In view of (3.30), there are $L > 0$, $\eta^* < \eta$ and $s > T$ such that $\eta^* > \Lambda(t)$ for $t \ge s$ and $\varrho(t)\,\mathrm{e}^{-\eta t} < L$ for $t \ge s$. From (3.23) we get

$$\begin{aligned}
\mu|z(t)| &\le V(s)\,\mathrm{e}^{\eta^*(t-s)} + L\int_s^t \mathrm{e}^{\eta\tau}\,\mathrm{e}^{\eta^*(t-\tau)}\,d\tau \le \\
&\le V(s)\,\mathrm{e}^{\eta^*(t-s)} + L\,\mathrm{e}^{\eta^* t}\,\frac{\mathrm{e}^{(\eta-\eta^*)t} - \mathrm{e}^{(\eta-\eta^*)s}}{\eta-\eta^*} \le \\
&\le V(s)\,\mathrm{e}^{\eta^*(t-s)} + \frac{L}{\eta-\eta^*}\,\mathrm{e}^{\eta t} = O(\mathrm{e}^{\eta t}).
\end{aligned} \tag{3.31}$$

□

*Remark* 3.5. If $\varrho(t) \equiv 0$, we can take $L = 0$ in the proof of Corollary 3.4 and taking the inequalities (3.31) into account we obtain the following statement: there is an $\eta^* < \eta_0 < \eta$ such that $z(t) = o(\mathrm{e}^{\eta_0 t})$ holds for the solution $z(t)$ of (3.2).

## 3.3 The case $\liminf_{t\to\infty}(|\operatorname{Im} a(t)| - |b(t)|) > 0$

Instead of the case $\liminf_{t\to\infty}(|a(t)| - |b(t)|) > 0$ investigated in Section 3.2, in this section we consider the equation (3.2) in the case when

$$\liminf_{t\to\infty}\big(|\operatorname{Im} a(t)| - |b(t)|\big) > 0 \tag{3.32}$$

and study the behavior of solutions of (3.2) under this assumption. This situation corresponds to the case when the autonomous system (1.3) has the singular point 0 of type centre or focus.

Obviously, this case is included in the case $\liminf_{t\to\infty}\big(|a(t)| - |b(t)|\big) > 0$ considered in Section 3.2, but in this special case we are able to derive more useful results as we will see later in an example. The idea is based upon the well known result that the condition $|a| > |b|$ in an autonomous equation $z' = az + b\bar{z}$ ensures that zero is a focus, a centre or a node while under the condition $|\operatorname{Im} a| > |b|$ zero can be just a focus or a centre. Details are contained in [39].

A simple example which is situated after Corollary 3.5 in Subsection 3.3.3 shows that, in some cases, the results of this section can be applied more suitable than those given in Section 3.2.

### 3.3.1 Assumptions

Regarding (3.32) and since the delay functions $\theta_k$ satisfy $\lim_{t\to\infty}\theta_k(t) = \infty$, there are numbers $T_1 \geq t_0$, $T \geq T_1$ and $\mu > 0$ such that

$$|\operatorname{Im} a(t)| > |b(t)| + \mu \text{ for } t \geq T_1, \quad t \geq \theta_k(t) \geq T_1 \text{ for } t \geq T \ (k = 1, \dots, m). \tag{3.33}$$

Denote

$$\tilde{\gamma}(t) = \operatorname{Im} a(t) + \sqrt{(\operatorname{Im} a(t))^2 - |b(t)|^2}\operatorname{sgn}(\operatorname{Im} a(t)), \quad \tilde{c}(t) = -ib(t). \tag{3.34}$$

Notice that, unlike the function $\gamma$ from Section 3.2, the function $\tilde{\gamma}$ defined in this section need not be positive.

Since $|\tilde{\gamma}(t)| > |\operatorname{Im} a(t)|$ and $|\tilde{c}(t)| = |b(t)|$, the inequality

$$|\tilde{\gamma}(t)| > |\tilde{c}(t)| + \mu \tag{3.35}$$

holds for all $t \geq T_1$. It is easy to verify that $\tilde{\gamma}, \tilde{c} \in AC_{\mathrm{loc}}([T_1, \infty), \mathbb{C})$.

For the purpose of this section denote

$$\tilde{\vartheta}(t) = \frac{\operatorname{Re}(\tilde{\gamma}(t)\tilde{\gamma}'(t) - \bar{\tilde{c}}(t)\tilde{c}'(t)) + |\tilde{\gamma}(t)\tilde{c}'(t) - \tilde{\gamma}'(t)\tilde{c}(t)|}{\tilde{\gamma}^2(t) - |\tilde{c}(t)|^2}. \tag{3.36}$$

The stability and asymptotic stability are studied subject to the following assumptions:

(i) The numbers $T_1 \geq t_0$, $T \geq T_1$ and $\mu > 0$ are such that (3.33) holds.

(ii) There exist functions $\tilde{\varkappa}, \tilde{\kappa}_k, \tilde{\varrho}\colon [T, \infty) \to \mathbb{R}$ such that

$$|\tilde{\gamma}(t)g(t, z, w_1, \dots, w_m) + \tilde{c}(t)\bar{g}(t, z, w_1, \dots, w_m)| \leq \tilde{\varkappa}(t)|\tilde{\gamma}(t)z + \tilde{c}(t)\bar{z}| + \sum_{k=1}^{m} \tilde{\kappa}_k(t)|\tilde{\gamma}(\theta_k(t))w_k + \tilde{c}(\theta_k(t))\bar{w}_k| + \tilde{\varrho}(t)$$

for $t \geq T$, $z, w_k \in \mathbb{C}$ $(k = 1, \dots, m)$, where $\tilde{\varkappa}$, $\tilde{\varrho} \in L_{\mathrm{loc}}([T, \infty), \mathbb{R})$.

(iii) $\tilde{\beta} \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}^0_+)$ is a function satisfying

$$\theta_k'(t)\tilde{\beta}(t) \geq \tilde{\lambda}_k(t) \quad \text{a. e. on } [T,\infty), \tag{3.37}$$

where $\tilde{\lambda}_k$ is defined for $t \geq T$ by

$$\tilde{\lambda}_k(t) = \tilde{\kappa}_k(t) + (|A_k(t)| + |B_k(t)|)\frac{|\tilde{\gamma}(t)| + |\tilde{c}(t)|}{|\tilde{\gamma}(\theta_k(t))| - |\tilde{c}(\theta_k(t))|}. \tag{3.38}$$

(iv) There exists a function $\tilde{\Lambda} \in L_{\mathrm{loc}}([T,\infty),\mathbb{R})$ which satisfies the inequalities $\tilde{\beta}'(t) \leq \tilde{\Lambda}(t)\tilde{\beta}(t)$, $\tilde{\Theta}(t) \leq \tilde{\Lambda}(t)$ for almost all $t \in [T,\infty)$, where the function $\tilde{\Theta}$ is defined by

$$\tilde{\Theta}(t) = \operatorname{Re} a(t) + \tilde{\vartheta}(t) + \tilde{\varkappa}(t) + m\tilde{\beta}(t). \tag{3.39}$$

If $A_k$, $B_k$, $\tilde{\kappa}_k$, $\theta_k'$ are locally absolutely continuous on $[T,\infty)$ and $\tilde{\lambda}_k(t) \geq 0$, $\theta_k'(t) > 0$ on $[T,\infty)$, the choice $\tilde{\beta}(t) = \max\limits_{k=1,\dots,m}[\tilde{\lambda}_k(t)(\theta_k'(t))^{-1}]$ is admissible in (iii).

Under the assumption (i), we can estimate

$$\begin{aligned} |\tilde{\vartheta}| &\leq \frac{|\operatorname{Re}(\tilde{\gamma}\tilde{\gamma}' - \bar{\tilde{c}}\tilde{c}')| + |\tilde{\gamma}\tilde{c}' - \tilde{\gamma}'\tilde{c}|}{\tilde{\gamma}^2 - |\tilde{c}|^2} \leq \frac{(|\tilde{\gamma}'| + |\tilde{c}'|)(|\tilde{\gamma}| + |\tilde{c}|)}{\tilde{\gamma}^2 - |\tilde{c}|^2} = \\ &= \frac{|\tilde{\gamma}'| + |\tilde{c}'|}{|\tilde{\gamma}| - |\tilde{c}|} \leq \frac{1}{\mu}(|\tilde{\gamma}'| + |\tilde{c}'|), \end{aligned}$$

hence the function $\tilde{\vartheta}$ is locally Lebesgue integrable on $[T,\infty)$. Moreover, if $\tilde{\beta} \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_+)$ and $\tilde{\varkappa} \in L_{\mathrm{loc}}([T,\infty)$, then we can choose

$$\tilde{\Lambda}(t) = \max\Big(\tilde{\Theta}(t), \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)}\Big) \tag{3.40}$$

in (iv).

Finally, if $\tilde{\varrho}(t) \equiv 0$ in (ii), then the equation (3.2) has the trivial solution $z(t) \equiv 0$. Notice that in this case the condition (ii) implies that the functions $\tilde{\varkappa}(t)$, $\tilde{\kappa}_k(t)$ are nonnegative on $[T,\infty)$ for $k = 1,\dots,m$, and due to this, $\tilde{\lambda}_k(t) \geq 0$ on $[T,\infty)$. The case $\tilde{\varrho}(t) < 0$ is omitted since it can be replaced by $\tilde{\varrho}(t) \equiv 0$.

### 3.3.2 Main results

The aim is to generalize the results for ordinary differential equations recalled in Chapter 2 as well as the results contained in [23] (one constant delay), [41] (a finite number of constant delays) and [43] (one nonconstant delay). The proof of Theorem 3.4 is similar to the proof of Theorem 3.1. In the proof we use Lemma 3.1 again.

**Theorem 3.4.** *Let the conditions* (i), (ii), (iii) *and* (iv) *hold and* $\tilde{\varrho}(t) \equiv 0$.

*a) If*

$$\limsup_{t\to\infty} \int^{t} \tilde{\Lambda}(s)ds < \infty, \tag{3.41}$$

*then the trivial solution of* (3.2) *is stable on* $[T, \infty)$*;*

*b) if*

$$\lim_{t\to\infty} \int^{t} \tilde{\Lambda}(s)ds = -\infty, \tag{3.42}$$

*then the trivial solution of* (3.2) *is asymptotically stable on* $[T, \infty)$*.*

*Proof.* Proceeding similarly as in the proof of Theorem 3.1, we obtain

$$\begin{aligned}
&UU' = U\big(\sqrt{(\tilde{\gamma}z + \tilde{c}\bar{z})(\bar{\tilde{\gamma}}\bar{z} + \bar{\tilde{c}}z)}\big)' = \operatorname{Re}\big[(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma}'z + \tilde{\gamma}z' + \tilde{c}'\bar{z} + \tilde{c}\bar{z}')\big] = \\
&= \operatorname{Re}\Big\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)\Big[\tilde{\gamma}'z + \tilde{c}'\bar{z} + \tilde{\gamma}\Big(az + b\bar{z} + \sum_{k=1}^{m}\big(A_k w_k + B_k \bar{w}_k\big) + g\Big) + \\
&+ \tilde{c}\Big(\bar{a}\bar{z} + \bar{b}z + \sum_{k=1}^{m}\big(\bar{A}_k \bar{w}_k + \bar{B}_k w_k\big) + \bar{g}\Big)\Big]\Big\} = \\
&= \operatorname{Re}\Big\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)\Big[\tilde{\gamma}'z + \tilde{c}'\bar{z} + (\tilde{\gamma}a + \tilde{c}\bar{b})z + (\tilde{\gamma}b + \tilde{c}\bar{a})\bar{z} + \tilde{\gamma}\Big(\sum_{k=1}^{m}\big(A_k w_k + B_k \bar{w}_k\big) + g\Big) + \\
&+ \tilde{c}\Big(\sum_{k=1}^{m}\big(\bar{A}_k \bar{w}_k + \bar{B}_k w_k\big) + \bar{g}\Big)\Big]\Big\}
\end{aligned}$$

for almost all $t \in \mathcal{K}$.

Short computation gives $(\tilde{\gamma}a + \tilde{c}\bar{b})\tilde{c} = (\tilde{\gamma}b + \tilde{c}\bar{a})\tilde{\gamma}$, and from this we get

$$\begin{aligned}
UU' \le\; & \operatorname{Re}\Big\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma}a + \tilde{c}\bar{b})\Big(z + \frac{\tilde{c}}{\tilde{\gamma}}\bar{z}\Big)\Big\} + \\
& + \operatorname{Re}\Big\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)\big[\tilde{\gamma}\sum_{k=1}^{m}\big(A_k w_k + B_k \bar{w}_k\big) + \tilde{c}\sum_{k=1}^{m}\big(\bar{A}_k \bar{w}_k + \bar{B}_k w_k\big)\big]\Big\} + \\
& + \operatorname{Re}[(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma}g + \tilde{c}\bar{g})] + \operatorname{Re}[(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma}'z + \tilde{c}'\bar{z})] \le \\
& \le U^2 \operatorname{Re}\Big(a + \frac{\tilde{c}}{\tilde{\gamma}}\bar{b}\Big) + U(|\tilde{\gamma}| + |\tilde{c}|)\Big(\sum_{k=1}^{m}|A_k w_k + B_k \bar{w}_k|\Big) + U|\tilde{\gamma}g + \tilde{c}\bar{g}| + U^2 \operatorname{Re}\frac{\tilde{\gamma}'z + \tilde{c}'\bar{z}}{\tilde{\gamma}z + \tilde{c}\bar{z}}.
\end{aligned}$$

for almost all $t \in \mathcal{K}$.

Applying Lemma 3.1 to the last term, we obtain

$$\operatorname{Re} \frac{\tilde{\gamma}' z + \tilde{c}' \bar{z}}{\tilde{\gamma} z + \tilde{c}\bar{z}} \leq \tilde{\vartheta}.$$

Using this inequality together with (3.38), the assumption (ii) and the relation $\operatorname{Re}(a + \frac{\bar{\tilde{c}}}{\tilde{\gamma}}\bar{b}) = \operatorname{Re} a$, we obtain

$$\begin{aligned}
UU' &\leq U^2(\operatorname{Re} a + \tilde{\vartheta} + \tilde{\varkappa}) + U \sum_{k=1}^{m} \big(\tilde{\kappa}_k |\tilde{\gamma}(\theta_k) w_k + \tilde{c}(\theta_k)\bar{w}_k|\big) + \\
&+ U(|\tilde{\gamma}| + |\tilde{c}|)\Big(\sum_{k=1}^{m} \frac{|A_k||w_k| + |B_k||\bar{w}_k|}{|\tilde{\gamma}(\theta_k)| - |\tilde{c}(\theta_k)|}(|\tilde{\gamma}(\theta_k)| - |\tilde{c}(\theta_k)|)\Big) \leq \\
&\leq U^2(\operatorname{Re} a + \tilde{\vartheta} + \tilde{\varkappa}) + \\
&+ U\Big\{\sum_{k=1}^{m} \Big[\tilde{\kappa}_k + (|A_k| + |B_k|)\frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}(\theta_k)| - |\tilde{c}(\theta_k)|}\Big] |\tilde{\gamma}(\theta_k) w_k + \tilde{c}(\theta_k)\bar{w}_k|\Big\} \leq \\
&\leq U^2(\operatorname{Re} a + \tilde{\vartheta} + \tilde{\varkappa}) + U \sum_{k=1}^{m} \tilde{\lambda}_k |\tilde{\gamma}(\theta_k) w_k + \tilde{c}(\theta_k)\bar{w}_k|
\end{aligned}$$

for almost all $t \in \mathcal{K}$.

Consequently,

$$U' \leq U(\operatorname{Re} a + \tilde{\vartheta} + \tilde{\varkappa}) + \sum_{k=1}^{m} \tilde{\lambda}_k |\tilde{\gamma}(\theta_k) w_k + \tilde{c}(\theta_k)\bar{w}_k| \tag{3.43}$$

for almost all $t \in \mathcal{K}$.

Recalling that $U'(t) = 0$ for almost all $t \in \mathcal{M}$, we can see that the inequality (3.43) is valid for almost all $t \geq t_1$ for which $z(t)$ is defined.

From (3.43) we have

$$\begin{aligned}
V' &\leq U(\operatorname{Re} a + \tilde{\vartheta} + \tilde{\varkappa} + m\tilde{\beta}) + \sum_{k=1}^{m} (\tilde{\lambda}_k - \theta_k' \tilde{\beta}) |\tilde{\gamma}(\theta_k) w_k + \tilde{c}(\theta_k)\bar{w}_k| + \\
&+ \tilde{\beta}' \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} |\tilde{\gamma}(s) z(s) + \tilde{c}(s)\bar{z}(s)| ds.
\end{aligned}$$

As $\tilde{\beta}(t)$ fulfills the condition (3.37), we obtain

$$V'(t) \leq U(t)\tilde{\Theta}(t) + \tilde{\beta}'(t) \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} |\tilde{\gamma}(s) z(s) + \tilde{c}(s)\bar{z}(s)| ds,$$

hence

$$V'(t) - \tilde{\Lambda}(t)V(t) \le 0 \tag{3.44}$$

for almost all $t \ge t_1$ for which the solution $z(t)$ exists.

The rest of the proof is identical to the corresponding part of the proof of Theorem 3.1. □

*Remark* 3.6. Theorem 3.4 represents a generalization of previous results.

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = t - r$, where $r > 0$, we get Theorem 4 from [23].

If we take $\theta_k(t) = t - r_k$, where $r_k > 0$, $k = 1, \ldots, m$, we obtain Theorem 1 from [41].

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = \theta(t)$, we get Theorem 1 from [43].

The next theorem is analogous to Theorem 3.2. The proof is identical to the proof of Theorem 3.2 and thus is omitted.

**Theorem 3.5.** *Let the assumptions* (i), (ii), (iii) *and* (iv) *hold and*

$$V(t) = |\tilde{\gamma}(t)z(t) + \tilde{c}(t)\bar{z}(t)| + \tilde{\beta}(t) \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} |\tilde{\gamma}(s)z(s) + \tilde{c}(s)\bar{z}(s)| ds, \tag{3.45}$$

*where $z(t)$ is any solution of* (3.2) *defined for $t \to \infty$. Then*

$$\mu|z(t)| \le V(s) \exp\left( \int_{s}^{t} \tilde{\Lambda}(\xi) d\xi \right) + \int_{s}^{t} \tilde{\varrho}(\xi) \exp\left( \int_{\xi}^{t} \tilde{\Lambda}(\sigma) d\sigma \right) d\xi \tag{3.46}$$

*for $t \ge s \ge t_1$ where $t_1 \ge T$.*

*Remark* 3.7. Theorem 3.5 generalizes theorems contained in previous papers.

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = t - r$, where $r > 0$, we get Theorem 5 from [23].

If we take $\theta_k(t) = t - r_k$, where $r_k > 0$, $k = 1, \ldots, m$, we obtain Theorem 2 from [41].

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = \theta(t)$, we get Theorem 2 from [43].

Theorem 3.6 contains analogous result to Theorem 3.3. The proof is not included since it is identical to the proof of Theorem 3.3.

**Theorem 3.6.** *Let the assumptions* (i), (ii), (iii) *and* (iv) *be satisfied. Let $\tilde{\Lambda}(t) \le 0$ a. e. on $[T^*, \infty)$, where $T^* \in [T, \infty)$. If*

$$\lim_{t \to \infty} \int^{t} \tilde{\Lambda}(s) ds = -\infty \quad \textit{and} \quad \tilde{\varrho}(t) = o(\tilde{\Lambda}(t)), \tag{3.47}$$

*then any solution $z(t)$ of the equation* (3.2) *existing for $t \to \infty$ satisfies*

$$\lim_{t\to\infty} z(t) = 0.$$

*Remark* 3.8. Theorem 3.6 is a generalization of results published in the papers [23], [41] and [43].

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \dots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \dots, m$, $\theta_1(t) = t - r$, where $r > 0$, we get Theorem 6 from [23].

If we take $\theta_k(t) = t - r_k$, where $r_k > 0$, $k = 1, \dots, m$, we obtain Theorem 3 from [41].

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \dots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \dots, m$, $\theta_1(t) = \theta(t)$, we get Theorem 3 from [43].

### 3.3.3 Corollaries and examples

From Theorem 3.4 we easily obtain several corollaries. We give an example which shows that it is worth to consider the case (3.32) as well.

**Corollary 3.5.** *Let $a(t) \equiv a \in \mathbb{C}$, $b(t) \equiv b \in \mathbb{C}$, $|\operatorname{Im} a| > |b|$. Suppose that $\lim_{t\to\infty} \theta_k(t) = \infty$, $\theta_k(t) \le t$ for $t \ge T_1$, where $T_1 \ge t_0$. Let $\rho_0, \rho_1, \dots, \rho_m \colon [T_1, \infty) \to \mathbb{R}$ be such that*

$$|g(t, z, w_1, \dots, w_m)| \le \rho_0(t)|z| + \sum_{k=1}^{m} \rho_k(t)|w_k| \tag{3.48}$$

*for $t \ge T_1$, $|z| < R$, $|w_k| < R$, $k = 1, \dots, m$, $R > 0$ and $\rho_0 \in L_{\mathrm{loc}}([T_1, \infty), \mathbb{R})$.*

*Let $\tilde\beta \in AC_{\mathrm{loc}}([T_1, \infty), \mathbb{R}_+)$ satisfy*

$$\theta_k'(t)\tilde\beta(t) \ge \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{\frac{1}{2}} \big(\rho_k(t) + |A_k(t)| + |B_k(t)|\big) \quad \text{a. e. on } [T_1, \infty) \text{ for } k = 1, \dots, m.$$

*If*

$$\limsup_{t\to\infty} \int^{t} \max\left(\operatorname{Re} a + \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{\frac{1}{2}} \rho_0(s) + m\tilde\beta(s), \frac{\tilde\beta'(s)}{\tilde\beta(s)}\right) ds < \infty, \tag{3.49}$$

*then the trivial solution of the equation* (3.2) *is stable. If*

$$\lim_{t\to\infty} \int^{t} \max\left(\operatorname{Re} a + \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{\frac{1}{2}} \rho_0(s) + m\tilde\beta(s), \frac{\tilde\beta'(s)}{\tilde\beta(s)}\right) ds = -\infty, \tag{3.50}$$

*then the trivial solution of* (3.2) *is asymptotically stable.*

*Proof.* First part of the proof is identical to the first part of the proof of Corollary 3.1. We continue with the idea that since

$$\frac{|\tilde{\gamma}|+|\tilde{c}|}{|\tilde{\gamma}|-|\tilde{c}|}=\frac{|\operatorname{Im}a|+\sqrt{|\operatorname{Im}a|^2-|b|^2}+|b|}{|\operatorname{Im}a|+\sqrt{|\operatorname{Im}a|^2-|b|^2}-|b|}=\left(\frac{|\operatorname{Im}a|+|b|}{|\operatorname{Im}a|-|b|}\right)^{\frac{1}{2}},$$

in view of (3.39) we obtain

$$\begin{aligned}\tilde{\lambda}_k(t)&=\left(\frac{|\operatorname{Im}a|+|b|}{|\operatorname{Im}a|-|b|}\right)^{\frac{1}{2}}\big\{\rho_k(t)+|A_k(t)|+|B_k(t)|\big\},\\ \tilde{\Theta}(t)&=\operatorname{Re}a+\frac{|\tilde{\gamma}|+|\tilde{c}|}{|\tilde{\gamma}|-|\tilde{c}|}\rho_0(t)+m\tilde{\beta}(t)=\operatorname{Re}a+\left(\frac{|\operatorname{Im}a|+|b|}{|\operatorname{Im}a|-|b|}\right)^{\frac{1}{2}}\rho_0(t)+m\tilde{\beta}(t),\end{aligned}$$

and the assertion follows from (3.40) and Theorem 3.4. □

Now we show an example that, under certain circumstances, Corollary 3.5 is more useful than Corollary 3.1.

**Example 3.2.** Consider the equation (3.2), where $a(t)\equiv-\sqrt{5}+2i$, $b(t)\equiv 1$, $A_k(t)\equiv 0$, $B_k(t)\equiv 0$ for $k=1,\dots,m$,

$$g(t,z,w_1,\dots,w_m)=\frac{2}{\sqrt{3}}\,\mathrm{e}^{it}\,z+\sum_{k=1}^{m}\frac{k}{2mt}(\sqrt{15}-\sqrt{14})\,\mathrm{e}^{-t}\,w_k.$$

Assume that $t_0=m$ and $R=\infty$, $\theta_k(t)=k\ln t$. Put $T=\mathrm{e}^{t_0}=\mathrm{e}^m$. Then $\rho_0(t)\equiv\frac{2}{\sqrt{3}}$, $\rho_k(t)=\frac{k}{2mt}(\sqrt{15}-\sqrt{14})\,\mathrm{e}^{-t}$. We have

$$\begin{aligned}&\max\left(\frac{|a|-|b|}{|a|}\operatorname{Re}a+\left(\frac{|a|+|b|}{|a|-|b|}\right)^{\frac{1}{2}}\rho_0(t)+m\beta(t),\frac{\beta'(t)}{\beta(t)}\right)=\\ &\quad=\max\left(-\frac{2}{3}\sqrt{5}+\sqrt{2}\frac{2}{\sqrt{3}}+m\beta(t),\frac{\beta'(t)}{\beta(t)}\right)\geq\frac{2}{3}(\sqrt{6}-\sqrt{5})>0\end{aligned}$$

for

$$\theta_k'(t)\beta(t)=\frac{k}{t}\beta(t)\geq\left(\frac{|a|+|b|}{|a|-|b|}\right)^{\frac{1}{2}}\big\{\rho_k(t)+|A_k(t)|+|B_k(t)|\big\}=\frac{k}{mt\sqrt{2}}(\sqrt{15}-\sqrt{14})\,\mathrm{e}^{-t},$$

where $k\in\{1,\dots,m\}$, hence we cannot apply Corollary 3.1.

On the other hand, if we use

$$\theta_k'(t)\tilde{\beta}(t)=\frac{k}{t}\tilde{\beta}(t)=\frac{k\sqrt{3}}{2mt}(\sqrt{15}-\sqrt{14})\,\mathrm{e}^{-t}\geq\left(\frac{|\operatorname{Im}a|+|b|}{|\operatorname{Im}a|-|b|}\right)^{\frac{1}{2}}\big\{\rho_k(t)+|A_k(t)|+|B_k(t)|\big\},$$

where $k \in \{1, \dots, m\}$, we have

$$\max\left(\operatorname{Re} a + \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{\frac{1}{2}} \rho_0(t) + m\tilde{\beta}(t), \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)}\right) =$$
$$= \max\Big(-\sqrt{5} + 2 + m\frac{\sqrt{3}}{2m}(\sqrt{15} - \sqrt{14})\,\mathrm{e}^{-t}, -1\Big) \le$$
$$\le -\sqrt{5} + 2 + \frac{\sqrt{3}}{2}(\sqrt{15} - \sqrt{14}) < -\frac{12}{100} < 0,$$

thus Corollary 3.5 guarantees the stability and also asymptotic stability of the trivial solution of the considered equation.

The following corollary gives sufficient conditions for stability of the trivial solution of (3.2). It is analogous to Corollary 3.2. The proof is identical to the proof of Corollary 3.2.

**Corollary 3.6.** *Assume that the conditions* (i), (ii) *and* (iii) *are valid with* $\tilde{\varrho}(t) \equiv 0$. *If* $\tilde{\beta}(t)$ *is monotone and bounded on* $[T, \infty)$ *and if*

$$\limsup_{t\to\infty} \int^{t} [\tilde{\Theta}(s)]_+ ds < \infty,$$

*where* $[\tilde{\Theta}(t)]_+ = \max\{\tilde{\Theta}(t), 0\}$, *then the trivial solution of* (3.2) *is stable.*

Corollaries which can be derived from Theorem 3.5 are the same type as corollaries obtained from Theorem 3.2. The proofs are identical to the proofs of corresponding corollaries of Theorem 3.2.

**Corollary 3.7.** *Let the conditions* (i), (ii), (iii) *and* (iv) *be fulfilled and*

$$\limsup_{t\to\infty} \int_s^t \tilde{\varrho}(\xi) \exp\left(-\int_s^{\xi} \tilde{\Lambda}(\sigma) d\sigma\right) d\xi < \infty$$

*for some* $s \ge T$.

*If* $z(t)$ *is any solution of* (3.2) *existing for* $t \to \infty$, *then*

$$z(t) = O\left[\exp\left(\int_s^t \tilde{\Lambda}(\xi) d\xi\right)\right].$$

**Corollary 3.8.** *Let the assumptions* (i), (ii), (iii) *and* (iv) *hold and let*

$$\limsup_{t\to\infty} \tilde{\Lambda}(t) < \infty \quad \text{and} \quad \tilde{\varrho}(t) = O(\mathrm{e}^{\eta t}), \tag{3.51}$$

*where* $\eta > \limsup_{t\to\infty} \tilde{\Lambda}(t)$. *If* $z(t)$ *is any solution of* (3.2) *existing for* $t \to \infty$, *then* $z(t) = O(\mathrm{e}^{\eta t})$.

# Chapter 4

# Differential systems with delay: The unstable case

This chapter covers the situation analogous to the case when the singular point 0 of the autonomous system (1.3) is unstable. The transformation of the system (3.1) to the equation (3.2) is described in Section 3.1.

## 4.1 The case $\liminf\limits_{t\to\infty}(|a(t)| - |b(t)|) > 0$

In this section we consider the equation (3.2) in the case when

$$\liminf_{t\to\infty}\big(|a(t)| - |b(t)|\big) > 0 \tag{4.1}$$

and study the behavior of solutions of (3.2) under this assumption. This situation corresponds to the case when the singular point 0 of the system (1.3) is a centre, a focus or a node. In fact, this section is an unstable analogy of Section 3.2.

### 4.1.1 Assumptions

Regarding (4.1) and since the delay functions $\theta_k$ satisfy $\lim\limits_{t\to\infty}\theta_k(t) = \infty$, there are numbers $T_1 \ge t_0$, $T \ge T_1$ and $\mu > 0$ such that

$$|a(t)| > |b(t)| + \mu \text{ for } t \ge T_1, \quad t \ge \theta_k(t) \ge T_1 \text{ for } t \ge T \ (k = 1, \dots, m). \tag{4.2}$$

Denote

$$\gamma(t) = |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2}, \quad c(t) = \frac{\bar{a}(t)b(t)}{|a(t)|}. \tag{4.3}$$

Since $\gamma(t) > |a(t)|$ and $|c(t)| = |b(t)|$, the inequality

$$\gamma(t) > |c(t)| + \mu \tag{4.4}$$

is valid for $t \geq T_1$. It can be easily verified that $\gamma, c \in AC_{\mathrm{loc}}([T_1, \infty), \mathbb{C})$.

For the rest of this section we shall denote

$$
\begin{aligned}
\alpha(t) &= 1 - \left|\frac{b(t)}{a(t)}\right| \operatorname{sgn} \operatorname{Re} a(t), \\
\vartheta(t) &= \frac{\operatorname{Re}(\gamma(t)\gamma'(t) - \bar{c}(t)c'(t)) - |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2}.
\end{aligned} \tag{4.5}
$$

The instability properties mentioned in the title of the chapter and other asymptotic properties are studied under the following assumptions:

(i) The numbers $T_1 \geq t_0$, $T \geq T_1$ and $\mu > 0$ are such that (4.2) holds.

(ii) There exist functions $\varkappa, \kappa_k, \varrho : [T, \infty) \to \mathbb{R}$ such that

$$
\begin{aligned}
|\gamma(t)g(t, z, w_1, \ldots, w_m) + c(t)\bar{g}(t, z, w_1, \ldots, w_m)| \leq{}& \varkappa(t)|\gamma(t)z + c(t)\bar{z}| \\
&+ \sum_{k=1}^{m} \kappa_k(t)|\gamma(\theta_k(t))w_k + c(\theta_k(t))\bar{w}_k| + \varrho(t)
\end{aligned}
$$

for $t \geq T$, $z, w_k \in \mathbb{C}$ $(k = 1, \ldots, m)$, where $\varrho$ is continuous on $[T, \infty)$.

(ii$_\mathrm{n}$) There exist numbers $R_n \geq 0$ and functions $\varkappa_n, \kappa_{nk} : [T, \infty) \to \mathbb{R}$ such that

$$
\begin{aligned}
|\gamma(t)g(t, z, w_1, \ldots, w_m) + c(t)\bar{g}(t, z, w_1, \ldots, w_m)| \leq{}& \varkappa_n(t)|\gamma(t)z + c(t)\bar{z}| \\
&+ \sum_{k=1}^{m} \kappa_{nk}(t)|\gamma(\theta_k(t))w_k + c(\theta_k(t))\bar{w}_k|
\end{aligned}
$$

for $t \geq \tau_n \geq T$, $|z| + \sum\limits_{k=1}^{m} |w_k| > R_n$.

(iii) $\beta \in AC_{\mathrm{loc}}([T, \infty), \mathbb{R}_-^0)$ is a function satisfying

$$
\theta_k'(t)\beta(t) \leq -\lambda_k(t) \quad \text{a. e. on } [T, \infty), \tag{4.6}
$$

where $\lambda_k$ is defined for $t \geq T$ by

$$
\lambda_k(t) = \kappa_k(t) + (|A_k(t)| + |B_k(t)|)\frac{\gamma(t) + |c(t)|}{\gamma(\theta_k(t)) - |c(\theta_k(t))|}. \tag{4.7}
$$

(iii$_\mathrm{n}$) $\beta_n \in AC_{\mathrm{loc}}[T, \infty), \mathbb{R}_-^0)$ is a function satisfying

$$
\theta_k'(t)\beta_n(t) \leq -\lambda_{nk}(t) \quad \text{a. e. on } [\tau_n, \infty), \tag{4.8}
$$

where $\lambda_{nk}$ is defined for $t \geq T$ by

$$
\lambda_{nk}(t) = \kappa_{nk}(t) + (|A_k(t)| + |B_k(t)|)\frac{\gamma(t) + |c(t)|}{\gamma(\theta_k(t)) - |c(\theta_k(t))|}. \tag{4.9}
$$

(iv$_n$) $\Lambda_n$ is a real locally Lebesgue integrable function satisfying the inequalities $\beta_n'(t) \geq \Lambda_n(t)\beta_n(t)$, $\Theta_n(t) \geq \Lambda_n(t)$ for almost all $t \in [\tau_n, \infty)$, where $\Theta_n$ is defined by

$$\Theta_n(t) = \alpha(t) \operatorname{Re} a(t) + \vartheta(t) - \varkappa_n(t) + m\beta_n(t). \tag{4.10}$$

Obviously, if $A_k$, $B_k$, $\kappa_k$, $\theta_k'$ are locally absolutely continuous on $[T, \infty)$ and $\lambda_k(t) \geq 0$, $\theta_k'(t) > 0$, the choice $\beta(t) = -\max\limits_{k=1,\dots,m} [\lambda_k(t)(\theta_k'(t))^{-1}]$ is admissible in (iii). Similarly, if $A_k$, $B_k$, $\kappa_{nk}$, $\theta_k'$ are locally absolutely continuous on $[T, \infty)$ and $\lambda_{nk}(t) \geq 0$, $\theta_k'(t) > 0$, the choice $\beta_n(t) = -\max\limits_{k=1,\dots,m} [\lambda_{nk}(t)(\theta_k'(t))^{-1}]$ is admissible in (iii$_n$).

Denote

$$\Theta(t) = \alpha(t) \operatorname{Re} a(t) + \vartheta(t) - \varkappa(t). \tag{4.11}$$

From the assumption (i) it follows that

$$\begin{aligned} |\vartheta| &\leq \frac{|\operatorname{Re}(\gamma\gamma' - \bar{c}c')| + |\gamma c' - \gamma' c|}{\gamma^2 - |c|^2} \leq \frac{(|\gamma'| + |c'|)(|\gamma| + |c|)}{\gamma^2 - |c|^2} \\ &= \frac{|\gamma'| + |c'|}{\gamma - |c|} \leq \frac{1}{\mu}(|\gamma'| + |c'|), \end{aligned}$$

therefore the function $\vartheta$ is locally Lebesgue integrable on $[T, \infty)$, assuming that (i) holds true. If relations $\beta_n \in AC_{\mathrm{loc}}([T, \infty), \mathbb{R}_-)$, $\varkappa_n \in L_{\mathrm{loc}}([T, \infty), \mathbb{R})$ and $\beta_n'(t)/\beta_n(t) \leq \Theta_n(t)$ for almost all $t \geq \tau_n$ together with the conditions (i), (ii$_n$) are fulfilled, then we can choose $\Lambda_n(t) = \Theta_n(t)$ for $t \in [T, \infty)$ in (iv$_n$).

### 4.1.2 Main results

The aim is to generalize the results for ordinary differential equations recalled in Chapter 2 as well as the results contained in [19] (one constant delay) and [21] (one nonconstant delay).

The main result concerning instability is Theorem 4.1. In the proof of Theorem 4.1 below, the following Lemma 4.1 will be utilized.

**Lemma 4.1.** *Let $a_1$, $a_2$, $b_1$, $b_2 \in \mathbb{C}$, $|a_2| > |b_2|$. Then*

$$\operatorname{Re} \frac{a_1 z + b_1 \bar{z}}{a_2 z + b_2 \bar{z}} \geq \frac{\operatorname{Re}(a_1 \bar{a}_2 - b_1 \bar{b}_2) - |a_1 b_2 - a_2 b_1|}{|a_2|^2 - |b_2|^2}$$

*for $z \in \mathbb{C}$, $z \neq 0$.*

The proof is analogous to that of Lemma 3.1 or to the proof of Lemma in [39], p. 131.

**Theorem 4.1.** *Let the assumptions (i), (ii$_0$), (iii$_0$), (iv$_0$) be fulfilled for some $\tau_0 \geq T$. Suppose there exist $t_1 \geq \tau_0$ and $\nu \in (-\infty, \infty)$ such that*

$$\inf_{t \geq t_1} \left[ \int_{t_1}^{t} \Lambda_0(s)\, ds - \ln(\gamma(t) + |c(t)|) \right] \geq \nu. \tag{4.12}$$

*If $z(t)$ is any solution of* (3.1) *satisfying*

$$\min_{\theta(t_1)\le s\le t_1}|z(s)|>R_0,\qquad \Delta(t_1)>R_0e^{-\nu}, \tag{4.13}$$

*where*

$$\theta(t)=\min_{k=1,\dots,m}\theta_k(t),$$

$$\Delta(t)=(\gamma(t)-|c(t)|)|z(t)|+\beta_0(t)\max_{\theta(t)\le s\le t}|z(s)|\sum_{k=1}^{m}\int\limits_{\theta_k(t)}^{t}(\gamma(s)+|c(s)|)\,ds,$$

*then*

$$|z(t)|\ge\frac{\Delta(t_1)}{\gamma(t)+|c(t)|}\exp\left[\int_{t_1}^{t}\Lambda_0(s)\,ds\right] \tag{4.14}$$

*for all $t\ge t_1$ for which $z(t)$ is defined.*

*Proof.* Let $z(t)$ be any solution of (3.2) satisfying (4.13). Consider the Lyapunov functional

$$V(t)=U(t)+\beta_0(t)\sum_{k=1}^{m}\int\limits_{\theta_k(t)}^{t}U(s)\,ds, \tag{4.15}$$

where

$$U(t)=|\gamma(t)z(t)+c(t)\bar z(t)|. \tag{4.16}$$

For brevity we shall denote $w_k(t)=z(\theta_k(t))$ and we shall write the function of variable $t$ simply without indicating the variable $t$, for example, $\gamma$ instead of $\gamma(t)$.

In view of (4.15) we have

$$V'=U'+\beta_0'\sum_{k=1}^{m}\int\limits_{\theta_k(t)}^{t}U(s)ds+m\beta_0|\gamma z+c\bar z|-\sum_{k=1}^{m}\theta_k'\beta_0|\gamma(\theta_k(t))w_k+c(\theta_k(t))\bar w_k| \tag{4.17}$$

for almost all $t\ge t_1$ for which $z(t)$ is defined and $U'(t)$ exists. Put $\mathcal{K}=\{t\ge t_1: z(t)\text{ exists}, |z(t)|>R_0\}$. Clearly $U(t)\ne 0$ for $t\in\mathcal{K}$. The derivative $U'(t)$ exists for almost all $t\in\mathcal{K}$.

Since

$$az+b\bar z=\frac{a}{2|a|}(\gamma z+c\bar z)+\frac{b}{2\gamma}(\gamma\bar z+\bar c z),$$

the equation (3.2) can be written in the form

$$z'=\frac{a}{2|a|}(\gamma z+c\bar z)+\frac{b}{2\gamma}(\gamma\bar z+\bar c z)+\sum_{k=1}^{m}\bigl(A_kw_k+B_k\bar w_k\bigr)+g(t,z,w_1,\dots,w_m). \tag{4.18}$$

Short computation leads to

$$\operatorname{Re}\left[\frac{\gamma a}{2|a|}+\frac{c\bar{b}}{2\gamma}\right]=\operatorname{Re}a, \qquad \frac{b}{2}+\frac{c\bar{a}}{2|a|}=b\frac{\operatorname{Re}a}{a}. \tag{4.19}$$

In view of (4.18) and (4.19) we have

$$\begin{aligned}
UU' &= U\big(\sqrt{(\gamma z+c\bar{z})(\bar{\gamma}\bar{z}+\bar{c}z)}\big)' = \operatorname{Re}\big[(\gamma\bar{z}+\bar{c}z)(\gamma' z+\gamma z'+c'\bar{z}+c\bar{z}')\big] = \\
&= \operatorname{Re}\Big\{(\gamma\bar{z}+\bar{c}z)\Big[\gamma' z+c'\bar{z}+\gamma\Big(\frac{a}{2|a|}(\gamma z+c\bar{z})+\frac{b}{2\gamma}(\gamma\bar{z}+\bar{c}z)+\sum_{k=1}^{m}\big(A_k w_k+B_k\bar{w}_k\big)+g\Big)+ \\
&+ c\Big(\frac{\bar{a}}{2|a|}(\gamma\bar{z}+\bar{c}z)+\frac{\bar{b}}{2\gamma}(\gamma z+c\bar{z})+\sum_{k=1}^{m}\big(\bar{A}_k\bar{w}_k+\bar{B}_k w_k\big)+\bar{g}\Big)\Big]\Big\} \geq \\
&\geq |\gamma z+c\bar{z}|^2\Big(\operatorname{Re}a-|b|\frac{|\operatorname{Re}a|}{|a|}\Big)+\operatorname{Re}\Big\{(\gamma\bar{z}+\bar{c}z)\Big[\gamma' z+c'\bar{z}+\gamma\Big(\sum_{k=1}^{m}\big(A_k w_k+B_k\bar{w}_k\big)+g\Big)+ \\
&+ c\Big(\sum_{k=1}^{m}\big(\bar{A}_k\bar{w}_k+\bar{B}_k w_k\big)+\bar{g}\Big)\Big]\Big\}
\end{aligned}$$

for almost all $t\in\mathcal{K}$.

If we recall the definition of $\alpha(t)$ in (4.5), then

$$\begin{aligned}
UU' &\geq U^2\alpha\operatorname{Re}a+\operatorname{Re}\Big\{(\gamma\bar{z}+\bar{c}z)\big[\gamma\sum_{k=1}^{m}\big(A_k w_k+B_k\bar{w}_k\big)+c\sum_{k=1}^{m}\big(\bar{A}_k\bar{w}_k+\bar{B}_k w_k\big)\big]\Big\}+ \\
&+\operatorname{Re}[(\gamma\bar{z}+\bar{c}z)(\gamma g+c\bar{g})]+\operatorname{Re}[(\gamma\bar{z}+\bar{c}z)(\gamma' z+c'\bar{z})] \geq \\
&\geq U^2\alpha\operatorname{Re}a-U(\gamma+|c|)\Big(\sum_{k=1}^{m}|A_k w_k+B_k\bar{w}_k|\Big)-U|\gamma g+c\bar{g}|+U^2\operatorname{Re}\frac{\gamma' z+c'\bar{z}}{\gamma z+c\bar{z}}.
\end{aligned}$$

Applying Lemma 4.1 to the last term, we obtain

$$\operatorname{Re}\frac{\gamma' z+c'\bar{z}}{\gamma z+c\bar{z}} \geq \vartheta.$$

Using this inequality together with (4.9), taken for $n=0$, and assumption (ii$_0$) we get

$$
\begin{aligned}
UU' \geq & \, U^2(\alpha \operatorname{Re} a + \vartheta - \varkappa_0) - U \sum_{k=1}^{m} \big(\kappa_{0k} |\gamma(\theta_k) w_k + c(\theta_k) \bar{w}_k|\big) - \\
& - U(\gamma + |c|)\Big(\sum_{k=1}^{m} \frac{|A_k||w_k| + |B_k||\bar{w}_k|}{\gamma(\theta_k) - |c(\theta_k)|} (\gamma(\theta_k) - |c(\theta_k)|)\Big) \geq \\
& \geq U^2(\alpha \operatorname{Re} a + \vartheta - \varkappa_0) - \\
& - U \Big\{ \sum_{k=1}^{m} \Big[ \kappa_{0k} + (|A_k| + |B_k|) \frac{\gamma + |c|}{\gamma(\theta_k) - |c(\theta_k)|} \Big] |\gamma(\theta_k) w_k + c(\theta_k) \bar{w}_k| \Big\} \geq \\
& \geq U^2(\alpha \operatorname{Re} a + \vartheta - \varkappa_0) - U \sum_{k=1}^{m} \lambda_{0k} |\gamma(\theta_k) w_k + c(\theta_k) \bar{w}_k|
\end{aligned}
$$

for almost all $t \in \mathcal{K}$.

Consequently,

$$
U' \geq U(\alpha \operatorname{Re} a + \vartheta - \varkappa_0) - \sum_{k=1}^{m} \lambda_{0k} |\gamma(\theta_k) w_k + c(\theta_k) \bar{w}_k| \tag{4.20}
$$

for almost all $t \in \mathcal{K}$. The inequality (4.20) together with the relation (4.17) gives

$$
\begin{aligned}
V' \geq & \, U(\alpha \operatorname{Re} a + \vartheta - \varkappa_0 + m\beta_0) - \sum_{k=1}^{m} (\lambda_{0k} + \theta_k' \beta_0) |\gamma(\theta_k) w_k + c(\theta_k) \bar{w}_k| + \\
& + \beta_0' \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} |\gamma(s) z(s) + c(s) \bar{z}(s)| ds.
\end{aligned}
$$

Using (4.8) and (4.10) for $n = 0$, we obtain

$$
V'(t) \geq U(t) \Theta_0(t) + \beta_0'(t) \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} U(s)\, ds.
$$

Hence, in view of (iv$_0$)

$$
V'(t) - \Lambda_0(t) V(t) \geq 0 \tag{4.21}
$$

for almost all $t \in \mathcal{K}$.

Multiplying (4.21) by $\exp\Big[-\int_{t_1}^{t} \Lambda_0(s)\, ds\Big]$ and integrating over $[t_1, t]$, we get

$$
V(t) \exp\Big[-\int_{t_1}^{t} \Lambda_0(s)\, ds\Big] - V(t_1) \geq 0
$$

on any interval $[t_1, \omega)$ where the solution $z(t)$ exists and satisfies the inequality $|z(t)| > R_0$. Now, with respect to (4.15), (4.16) and $\beta_0 \le 0$, we have

$$(\gamma(t) + |c(t)|)|z(t)| \ge V(t) \ge V(t_1) \exp\left[\int_{t_1}^{t} \Lambda_0(s)\, ds\right] \ge \Delta(t_1) \exp\left[\int_{t_1}^{t} \Lambda_0(s)\, ds\right].$$

If (4.13) is fulfilled, there is a $R > R_0$ such that $\Delta(t_1) > Re^{-\nu}$. By virtue of (4.12) and (4.13) we can easily see that

$$|z(t)| \ge \frac{\Delta(t_1)}{\gamma(t) + |c(t)|} \exp\left[\int_{t_1}^{t} \Lambda_0(s)\, ds\right] \ge Re^{-\nu} e^{\nu} = R$$

for all $t \ge t_1$ for which $z(t)$ is defined. □

*Remark* 4.1. Theorem 4.1 represents a generalization of previous results.

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \dots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \dots, m$, $\theta_1(t) = t - r_0$, where $r_0 > 0$, we get a slight generalization of Theorem 2 of [19]. Notice that in the case $r_0 = 0$ (i. e. $\theta_1(t) \equiv t$) the condition (4.13) takes the form

$$|z(t_1)| > R_0 \max\left\{1, \frac{1}{(\gamma(t_1) - |c(t_1)|)e^{\nu}}\right\}.$$

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \dots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \dots, m$, $\theta_1(t) = \theta(t)$, we get Theorem 1 from [21].

To obtain results on the existence of bounded solutions, we shall suppose that (3.2) satisfies the uniqueness property of solutions. Moreover, we suppose that the delays are bounded, i. e., that the functions $\theta_k$ satisfy the condition

$$t - r \le \theta_k(t) \le t \quad \text{for} \quad t \ge t_0 + r,$$

where $r > 0$ is a constant. Our assumptions imply the existence of numbers $T_1 = t_0 + r$, $T \ge T_1$ and $\mu > 0$ such that

$$|a(t)| > |b(t)| + \mu \text{ for } t \ge T_1, \qquad t \ge \theta_k(t) \ge t - r \text{ for } t \ge T \ (k = 1, \dots, m). \tag{4.2'}$$

In view of this, we replace (4.2) in the assumption (i) with (4.2'). All other assumptions we keep in validity.

In the proof of the following theorem we shall utilize Ważewski topological principle for retarded functional differential equations of Carathéodory type. For details of this theory see the results of K. P. Rybakowski [45].

**Theorem 4.2.** *Let the conditions (i), (ii), (iii) be fulfilled and $\Lambda$, $\theta_k'$ $(k = 1, \dots, m)$ be continuous functions such that the inequality $\Lambda(t) \le \Theta(t)$ holds a. e. on $[T, \infty)$, where $\Theta$ is defined by (4.11). Suppose that $\xi : [T - r, \infty) \to \mathbb{R}$ is a continuous function such that*

$$\Lambda(t) + \beta(t) \sum_{k=1}^{m} \theta_k'(t) \exp\left[ - \int_{\theta_k(t)}^{t} \xi(s)\, ds \right] - \xi(t) > \varrho(t) C^{-1} \exp\left[ - \int_{T}^{t} \xi(s)\, ds \right] \tag{4.22}$$

*for $t \in [T, \infty]$ and some constant $C > 0$. Then there exists a $t_2 > T$ and a solution $z_0(t)$ of* (3.2) *satisfying*

$$|z_0(t)| \le \frac{C}{\gamma(t) - |c(t)|} \exp\left[ \int_{T}^{t} \xi(s)\, ds \right] \tag{4.23}$$

*for $t \ge t_2$.*

*Proof.* Write the equation (3.2) in the form

$$z' = F(t, z_t), \tag{3.2$'$}$$

where $F : J \times \mathcal{C} \to \mathbb{C}$ is defined by

$$\begin{aligned} F(t,\psi) &= a(t)\psi(0) + b(t)\bar{\psi}(0) \\ &+ \sum_{k=1}^{m} [A_k(t)\psi(\theta_k(t) - t) + B_k(t)\bar{\psi}(\theta_k(t) - t)] + g(t, \psi(0), \psi(\theta_1(t) - t), \dots, \psi(\theta_m(t) - t)) \end{aligned}$$

and $z_t$ is the element of $\mathcal{C}$ defined by the relation $z_t(\tilde{\theta}) = z(t + \tilde{\theta})$, $\tilde{\theta} \in [-r, 0]$. Let $\tau > T$. Put

$$\begin{aligned} \widetilde{U}(t, z, \bar{z}) &= |\gamma(t) z + c(t) \bar{z}| - \varphi(t), \\ \varphi(t) &= C \exp\left[ \int_{T}^{t} \xi(s)\, ds \right], \\ \Omega^0 &= \{(t, z) \in (\tau, \infty) \times \mathbb{C} : \widetilde{U}(t, z, \bar{z}) < 0\}, \\ \Omega_{\widetilde{U}} &= \{(t, z) \in (\tau, \infty) \times \mathbb{C} : \widetilde{U}(t, z, \bar{z}) = 0\}. \end{aligned}$$

It can be easily verified that $\Omega^0$ is a polyfacial set generated by the functions $\hat{U}(t) = \tau - t$, $\widetilde{U}(t, z, \bar{z})$ (see Rybakowski [45, p. 134]). It holds that $\Omega_{\widetilde{U}} \subset \partial \Omega^0$. As $(\gamma(t) + |c(t)|)|z(t)| \ge |\gamma(t) z + c(t) \bar{z}|$, we have

$$|z| \ge \frac{\varphi(t)}{\gamma(t) + |c(t)|} = \frac{C}{\gamma(t) + |c(t)|} \exp\left[ \int_{T}^{t} \xi(s)\, ds \right] > 0$$

for $(t, z) \in \Omega_{\widetilde{U}}$. It holds that

$$D^+ \hat{U}(t) = \frac{\partial}{\partial t} (\tau - t) = -1 < 0.$$

Let $(t^*,\zeta) \in \Omega_{\widetilde U}$ and $\phi \in \mathcal{C}$ be such that $\phi(0)=\zeta$ and $(t^*+\tilde\theta, \phi(\tilde\theta)) \in \Omega^0$ for all $\tilde\theta \in [-r,0)$. If $(t,\psi) \in (\tau,\infty)\times\mathcal{C}$, then

$$
\begin{aligned}
&D^+\widetilde U(t,\psi(0),\bar\psi(0)) := \\
&= \limsup_{h\to 0+}(1/h)[\widetilde U(t+h,\psi(0)+hF(t,\psi),\bar\psi(0)+h\bar F(t,\psi)) - \widetilde U(t,\psi(0),\bar\psi(0))] \\
&= \frac{\partial \widetilde U(t,\psi(0),\bar\psi(0))}{\partial t} + \frac{\partial \widetilde U(t,\psi(0),\bar\psi(0))}{\partial z}F(t,\psi) + \frac{\partial \widetilde U(t,\psi(0),\bar\psi(0))}{\partial \bar z}\bar F(t,\psi).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
D^+\widetilde U(t,\psi(0),\bar\psi(0)) &= |\gamma(t)\psi(0)+c(t)\bar\psi(0)| \operatorname{Re}\frac{\gamma'(t)\psi(0)+c'(t)\bar\psi(0)}{\gamma(t)\psi(0)+c(t)\bar\psi(0)} - \varphi'(t) \\
&+ \tfrac12 |\gamma(t)\psi(0)+c(t)\bar\psi(0)|^{-1}\{[\gamma(t)(\gamma(t)\bar\psi(0)+\bar c(t)\psi(0)) \\
&\quad + (\gamma(t)\psi(0)+c(t)\bar\psi(0))\bar c(t)]F(t,\psi) \\
&\qquad + [c(t)(\gamma(t)\bar\psi(0)+\bar c(t)\psi(0)) + \gamma(t)(\gamma(t)\psi(0)+c(t)\bar\psi(0))]\bar F(t,\psi)\}
\end{aligned}
$$

provided that the derivatives $\gamma'(t)$, $c'(t)$ exist and that $\psi(0)\neq 0$. Thus

$$
\begin{aligned}
D^+\widetilde U(t,\psi(0),\bar\psi(0)) &= |\gamma(t)\psi(0)+c(t)\bar\psi(0)| \operatorname{Re}\frac{\gamma'(t)\psi(0)+c'(t)\bar\psi(0)}{\gamma(t)\psi(0)+c(t)\bar\psi(0)} - \varphi'(t) \\
&+ |\gamma(t)\psi(0)+c(t)\bar\psi(0)|^{-1}\operatorname{Re}\{\gamma(t)(\gamma(t)\bar\psi(0)+\bar c(t)\psi(0))F(t,\psi) \\
&\qquad + c(t)(\gamma(t)\bar\psi(0)+\bar c(t)\psi(0))\bar F(t,\psi)\} \\
&= |\gamma(t)\psi(0)+c(t)\bar\psi(0)| \operatorname{Re}\frac{\gamma'(t)\psi(0)+c'(t)\bar\psi(0)}{\gamma(t)\psi(0)+c(t)\bar\psi(0)} - \varphi'(t) \\
&+ |\gamma(t)\psi(0)+c(t)\bar\psi(0)|^{-1}\operatorname{Re}\{(\gamma(t)\bar\psi(0)+\bar c(t)\psi(0))(\gamma(t)F(t,\psi)+c(t)\bar F(t,\psi))\}.
\end{aligned}
$$

Using (4.18), (4.19) and (ii), similarly to the proof of Theorem 4.1, we obtain

$$
\begin{aligned}
&D^+\widetilde U(t,\psi(0),\bar\psi(0)) \ge |\gamma(t)\psi(0)+c(t)\bar\psi(0)|\alpha(t)\operatorname{Re} a(t) \\
&- \sum_{k=1}^m |A_k(t)\psi(\theta_k(t)-t)+B_k(t)\bar\psi(\theta_k(t)-t)|(\gamma(t)+|c(t)|) - \varkappa(t)|\gamma(t)\psi(0)+c(t)\bar\psi(0)| \\
&- \sum_{k=1}^m \kappa_k(t)|\gamma(\theta_k(t))\psi(\theta_k(t)-t)+c(\theta_k(t))\bar\psi(\theta_k(t)-t)| \\
&\qquad\qquad + \vartheta(t)|\gamma(t)\psi(0)+c(t)\bar\psi(0)| - \varrho(t) - \varphi'(t)
\end{aligned}
$$

and consequently, with respect to (iii),

$$\begin{aligned}
D^+\widetilde{U}(t,\psi(0),\bar{\psi}(0)) \geq (\alpha(t)\operatorname{Re} a(t) + \vartheta(t) - \varkappa(t))|\gamma(t)\psi(0) + c(t)\bar{\psi}(0)| \\
- \sum_{k=1}^{m} \lambda_k(t)|\gamma(\theta_k(t))\psi(\theta_k(t)-t) + c(\theta_k(t))\bar{\psi}(\theta_k(t)-t)| - \varrho(t) - \varphi'(t) \geq \\
\Theta(t)|\gamma(t)\psi(0)+c(t)\bar{\psi}(0)| + \beta(t)\sum_{k=1}^{m}\theta_k'(t)|\gamma(\theta_k(t))\psi(\theta_k(t)-t)+c(\theta_k(t))\bar{\psi}(\theta_k(t)-t)| \\
- \varrho(t) - \varphi'(t) \geq \\
\Lambda(t)|\gamma(t)\psi(0)+c(t)\bar{\psi}(0)| + \beta(t)\sum_{k=1}^{m}\theta_k'(t)|\gamma(\theta_k(t))\psi(\theta_k(t)-t) + c(\theta_k(t))\bar{\psi}(\theta_k(t)-t)| \\
- \varrho(t) - \varphi'(t)
\end{aligned}$$

for almost all $t \in (\tau,\infty)$ and for $\psi \in \mathcal{C}$ sufficiently close to $\phi$. Replacing $t$ and $\psi$ by $t^*$ and $\phi$, respectively, in the last expression, we get

$$\begin{aligned}
&\Lambda(t^*)|\gamma(t^*)\phi(0) + c(t^*)\bar{\phi}(0)| \\
&\quad + \beta(t^*)\sum_{k=1}^{m}\theta_k'(t^*)|\gamma(\theta_k(t^*))\phi(\theta_k(t^*)-t^*) + c(\theta_k(t^*))\bar{\phi}(\theta_k(t^*)-t^*)| - \varrho(t^*) - \varphi'(t^*) \\
&\geq \Lambda(t^*)|\gamma(t^*)\zeta + c(t^*)\bar{\zeta}| + \beta(t^*)\sum_{k=1}^{m}\theta_k'(t^*)\varphi(\theta_k(t^*)) - \varrho(t^*) - \varphi'(t^*) \\
&\geq \Lambda(t^*)\varphi(t^*) + \beta(t^*)\sum_{k=1}^{m}\theta_k'(t^*)\varphi(\theta_k(t^*)) - \varrho(t^*) - \varphi'(t^*) \\
&= \Lambda(t^*)C\exp\left[\int_T^{t^*}\xi(s)\,ds\right] + \beta(t^*)\sum_{k=1}^{m}\theta_k'(t^*)C\exp\left[\int_T^{\theta_k(t^*)}\xi(s)\,ds\right] \\
&\qquad - \varrho(t^*) - C\xi(t^*)\exp\left[\int_T^{t^*}\xi(s)\,ds\right] \\
&= \left\{\Lambda(t^*) + \beta(t^*)\sum_{k=1}^{m}\theta_k'(t^*)\exp\left[-\int_{\theta_k(t^*)}^{t^*}\xi(s)\,ds\right] - \xi(t^*)\right\} C\exp\left[\int_T^{t^*}\xi(s)\,ds\right] \\
&\quad - \varrho(t^*) > 0.
\end{aligned}$$

Therefore, in view of the continuity, $D^+\widetilde{U}(t,\psi(0),\bar{\psi}(0)) > 0$ holds for $\psi$ sufficiently close to $\phi$ and almost all $t$ sufficiently close to $t^*$. Hence $\Omega^0$ is a regular polyfacial set with respect to (3.2′).

Choose $Z = \{(t_2, z) \in \Omega^0 \cup \Omega_{\widetilde{U}}\}$, where $t_2 > \tau + r$ is fixed. It can be easily verified that $Z \cap \Omega_{\widetilde{U}}$ is a retract of $\Omega_{\widetilde{U}}$, but $Z \cap \Omega_{\widetilde{U}}$ is not a retract of $Z$. Let $\eta \in \mathcal{C}$ be such that

$\eta(0) = 1$ and $0 \leq \eta(\theta) < 1$ for $\theta \in [-r, 0)$. Define the mapping $p : Z \to \mathcal{C}$ for $(t_2, z) \in Z$ by the relation

$$p(t_2, z)(\theta) = \frac{\varphi(t_2+\theta)\eta(\theta)}{(\gamma^2(t_2+\theta) - |c(t_2+\theta)|^2)\varphi(t_2)}[(\gamma(t_2)\gamma(t_2+\theta) - \bar{c}(t_2)c(t_2+\theta))z + (\gamma(t_2+\theta)c(t_2) - \gamma(t_2)c(t_2+\theta))\bar{z}].$$

The mapping $p$ is continuous and it holds that

$$p(t_2, z)(0) = z \quad \text{for } (t_2, z) \in Z, \qquad p(t_2, 0)(\theta) = 0 \quad \text{for } \theta \in [-r, 0].$$

Since

$$\gamma(t_2+\theta)p(t_2, z)(\theta) + c(t_2+\theta)\overline{p(t_2, z)(\theta)} = \frac{\varphi(t_2+\theta)\eta(\theta)}{\varphi(t_2)}(\gamma(t_2)z + c(t_2)\bar{z}),$$

we have

$$|\gamma(t_2)z + c(t_2)\bar{z}| < \varphi(t_2)$$

and

$$|\gamma(t_2+\theta)p(t_2, z)(\theta) + c(t_2+\theta)\overline{p(t_2, z)(\theta)}| < \varphi(t_2+\theta) \tag{4.24}$$

for $(t_2, z) \in Z \cap \Omega^0$ and $\theta \in [-r, 0]$. Clearly, the inequality (4.24) holds also for $(t_2, z) \in Z \cap \Omega_{\tilde{U}}$ and $\theta \in [-r, 0)$.

Using the topological principle for retarded functional differential equations (see Rybakowski [45, Theorem 2.1]), we infer that there is a solution $z_0(t)$ of (3.2) such that $(t, z_0(t)) \in \Omega^0$ for all $t \geq t_2$ for which the solution $z_0(t)$ exists. Obviously $z_0(t)$ exists for all $t \geq t_2$ and

$$(\gamma(t) - |c(t)|)|z_0(t)| \leq |\gamma(t)z_0(t) + c(t)\bar{z}_0(t)| \leq \varphi(t) \qquad \text{for } t \geq t_2.$$

Hence

$$|z_0(t)| \leq \frac{\varphi(t)}{\gamma(t) - |c(t)|} \qquad \text{for } t \geq t_2.$$

□

*Remark* 4.2. If $\theta'_k(t) \geq 0$ for $k = 1, \dots, m$, $\eta_1(t)\Lambda(t) > |\beta(t)| \sum_{k=1}^{m} \theta'_k(t) + C^{-1}\varrho(t) > 0$, where $0 < \eta_1(t) \leq 1$, the functions $\eta_1$, $\Lambda$ and $\theta'_k$ are continuous on $[T, \infty)$ and $\Lambda(t) \leq \Theta(t)$ a. e. on $[T, \infty)$, then the choice of $\xi$ is possible in (4.22) such that $\xi(t) = \eta_1(t)\Lambda(t) + \beta(t) \sum_{k=1}^{m} \theta'_k(t) - C^{-1}\varrho(t)$ on $[T, \infty)$. Moreover, in some cases, the condition $|\beta(t)| \sum_{k=1}^{m} \theta'_k(t) + C^{-1}\varrho(t) > 0$ can be omitted if Theorem 4.2 is used. For instance, the identity $|\beta(t)| \sum_{k=1}^{m} \theta'_k(t) + C^{-1}\varrho(t) \equiv 0$ implies $\beta(t) \sum_{k=1}^{m} \theta'_k(t) \equiv 0$, $\varrho(t) \equiv 0$ and consequently, in view of (4.6), (4.7), (ii), we have $\lambda_k(t) \equiv 0$, $\kappa_k(t) \equiv 0$, $A_k(t) \equiv 0$, $B_k(t) \equiv 0$, $g(t, 0, 0, \dots, 0) \equiv 0$. Thus the equation (3.2) has the trivial solution $z_0(t) \equiv 0$ in this case.

*Remark* 4.3. Theorem 4.2 generalizes theorems contained in previous papers.

Taking $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \dots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \dots, m$, $\theta_1(t) \equiv t - r_0$, where $0 \le r_0 \le r$, in Theorem 4.2, we obtain a generalization of Theorem 5 of [19].

Taking $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \dots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \dots, m$, $\theta_1(t) = \theta(t)$, in Theorem 4.2, we get Theorem 4 of [21].

**Theorem 4.3.** *Suppose that the hypotheses (i), (ii), (ii$_n$), (iii), (iii$_n$), (iv$_n$) are fulfilled for $\tau_n \ge T$ and $n \in \mathbb{N}$, where $R_n > 0$, $\inf_{n\in\mathbb{N}} R_n = 0$. Let $\Lambda$, $\theta_k'$ be continuous functions satisfying the inequality $\Lambda(t) \le \Theta(t)$ a. e. on $[T, \infty)$, where $\Theta$ is defined by (4.11). Assume that $\xi : [T - r, \infty) \to \mathbb{R}$ is a continuous function such that*

$$\Lambda(t) + \beta(t) \sum_{k=1}^{m} \theta_k'(t) \exp\left[ - \int_{\theta_k(t)}^{t} \xi(s)\, ds \right] - \xi(t) > \varrho(t) C^{-1} \exp\left( - \int_{T}^{t} \xi(s)\, ds \right) \quad (4.25)$$

*for $t \in [T, \infty)$ and some constant $C > 0$. Suppose*

$$\limsup_{t\to\infty} \left[ \int_T^t (\Lambda_n(s) - \xi(s))\, ds + \ln \frac{\gamma(t) - |c(t)|}{\gamma(t) + |c(t)|} \right] = \infty, \quad (4.26)$$

$$\lim_{t\to\infty} \left[ \beta_n(t) \max_{\theta(t)\le s\le t} \frac{\exp\left[\int_T^s \xi(\sigma)\, d\sigma\right]}{\gamma(s) - |c(s)|} \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} (\gamma(s) + |c(s)|)\, ds \right] = 0, \quad (4.27)$$

$$\inf_{\tau_n \le s \le t < \infty} \left[ \int_s^t \Lambda_n(\sigma)\, d\sigma - \ln(\gamma(t) + |c(t)|) \right] \ge \nu \quad (4.28)$$

*for $n \in \mathbb{N}$, where $\theta(t) = \min_{k=1,\dots,m} \theta_k(t)$ and $\nu \in (-\infty, \infty)$. Then there exists a solution $z_0(t)$ of (3.2) such that*

$$\lim_{t\to\infty} \min_{\theta(t)\le s\le t} |z_0(s)| = 0. \quad (4.29)$$

*Proof.* By the use of Theorem 4.2 we observe that there is a $t_2 \ge T$ and a solution $z_0(t)$ of (3.2) with property

$$|z_0(t)| \le \frac{C}{\gamma(t) - |c(t)|} \exp\left[ \int_T^t \xi(s)\, ds \right] \quad (4.30)$$

for $t \ge t_2$. Suppose that (4.29) is not satisfied. Then there is $\varepsilon_0 > 0$ such that

$$\limsup_{t\to\infty} \min_{\theta(t)\le s\le t} |z_0(s)| > \varepsilon_0.$$

Choose $N \in \mathbb{N}$ such that

$$\max\left\{ R_N, \frac{2}{\mu} R_N e^{-\nu} \right\} < \varepsilon_0.$$

It holds that

$$\min_{\theta(\tau)\le s\le \tau} |z_0(s)| > \max\left\{ R_N, \frac{2}{\mu} R_N e^{-\nu} \right\} \quad (4.31)$$

for some $\tau > \max\{T, \tau_N, t_2\}$. In view of (4.27) we can suppose that

$$|\beta_N(\tau)| C \max_{\theta(\tau) \le s \le \tau} \frac{\exp\left[\int_T^s \xi(\sigma)\, d\sigma\right]}{\gamma(s) - |c(s)|} \sum_{k=1}^m \int_{\theta_k(\tau)}^{\tau} (\gamma(s) + |c(s)|)\, ds < \frac{1}{2} R_N e^{-\nu}. \qquad (4.32)$$

Therefore, taking into account (4.4), (4.30), (4.31), (4.32) and the nonpositiveness of $\beta_N$, we have

$$\begin{aligned}
&(\gamma(\tau) - |c(\tau)|)|z_0(\tau)| + \beta_N(\tau) \max_{\theta(\tau) \le s \le \tau} |z_0(s)| \sum_{k=1}^m \int_{\theta_k(\tau)}^{\tau} (\gamma(s) + |c(s)|)\, ds \\
&\ge (\gamma(\tau) - |c(\tau)|)|z_0(\tau)| + \beta_N(\tau) C \max_{\theta(\tau) \le s \le \tau} \frac{\exp\left[\int_T^s \xi(\sigma)\, d\sigma\right]}{\gamma(s) - |c(s)|} \sum_{k=1}^m \int_{\theta_k(\tau)}^{\tau} (\gamma(s) + |c(s)|)\, ds \\
&\ge \mu \frac{2}{\mu} R_N e^{-\nu} - \frac{1}{2} R_N e^{-\nu} > R_N e^{-\nu}.
\end{aligned}$$

Moreover (4.28) implies

$$\inf_{\tau \le t < \infty} \left[\int_\tau^t \Lambda_N(s)\, ds - \ln(\gamma(t) + |c(t)|)\right] \ge \nu > -\infty.$$

By Theorem 4.1 we obtain an estimation

$$|z_0(t)| \ge \frac{\Psi(\tau)}{\gamma(t) + |c(t)|} \exp\left[\int_\tau^t \Lambda_N(s)\, ds\right] \qquad (4.33)$$

for all $t \ge \tau$, $\Psi$ being defined by

$$\Psi(\tau) = (\gamma(\tau) - |c(\tau)|)|z_0(\tau)| + \beta_N(\tau) \max_{\theta(\tau) \le s \le \tau} |z_0(s)| \sum_{k=1}^m \int_{\theta_k(\tau)}^{\tau} (\gamma(s) + |c(s)|)\, ds.$$

The relation (4.30) together with (4.33) yield

$$\frac{\Psi(\tau)}{\gamma(t) + |c(t)|} \exp\left[\int_\tau^t \Lambda_N(s)\, ds\right] \le \frac{C}{\gamma(t) - |c(t)|} \exp\left[\int_T^t \xi(s)\, ds\right],$$

i. e.

$$\int_T^t [\Lambda_N(s) - \xi(s)]\, ds + \ln \frac{\gamma(t) - |c(t)|}{\gamma(t) + |c(t)|} \le \int_T^\tau \Lambda_N(s)\, ds - \ln[C^{-1}\Psi(\tau)]$$

for $t \ge \tau$. However, the last inequality contradicts (4.26) and Theorem 4.3 is proved. □

*Remark* 4.4. Theorem 4.3 is a generalization of results published in the papers [19] and [21].

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = t - r_0$, where $0 \le r_0 \le r$, we obtain a generalization of Theorem 8 of [19]. Notice that in the case $r_0 = 0$ (i. e. $\theta_1(t) \equiv t$) the condition (4.27) can be omitted and (4.29) is of the form $\lim_{t\to\infty} |z(t)| = 0$.

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = \theta(t)$, we get Theorem 6 from [21].

### 4.1.3 Corollaries and examples

From Theorem 4.1 we easily obtain several corollaries.

**Corollary 4.1.** *Let the assumptions of Theorem 4.1 be fulfilled with* $R_0 > 0$. *If*

$$\liminf_{t\to\infty}\left[\int_{t_1}^{t} \Lambda_0(s)\,ds - \ln(\gamma(t)+|c(t)|)\right] = \varsigma > \nu, \tag{4.34}$$

*then to any* $\varepsilon$, $0<\varepsilon<R_0e^{\varsigma-\nu}$, *there is a* $t_2 \ge t_1$ *such that*

$$|z(t)| > \varepsilon \tag{4.35}$$

*for all* $t\ge t_2$ *for which* $z(t)$ *is defined.*

*Proof.* Without loss of generality we can assume $\varepsilon > R_0$. Choose $\chi$, $0<\chi<1$ such that $R_0<\varepsilon<\chi R_0 e^{\varsigma-\nu}$. In view of (4.34) there is $t_2\ge t_1$ such that

$$\int_{t_1}^{t} \Lambda_0(s)\,ds - \ln(\gamma(t)+|c(t)|) > \varsigma + \ln\chi$$

for $t \ge t_2$. Hence

$$\int_{t_1}^{t} \Lambda_0(s)\,ds - \ln(\gamma(t)+|c(t)|) > \nu + \ln\frac{\varepsilon}{R_0}$$

for $t\ge t_2$. The estimation (4.14) together with (4.13) now yield

$$|z(t)| > R_0 e^{-\nu} e^{\nu}\frac{\varepsilon}{R_0} = \varepsilon$$

for all $t\ge t_2$ for which $z(t)$ is defined. □

**Corollary 4.2.** *Let the assumptions of Theorem 4.1 be fulfilled with* $R_0>0$. *If*

$$\lim_{t\to\infty}\left[\int_{t_1}^{t} \Lambda_0(s)\,ds - \ln(\gamma(t)+|c(t)|)\right] = \infty, \tag{4.36}$$

*then for any* $\varepsilon>0$ *there exists a* $t_2\ge t_1$ *such that* (4.35) *holds for all* $t\ge t_2$ *for which* $z(t)$ *is defined.*

The efficiency of Theorem 4.1 and Corollary 4.2 is demonstrated in the following example.

**Example 4.1.** Consider the equation (3.2) where $a(t)\equiv 4+3i$, $b(t)\equiv 2$, $A_k(t)\equiv 0$, $B_k(t)\equiv 0$, $\theta_k(t) = k\ln t$ for $k=1,\dots,m$, $g(t,z,w_1,\dots,w_m) = z+\sum_{k=1}^{m}\frac{k}{mt}\,\mathrm{e}^{-t}\,w_k$. Suppose $t_0=m$ and $T\ge \mathrm{e}^m$. Then $\gamma(t) = |a(t)|+\sqrt{|a(t)|^2-|b(t)|^2}\equiv 5+\sqrt{21}$, $c(t) = \bar a(t)b(t)/|a(t)|\equiv \frac{8-6i}{5}$. Further,

$$|\gamma(t)g(t,z,w_1,\dots,w_m)+c(t)\bar g(t,z,w_1,\dots,w_m)| \le |\gamma(t)z+c(t)\bar z| + \sum_{k=1}^{m}\frac{k}{mt}\,\mathrm{e}^{-t}\,|\gamma(\theta(t))w_k+c(\theta(t))\bar w_k|.$$

Taking $\varkappa_0(t)\equiv 1$, $\kappa_{0k}(t)=\frac{k}{mt}\,\mathrm{e}^{-t}$, $\vartheta(t)\equiv 0$, $\alpha(t)\equiv \frac{3}{5}$, $\tau_0=T$, $R_0=0$, $\beta_0(t)=-\frac{1}{m}\,\mathrm{e}^{-t}$, $\Lambda_0(t)=\Theta_0(t)=\frac{3}{5}\cdot 4-1+m\beta_0(t)=\frac{7}{5}-m\frac{1}{m}\,\mathrm{e}^{-t}=\frac{7}{5}-\mathrm{e}^{-t}$ $(>0)$ in Theorem 4.1, we have $\theta_k'(t)\beta_0(t)\le -\lambda_{0k}(t)$, $\beta_0'(t)\ge \Theta_0(t)\beta_0(t)$ for $t\in[T,\infty)$ and we are able to apply Theorem 4.1 and Corollary 4.2 to the considered equation.

The following corollary is a consequence of Theorem 4.2.

**Corollary 4.3.** *Let the assumptions of Theorem 4.2 be satisfied. If*

$$\limsup_{t\to\infty}\left[\frac{1}{\gamma(t)-|c(t)|}\exp\left(\int_T^t \xi(s)\,ds\right)\right]<\infty,$$

*then there is a bounded solution $z_0(t)$ of* (3.2)*. If*

$$\lim_{t\to\infty}\left[\frac{1}{\gamma(t)-|c(t)|}\exp\left(\int_T^t \xi(s)\,ds\right)\right]=0,$$

*then there is a solution $z_0(t)$ of* (3.2) *such that*

$$\lim_{t\to\infty} z_0(t)=0.$$

The next example shows how Theorem 4.2 and Corollary 4.3 (namely the first part) can be used.

**Example 4.2.** Consider the equation (3.2) where $a(t)\equiv 4+3i$, $b(t)\equiv i$, $A_k(t)\equiv 0$, $B_k(t)\equiv 0$, $\theta_k(t)=t-\mathrm{e}^{-kt}$ for $k=1,\dots,m$, $g(t,z,w_1,\dots,w_m)=\frac{1}{2}z+\sum\limits_{k=1}^{m}\frac{1}{4m}w_k+\mathrm{e}^{-t}$. Obviously $t-1\le\theta_k(t)\le t$ and $\theta_k'(t)=1+k\,\mathrm{e}^{-kt}\ge 1>0$ for $t\ge 0$. Suppose $t_0=1$ and $T\ge 2$. Then $\gamma(t)=|a(t)|+\sqrt{|a(t)|^2-|b(t)|^2}\equiv 5+2\sqrt{6}$, $c(t)=\bar a(t)b(t)/|a(t)|\equiv\frac{3+4i}{5}$. Further,

$$\begin{aligned}|\gamma(t)g(t,z,&w_1,\dots,w_m)+c(t)\bar g(t,z,w_1,\dots,w_m)|\le\\ &\le\frac{\gamma+|c|}{\gamma-|c|}\frac{1}{2}|\gamma(t)z+c(t)\bar z|+\frac{\gamma+|c|}{\gamma-|c|}\sum_{k=1}^{m}\Big[\frac{1}{4m}|\gamma(\theta(t))w_k+c(\theta(t))\bar w_k|\Big]+\mathrm{e}^{-t}=\\ &=\frac{\sqrt{3}}{\sqrt{2}}\frac{1}{2}|\gamma(t)z+c(t)\bar z|+\frac{\sqrt{3}}{\sqrt{2}}\sum_{k=1}^{m}\Big[\frac{1}{4m}|\gamma(\theta(t))w_k+c(\theta(t))\bar w_k|\Big]+\mathrm{e}^{-t}.\end{aligned}$$

If we take $\varkappa(t) \equiv \frac{\sqrt{3}}{2\sqrt{2}}$, $\kappa_k(t) = \frac{\sqrt{3}}{4m\sqrt{2}}$, $\vartheta(t) \equiv 0$, $\alpha(t) \equiv \frac{4}{5}$, $\beta(t) = -\frac{\sqrt{3}}{4m\sqrt{2}}$, $\Lambda(t) = \Theta(t) = \frac{4}{5} \cdot 4 - \frac{\sqrt{3}}{2\sqrt{2}} = \frac{16}{5} - \frac{\sqrt{3}}{2\sqrt{2}}$ in Theorem 4.2, we observe that $\theta_k'(t)\beta(t) = -(1 + k\,\mathrm{e}^{-kt})\frac{\sqrt{3}}{4m\sqrt{2}} \leq -\frac{\sqrt{3}}{4m\sqrt{2}} = -\lambda_k(t)$ for $t \in [T, \infty)$. Then for $\xi \equiv 0$ and $C = 1$ we have

$$\Lambda(t) + \beta(t) \sum_{k=1}^{m} \theta_k'(t) \exp\left[-\int_{\theta_k(t)}^{t} \xi(s)\,ds\right] - \xi(t) = \frac{16}{5} - \frac{\sqrt{3}}{2\sqrt{2}} - \frac{\sqrt{3}}{4m\sqrt{2}} \sum_{k=1}^{m} (1 + k\,\mathrm{e}^{-kt}) \geq$$
$$\geq \frac{16}{5} - \frac{\sqrt{3}}{2\sqrt{2}} - \frac{\sqrt{3}}{4m\sqrt{2}} \cdot 2m = \frac{16}{5} - \frac{\sqrt{3}}{\sqrt{2}} > 1 > \mathrm{e}^{-t} = \varrho(t) C^{-1} \exp\left(-\int_{T}^{t} \xi(s)\,ds\right)$$

for $t \in [T, \infty)$, and

$$\limsup_{t\to\infty} \left[\frac{1}{\gamma(t) - |c(t)|} \exp\left(\int_T^t \xi(s)\,ds\right)\right] = \frac{1}{4 + 2\sqrt{6}} < \infty,$$

hence the assertions of Theorem 4.2 and the first part of Corollary 4.3 hold true.

## 4.2 The case $\liminf\limits_{t\to\infty}(|\operatorname{Im} a(t)| - |b(t)|) > 0$

Instead of the case $\liminf\limits_{t\to\infty}(|a(t)| - |b(t)|) > 0$ investigated in Section 4.1, we consider the equation (3.2) in the case when

$$\liminf_{t\to\infty}\big(|\operatorname{Im} a(t)| - |b(t)|\big) > 0 \tag{4.37}$$

and study the behavior of solutions of (3.2) under this assumption. This situation corresponds to the case when the singular point 0 of the system (1.3) is a centre or a focus. In fact, this section is an unstable analogy of Section 3.3.

A simple example which is situated after Corollary 4.5 in Subsection 4.2.3 shows that, in some cases, the results of this section can be applied more suitable than those given in Section 4.1.

### 4.2.1 Assumptions

Regarding (4.37) and since the delay functions $\theta_k$ satisfy $\lim\limits_{t\to\infty} \theta_k(t) = \infty$, there are numbers $T_1 \geq t_0$, $T \geq T_1$ and $\mu > 0$ such that

$$|\operatorname{Im} a(t)| > |b(t)| + \mu \text{ for } t \geq T_1, \quad t \geq \theta_k(t) \geq T_1 \text{ for } t \geq T\ (k = 1, \dots, m). \tag{4.38}$$

Denote

$$\tilde{\gamma}(t) = \operatorname{Im} a(t) + \sqrt{(\operatorname{Im} a(t))^2 - |b(t)|^2}\,\operatorname{sgn}(\operatorname{Im} a(t)), \quad \tilde{c}(t) = -ib(t). \tag{4.39}$$

Notice that, unlike the function $\gamma$ from Section 4.1, the above defined function $\tilde{\gamma}$ need not be positive.

Since $|\tilde{\gamma}(t)| > |\operatorname{Im} a(t)|$ and $|\tilde{c}(t)| = |b(t)|$, the inequality

$$|\tilde{\gamma}(t)| > |\tilde{c}(t)| + \mu \tag{4.40}$$

is valid for $t \geq T_1$. It can be easily verified that $\tilde{\gamma}, \tilde{c} \in AC_{\mathrm{loc}}([T_1, \infty), \mathbb{C})$.

For the rest of this section we shall denote

$$\tilde{\vartheta}(t) = \frac{\operatorname{Re}(\tilde{\gamma}(t)\tilde{\gamma}'(t) - \bar{\tilde{c}}(t)\tilde{c}'(t)) - |\tilde{\gamma}(t)\tilde{c}'(t) - \tilde{\gamma}'(t)\tilde{c}(t)|}{\tilde{\gamma}^2(t) - |\tilde{c}(t)|^2}. \tag{4.41}$$

The instability and boundedness of solutions are studied subject to suitable subsets of the following assumptions:

(i) The numbers $T_1 \geq t_0$, $T \geq T_1$ and $\mu > 0$ are such that (4.38) holds.

(ii) There exist functions $\tilde{\varkappa}, \tilde{\kappa}_k, \varrho : [T, \infty) \to \mathbb{R}$ such that

$$|\tilde{\gamma}(t)g(t, z, w_1, \ldots, w_m) + \tilde{c}(t)\bar{g}(t, z, w_1, \ldots, w_m)| \leq \tilde{\varkappa}(t)|\tilde{\gamma}(t)z + \tilde{c}(t)\bar{z}| + \sum_{k=1}^{m} \tilde{\kappa}_k(t)|\tilde{\gamma}(\theta_k(t))w_k + \tilde{c}(\theta_k(t))\bar{w}_k| + \varrho(t)$$

for $t \geq T$, $z, w_k \in \mathbb{C}$ $(k = 1, \ldots, m)$, where $\varrho$ is continuous on $[T, \infty)$.

(ii$_\mathrm{n}$) There exist numbers $R_n \geq 0$ and functions $\tilde{\varkappa}_n, \tilde{\kappa}_{nk} : [T, \infty) \to \mathbb{R}$ such that

$$|\tilde{\gamma}(t)g(t, z, w_1, \ldots, w_m) + \tilde{c}(t)\bar{g}(t, z, w_1, \ldots, w_m)| \leq \tilde{\varkappa}_n(t)|\tilde{\gamma}(t)z + \tilde{c}(t)\bar{z}| + \sum_{k=1}^{m} \tilde{\kappa}_{nk}(t)|\tilde{\gamma}(\theta_k(t))w_k + \tilde{c}(\theta_k(t))\bar{w}_k|$$

for $t \geq \tau_n \geq T$, $|z| + \sum_{k=1}^{m} |w_k| > R_n$.

(iii) $\tilde{\beta} \in AC_{\mathrm{loc}}([T, \infty), \mathbb{R}^0_-)$ is a function satisfying

$$\theta_k'(t)\tilde{\beta}(t) \leq -\tilde{\lambda}_k(t) \quad \text{a. e. on } [T, \infty), \tag{4.42}$$

where $\tilde{\lambda}_k$ is defined for $t \geq T$ by

$$\tilde{\lambda}_k(t) = \tilde{\kappa}_k(t) + (|A_k(t)| + |B_k(t)|)\frac{|\tilde{\gamma}(t)| + |\tilde{c}(t)|}{|\tilde{\gamma}(\theta_k(t))| - |\tilde{c}(\theta_k(t))|}. \tag{4.43}$$

(iii$_\mathrm{n}$) $\tilde{\beta}_n \in AC_{\mathrm{loc}}[T, \infty), \mathbb{R}^0_-)$ is a function satisfying

$$\theta_k'(t)\tilde{\beta}_n(t) \leq -\tilde{\lambda}_{nk}(t) \quad \text{a. e. on } [\tau_n, \infty), \tag{4.44}$$

where $\tilde{\lambda}_{nk}$ is defined for $t \geq T$ by

$$\tilde{\lambda}_{nk}(t) = \tilde{\kappa}_{nk}(t) + (|A_k(t)| + |B_k(t)|)\frac{|\tilde{\gamma}(t)| + |\tilde{c}(t)|}{|\tilde{\gamma}(\theta_k(t))| - |\tilde{c}(\theta_k(t))|}. \tag{4.45}$$

(iv$_\mathrm{n}$) $\tilde{\Lambda}_n$ is a real locally Lebesgue integrable function satisfying the inequalities $\tilde{\beta}_n'(t) \geq \tilde{\Lambda}_n(t)\tilde{\beta}_n(t)$, $\tilde{\Theta}_n(t) \geq \tilde{\Lambda}_n(t)$ for almost all $t \in [\tau_n, \infty)$, where $\tilde{\Theta}_n$ is defined by

$$\tilde{\Theta}_n(t) = \operatorname{Re} a(t) + \tilde{\vartheta}(t) - \tilde{\varkappa}_n(t) + m\tilde{\beta}_n(t). \tag{4.46}$$

Obviously, if $A_k$, $B_k$, $\tilde{\kappa}_k$, $\theta_k'$ are locally absolutely continuous on $[T, \infty)$ and $\tilde{\lambda}_k(t) \geq 0$, $\theta_k'(t) > 0$, the choice $\tilde{\beta}(t) = -\max\limits_{k=1,\dots,m}[\tilde{\lambda}_k(t)(\theta_k'(t))^{-1}]$ is admissible in (iii). Similarly, if $A_k$, $B_k$, $\tilde{\kappa}_{nk}$, $\theta_k'$ are locally absolutely continuous on $[T, \infty)$ and $\tilde{\lambda}_{nk}(t) \geq 0$, $\theta_k'(t) > 0$, the choice $\tilde{\beta}_n(t) = -\max\limits_{k=1,\dots,m}[\tilde{\lambda}_{nk}(t)(\theta_k'(t))^{-1}]$ is admissible in (iii$_\mathrm{n}$).

Denote

$$\tilde{\Theta}(t) = \operatorname{Re} a(t) + \tilde{\vartheta}(t) - \tilde{\varkappa}(t). \tag{4.47}$$

From the assumption (i) it follows that

$$\begin{aligned} |\tilde{\vartheta}| &\leq \frac{|\operatorname{Re}(\tilde{\gamma}\tilde{\gamma}' - \bar{\tilde{c}}\tilde{c}')| + |\tilde{\gamma}c' - \tilde{\gamma}'c|}{\tilde{\gamma}^2 - |\tilde{c}|^2} \leq \frac{(|\tilde{\gamma}'| + |\tilde{c}'|)(|\tilde{\gamma}| + |\tilde{c}|)}{\tilde{\gamma}^2 - |\tilde{c}|^2} \\ &= \frac{|\tilde{\gamma}'| + |\tilde{c}'|}{|\tilde{\gamma}| - |\tilde{c}|} \leq \frac{1}{\mu}(|\tilde{\gamma}'| + |\tilde{c}'|), \end{aligned}$$

therefore the function $\tilde{\vartheta}$ is locally Lebesgue integrable on $[T, \infty)$, assuming that (i) holds true. If the relations $\tilde{\beta}_n \in AC_{\mathrm{loc}}([T, \infty), \mathbb{R}_-)$, $\tilde{\varkappa}_n \in L_{\mathrm{loc}}([T, \infty), \mathbb{R})$ and $\tilde{\beta}_n'(t)/\tilde{\beta}_n(t) \leq \tilde{\Theta}_n(t)$ for almost all $t \geq \tau_n$ together with the conditions (i), (ii$_\mathrm{n}$) are fulfilled, then we can choose $\tilde{\Lambda}_n(t) = \tilde{\Theta}_n(t)$ for $t \in [T, \infty)$ in (iv$_\mathrm{n}$).

### 4.2.2 Main results

The aim is to generalize the results for ordinary differential equations recalled in Chapter 2 as well as the results contained in [20] (one constant delay) and [27] (one nonconstant delay).

The main result concerning instability is Theorem 4.4. The proof of this theorem is similar to the proof of Theorem 4.1. In the proof we use Lemma 4.1 again.

**Theorem 4.4.** *Let the assumptions (i), (ii$_0$), (iii$_0$), (iv$_0$) be fulfilled for some $\tau_0 \geq T$. Suppose there exist $t_1 \geq \tau_0$ and $\nu \in (-\infty, \infty)$ such that*

$$\inf_{t \geq t_1}\left[\int_{t_1}^{t} \tilde{\Lambda}_0(s)\, ds - \ln(|\tilde{\gamma}(t)| + |\tilde{c}(t)|)\right] \geq \nu. \tag{4.48}$$

*If $z(t)$ is any solution of (3.1) satisfying*

$$\min_{\theta(t_1) \leq s \leq t_1} |z(s)| > R_0, \qquad \Delta(t_1) > R_0 e^{-\nu}, \tag{4.49}$$

*where*

$$\theta(t) = \min_{k=1,\dots,m} \theta_k(t),$$

$$\varDelta(t) = (|\tilde{\gamma}(t)| - |\tilde{c}(t)|)|z(t)| + \tilde{\beta}_0(t) \max_{\theta(t)\le s\le t} |z(s)| \sum_{k=1}^{m} \int\limits_{\theta_k(t)}^{t} (|\tilde{\gamma}(s)| + |\tilde{c}(s)|)\, ds,$$

*then*

$$|z(t)| \ge \frac{\varDelta(t_1)}{|\tilde{\gamma}(t)| + |\tilde{c}(t)|} \exp\left[\int_{t_1}^{t} \tilde{\varLambda}_0(s)\, ds\right] \tag{4.50}$$

*for all $t \ge t_1$ for which $z(t)$ is defined.*

*Proof.* Let $z(t)$ be any solution of (3.2) satisfying (4.49). Consider the Lyapunov functional

$$V(t) = U(t) + \tilde{\beta}_0(t) \sum_{k=1}^{m} \int\limits_{\theta_k(t)}^{t} U(s)\, ds, \tag{4.51}$$

where

$$U(t) = |\tilde{\gamma}(t) z(t) + \tilde{c}(t)\bar{z}(t)|. \tag{4.52}$$

For brevity we shall denote $w_k(t) = z(\theta_k(t))$ and we shall write the function of variable $t$ simply without indicating the variable $t$, for example, $\tilde{\gamma}$ instead of $\tilde{\gamma}(t)$.

In view of (4.51) we have

$$V' = U' + \tilde{\beta}_0' \sum_{k=1}^{m} \int\limits_{\theta_k(t)}^{t} U(s) ds + m\tilde{\beta}_0 |\tilde{\gamma} z + \tilde{c}\bar{z}| - \sum_{k=1}^{m} \theta_k' \tilde{\beta}_0 |\tilde{\gamma}(\theta_k(t)) w_k + \tilde{c}(\theta_k(t))\bar{w}_k| \tag{4.53}$$

for almost all $t \ge t_1$ for which $z(t)$ is defined and $U'(t)$ exists. Put $\mathcal{K} = \{t \ge t_1 :$ $z(t)$ exists, $|z(t)| > R_0\}$. Clearly $U(t) \ne 0$ for $t \in \mathcal{K}$. The derivative $U'(t)$ exists for almost all $t \in \mathcal{K}$.

Since $z(t)$ is a solution of (3.2), we obtain

$$\begin{aligned}
UU' &= \mathrm{Re}[(\tilde{\gamma}\bar{z} + \bar{\tilde{c}} z)(\tilde{\gamma}' z + \tilde{\gamma} z' + \tilde{c}'\bar{z} + \tilde{c}\bar{z}')] \\
&= \mathrm{Re}\Big\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}} z)\Big[\tilde{\gamma}' z + \tilde{c}'\bar{z} + \tilde{\gamma}\big(az + b\bar{z} + \sum_{k=1}^{m}(A_k w_k + B_k \bar{w}_k) + g\big) \\
&\quad + \tilde{c}\big(\bar{a}\bar{z} + \bar{b} z + \sum_{k=1}^{m}(\bar{A}_k \bar{w}_k + \bar{B}_k w_k) + \bar{g}\big)\Big]\Big\} \\
&= \mathrm{Re}\Big\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}} z)\Big[\tilde{\gamma}' z + \tilde{c}'\bar{z} + (\tilde{\gamma} a + \tilde{c}\bar{b}) z + (\tilde{\gamma} b + \tilde{c}\bar{a})\bar{z} + \tilde{\gamma}\big(\sum_{k=1}^{m}(A_k w_k + B_k \bar{w}_k) + g\big) \\
&\quad + \tilde{c}\big(\sum_{k=1}^{m}(\bar{A}_k \bar{w}_k + \bar{B}_k w_k) + \bar{g}\big)\Big]\Big\}
\end{aligned}$$

for almost all $t \in \mathcal{K}$. Taking into account

$$(\tilde{\gamma}a + \tilde{c}\bar{b})\tilde{c} = (\tilde{\gamma}b + \tilde{c}\bar{a})\tilde{\gamma}, \tag{4.54}$$

we get

$$\begin{aligned} UU' \;\geq\;& \operatorname{Re}\Big\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma}a + \tilde{c}\bar{b})\Big(z + \frac{\tilde{c}}{\tilde{\gamma}}\bar{z}\Big)\Big\} + \\ & + \operatorname{Re}\Big\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)\big[\tilde{\gamma}\sum_{k=1}^{m}(A_k w_k + B_k \bar{w}_k) + \tilde{c}\sum_{k=1}^{m}(\bar{A}_k \bar{w}_k + \bar{B}_k w_k)\big]\Big\} + \\ & + \operatorname{Re}\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma}g + \tilde{c}\bar{g})\} + \operatorname{Re}\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma}'z + \tilde{c}'\bar{z})\} \\ \geq\;& U^2 \operatorname{Re}\Big(a + \frac{\tilde{c}}{\tilde{\gamma}}\bar{b}\Big) - U(|\tilde{\gamma}| + |\tilde{c}|)\big(\sum_{k=1}^{m}|A_k w_k + B_k \bar{w}_k|\big) - U|\tilde{\gamma}g + \tilde{c}\bar{g}| + U^2 \operatorname{Re}\frac{\tilde{\gamma}'z + \tilde{c}'\bar{z}}{\tilde{\gamma}z + \tilde{c}\bar{z}}. \end{aligned}$$

By the use of Lemma 4.1 we get

$$\operatorname{Re}\frac{\tilde{\gamma}'z + \tilde{c}'\bar{z}}{\tilde{\gamma}z + \tilde{c}\bar{z}} \geq \tilde{\vartheta}.$$

The last inequality together with (4.45), taken for $n = 0$, the assumption ($\mathrm{ii}_0$) and the relation

$$\operatorname{Re}\Big(a + \frac{\tilde{c}}{\tilde{\gamma}}\bar{b}\Big) = \operatorname{Re} a \tag{4.55}$$

yield

$$\begin{aligned} UU' \geq\;& U^2(\operatorname{Re} a + \tilde{\vartheta} - \tilde{\varkappa}_0) - U\sum_{k=1}^{m}\big(\tilde{\kappa}_{0k}|\tilde{\gamma}(\theta_k)w_k + \tilde{c}(\theta_k)\bar{w}_k|\big) - \\ & - U(|\tilde{\gamma}| + |\tilde{c}|)\Big(\sum_{k=1}^{m}\frac{|A_k||w_k| + |B_k||\bar{w}_k|}{|\tilde{\gamma}(\theta_k)| - |\tilde{c}(\theta_k)|}(|\tilde{\gamma}(\theta_k)| - |\tilde{c}(\theta_k)|)\Big) \geq \\ & \geq U^2(\operatorname{Re} a + \tilde{\vartheta} - \tilde{\varkappa}_0) - \\ & - U\Big\{\sum_{k=1}^{m}\Big[\tilde{\kappa}_{0k} + (|A_k| + |B_k|)\frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}(\theta_k)| - |\tilde{c}(\theta_k)|}\Big]|\tilde{\gamma}(\theta_k)w_k + \tilde{c}(\theta_k)\bar{w}_k|\Big\} \geq \\ & \geq U^2(\operatorname{Re} a + \tilde{\vartheta} - \tilde{\varkappa}_0) - U\sum_{k=1}^{m}\tilde{\lambda}_{0k}|\tilde{\gamma}(\theta_k)w_k + \tilde{c}(\theta_k)\bar{w}_k| \end{aligned}$$

for almost all $t \in \mathcal{K}$.

Consequently,

$$U' \geq U(\operatorname{Re} a + \tilde{\vartheta} - \tilde{\varkappa}_0) - \sum_{k=1}^{m}\tilde{\lambda}_{0k}|\tilde{\gamma}(\theta_k)w_k + \tilde{c}(\theta_k)\bar{w}_k| \tag{4.56}$$

for almost all $t \in \mathcal{K}$. The inequality (4.56) together with the relation (4.53) give

$$V' \geq U(\operatorname{Re} a + \tilde{\vartheta} - \tilde{\varkappa}_0 + m\tilde{\beta}_0) - \sum_{k=1}^{m}(\tilde{\lambda}_{0k} + \theta_k'\tilde{\beta}_0)|\tilde{\gamma}(\theta_k)w_k + \tilde{c}(\theta_k)\bar{w}_k| +$$

$$+ \tilde{\beta}_0' \sum_{k=1}^{m} \int\limits_{\theta_k(t)}^{t} |\tilde{\gamma}(s)z(s) + \tilde{c}(s)\bar{z}(s)| ds.$$

Using (4.44) and (4.46) for $n = 0$, we obtain

$$V'(t) \geq U(t)\tilde{\Theta}_0(t) + \tilde{\beta}_0'(t) \sum_{k=1}^{m} \int\limits_{\theta_k(t)}^{t} U(s)\, ds.$$

Hence, in view of ($\mathrm{iv}_0$)

$$V'(t) - \tilde{\Lambda}_0(t)V(t) \geq 0 \tag{4.57}$$

for almost all $t \in \mathcal{K}$.

The rest of the proof is identical to the corresponding part of the proof of Theorem 4.1. □

*Remark* 4.5. Theorem 4.4 represents a generalization of previous results.

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = t - r_0$, where $r_0 > 0$, we get a slight generalization of Theorem 1 of [20]. Notice that in the case $r_0 = 0$ (i. e. $\theta_1(t) \equiv t$) the condition (4.49) takes the form

$$|z(t_1)| > R_0 \max\left\{1, \frac{1}{(|\tilde{\gamma}(t_1)| - |\tilde{c}(t_1)|)e^{\nu}}\right\}.$$

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = \theta(t)$, we obtain Theorem 1 from [27].

Similarly to the section 4.1, to obtain results on the existence of bounded solutions, we shall suppose that (3.2) satisfies the uniqueness property of solutions. Again, we suppose that the delays are bounded, i. e., that the functions $\theta_k$ satisfy the condition

$$t - r \leq \theta_k(t) \leq t \quad \text{for} \quad t \geq t_0 + r,$$

where $r > 0$ is a constant. Our assumptions imply the existence of numbers $T_1 = t_0 + r$, $T \geq T_1$ and $\mu > 0$ such that

$$|a(t)| > |b(t)| + \mu \text{ for } t \geq T_1, \qquad t \geq \theta_k(t) \geq t - r \text{ for } t \geq T \ (k = 1, \ldots, m). \tag{4.38'}$$

In view of this, we replace (4.38) in the assumption (i) with (4.38'). All other assumptions we keep in validity.

Ważewski topological principle for retarded functional differential equations of Carathéodory type is used in the proof of the following theorem. Details of this theory can be found in the paper of K. P. Rybakowski [45].

**Theorem 4.5.** *Let the conditions (i), (ii), (iii) be fulfilled and $\tilde{\Lambda}$, $\theta_k'$ $(k = 1, \dots, m)$ be continuous functions such that the inequality $\tilde{\Lambda}(t) \le \tilde{\Theta}(t)$ holds a. e. on $[T, \infty)$, where $\tilde{\Theta}$ is defined by* (4.47)*. Suppose that $\xi : [T - r, \infty) \to \mathbb{R}$ is a continuous function such that*

$$\tilde{\Lambda}(t) + \tilde{\beta}(t) \sum_{k=1}^{m} \theta_k'(t) \exp\left[ -\int_{\theta_k(t)}^{t} \xi(s)\, ds \right] - \xi(t) > \varrho(t) C^{-1} \exp\left[ -\int_{T}^{t} \xi(s)\, ds \right] \tag{4.58}$$

*for $t \in [T, \infty]$ and some constant $C > 0$. Then there exists $t_2 > T$ and a solution $z_0(t)$ of* (3.2) *satisfying*

$$|z_0(t)| \le \frac{C}{|\tilde{\gamma}(t)| - |\tilde{c}(t)|} \exp\left[ \int_{T}^{t} \xi(s)\, ds \right] \tag{4.59}$$

*for $t \ge t_2$.*

*Proof.* Rewrite the equation (3.2) in the form (3.2′). Let $\tau > T$. Put

$$\begin{aligned}
\widetilde{U}(t, z, \bar{z}) &= |\tilde{\gamma}(t) z + \tilde{c}(t) \bar{z}| - \varphi(t), \\
\varphi(t) &= C \exp\left[ \int_{T}^{t} \xi(s)\, ds \right], \\
\Omega^0 &= \{(t, z) \in (\tau, \infty) \times \mathbb{C} : \widetilde{U}(t, z, \bar{z}) < 0\}, \\
\Omega_{\widetilde{U}} &= \{(t, z) \in (\tau, \infty) \times \mathbb{C} : \widetilde{U}(t, z, \bar{z}) = 0\}.
\end{aligned}$$

Repeating the approach used in the proof of Theorem 4.2, we get

$$\begin{aligned}
D^+ \widetilde{U}(t, \psi(0), \bar{\psi}(0)) &= |\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)| \operatorname{Re} \frac{\tilde{\gamma}'(t)\psi(0) + \tilde{c}'(t)\bar{\psi}(0)}{\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)} - \varphi'(t) \\
&\quad + |\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)|^{-1} \operatorname{Re}\{\tilde{\gamma}(t)(\tilde{\gamma}(t)\bar{\psi}(0) + \bar{\tilde{c}}(t)\psi(0))F(t, \psi) \\
&\qquad + \tilde{c}(t)(\tilde{\gamma}(t)\bar{\psi}(0) + \bar{\tilde{c}}(t)\psi(0))\bar{F}(t, \psi)\} \\
&= |\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)| \operatorname{Re} \frac{\tilde{\gamma}'(t)\psi(0) + \tilde{c}'(t)\bar{\psi}(0)}{\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)} - \varphi'(t) \\
&\quad + |\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)|^{-1} \operatorname{Re}\{(\tilde{\gamma}(t)\bar{\psi}(0) + \bar{\tilde{c}}(t)\psi(0))(\tilde{\gamma}(t)F(t, \psi) + \tilde{c}(t)\bar{F}(t, \psi))\}.
\end{aligned}$$

Using (4.54), (4.55) and (ii), similarly to the proof of Theorem 4.4, we obtain

$$\begin{aligned}
&D^+ \widetilde{U}(t, \psi(0), \bar{\psi}(0)) \ge |\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)| \operatorname{Re} a(t) \\
&- \sum_{k=1}^{m} |A_k(t)\psi(\theta_k(t) - t) + B_k(t)\bar{\psi}(\theta_k(t) - t)|(|\tilde{\gamma}(t)| + |\tilde{c}(t)|) - \tilde{\varkappa}(t)|\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)| \\
&- \sum_{k=1}^{m} \tilde{\kappa}_k(t)|\tilde{\gamma}(\theta_k(t))\psi(\theta_k(t) - t) + \tilde{c}(\theta_k(t))\bar{\psi}(\theta_k(t) - t)| \\
&\qquad + \tilde{\vartheta}(t)|\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)| - \varrho(t) - \varphi'(t)
\end{aligned}$$

and consequently

$$
\begin{aligned}
D^+\widetilde{U}(t,\psi(0),\bar{\psi}(0)) &\geq (\operatorname{Re} a(t)+\tilde{\vartheta}(t)-\tilde{\varkappa}(t))|\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\bar{\psi}(0)| \\
&\quad -\sum_{k=1}^{m}\tilde{\lambda}_k(t)|\tilde{\gamma}(\theta_k(t))\psi(\theta_k(t)-t)+\tilde{c}(\theta_k(t))\bar{\psi}(\theta_k(t)-t)|-\varrho(t)-\varphi'(t)\geq \\
&\tilde{\Theta}(t)|\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\bar{\psi}(0)|+\tilde{\beta}(t)\sum_{k=1}^{m}\theta_k'(t)|\tilde{\gamma}(\theta_k(t))\psi(\theta_k(t)-t)+\tilde{c}(\theta_k(t))\bar{\psi}(\theta_k(t)-t)| \\
&\quad -\varrho(t)-\varphi'(t)\geq \\
&\tilde{\Lambda}(t)|\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\bar{\psi}(0)|+\tilde{\beta}(t)\sum_{k=1}^{m}\theta_k'(t)|\tilde{\gamma}(\theta_k(t))\psi(\theta_k(t)-t)+\tilde{c}(\theta_k(t))\bar{\psi}(\theta_k(t)-t)| \\
&\quad -\varrho(t)-\varphi'(t)
\end{aligned}
$$

for almost all $t\in(\tau,\infty)$ and for $\psi\in\mathcal{C}$ sufficiently close to $\phi$.

Again recalling the steps of the proof of Theorem 4.2 and using a topological principle for retarded functional differential equations (see Rybakowski [45, Theorem 2.1]), we see that there is a solution $z_0(t)$ of (3.2) such that $(t,z_0(t))\in\Omega^0$ for all $t\geq t_2$ for which the solution $z_0(t)$ exists. Obviously $z_0(t)$ exists for all $t\geq t_2$ and

$$
(|\tilde{\gamma}(t)|-|\tilde{c}(t)|)|z_0(t)|\leq|\tilde{\gamma}(t)z_0(t)+\tilde{c}(t)\bar{z}_0(t)|\leq\varphi(t)\qquad\text{for } t\geq t_2.
$$

Hence

$$
|z_0(t)|\leq\frac{\varphi(t)}{|\tilde{\gamma}(t)|-|\tilde{c}(t)|}\qquad\text{for } t\geq t_2.
$$

□

*Remark* 4.6. If $\theta_k'(t)\geq 0$ for $k=1,\dots,m$, $\eta_1(t)\tilde{\Lambda}(t)>|\tilde{\beta}(t)|\sum\limits_{k=1}^{m}\theta_k'(t)+C^{-1}\varrho(t)>0$, where $0<\eta_1(t)\leq 1$, the functions $\eta_1$, $\tilde{\Lambda}$ and $\theta_k'$ are continuous on $[T,\infty)$ and $\tilde{\Lambda}(t)\leq\tilde{\Theta}(t)$ a. e. on $[T,\infty)$, then the choice of $\xi$ is possible in (4.58) such that $\xi(t)=\eta_1(t)\tilde{\Lambda}(t)+\tilde{\beta}(t)\sum\limits_{k=1}^{m}\theta_k'(t)-C^{-1}\varrho(t)$ on $[T,\infty)$. Moreover, in some cases, the condition $|\tilde{\beta}(t)|\sum\limits_{k=1}^{m}\theta_k'(t)+C^{-1}\varrho(t)>0$ can be omitted if Theorem 4.5 is used. For instance, the identity $|\tilde{\beta}(t)|\sum\limits_{k=1}^{m}\theta_k'(t)+C^{-1}\varrho(t)\equiv 0$ implies $\tilde{\beta}(t)\sum\limits_{k=1}^{m}\theta_k'(t)\equiv 0$, $\varrho(t)\equiv 0$ and consequently, in view of (4.42), (4.43), (ii), we have $\tilde{\lambda}_k(t)\equiv 0$, $\tilde{\kappa}_k(t)\equiv 0$, $A_k(t)\equiv 0$, $B_k(t)\equiv 0$, $g(t,0,0)\equiv 0$. Thus the equation (3.2) has the trivial solution $z_0(t)\equiv 0$ in this case.

*Remark* 4.7. Theorem 4.5 generalizes theorems contained in previous papers.

Taking $A_1(t)=A(t)$, $A_k\equiv 0$ for $k=2,\dots,m$, $B_1(t)=B(t)$, $B_k\equiv 0$ for $k=2,\dots,m$, $\theta_1(t)\equiv t-r_0$, where $0\leq r_0\leq r$, in Theorem 4.5, we obtain a generalization of Theorem 2 of [20].

Taking $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = \theta(t)$, in Theorem 4.5, we get Theorem 2 of [27].

**Theorem 4.6.** *Suppose that the hypotheses (i), (ii), (ii$_n$), (iii), (iii$_n$), (iv$_n$) are fulfilled for $\tau_n \geq T$ and $n \in \mathbb{N}$, where $R_n > 0$, $\inf\limits_{n\in\mathbb{N}} R_n = 0$. Let $\tilde{\Lambda}$, $\theta_k'$ be continuous functions satisfying the inequality $\tilde{\Lambda}(t) \leq \tilde{\Theta}(t)$ a. e. on $[T, \infty)$, where $\tilde{\Theta}$ is defined by (4.47). Assume that $\xi : [T - r, \infty) \to \mathbb{R}$ is a continuous function such that*

$$\tilde{\Lambda}(t) + \tilde{\beta}(t) \sum_{k=1}^{m} \theta_k'(t) \exp\left[ -\int_{\theta_k(t)}^{t} \xi(s)\, ds \right] - \xi(t) > \varrho(t) C^{-1} \exp\left( -\int_{T}^{t} \xi(s)\, ds \right) \quad (4.60)$$

*for $t \in [T, \infty)$ and some constant $C > 0$. Suppose*

$$\limsup_{t\to\infty} \left[ \int_T^t (\tilde{\Lambda}_n(s) - \xi(s))\, ds + \ln \frac{|\tilde{\gamma}(t)| - |\tilde{c}(t)|}{|\tilde{\gamma}(t)| + |\tilde{c}(t)|} \right] = \infty, \quad (4.61)$$

$$\lim_{t\to\infty} \left[ \tilde{\beta}_n(t) \max_{\theta(t)\leq s\leq t} \frac{\exp\left[\int_T^s \xi(\sigma)\, d\sigma\right]}{|\tilde{\gamma}(s)| - |\tilde{c}(s)|} \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} (|\tilde{\gamma}(s)| + |\tilde{c}(s)|)\, ds \right] = 0, \quad (4.62)$$

$$\inf_{\tau_n\leq s\leq t<\infty} \left[ \int_s^t \tilde{\Lambda}_n(\sigma)\, d\sigma - \ln(|\tilde{\gamma}(t)| + |\tilde{c}(t)|) \right] \geq \nu \quad (4.63)$$

*for $n \in \mathbb{N}$, where $\theta(t) = \min\limits_{k=1,\ldots,m} \theta_k(t)$ and $\nu \in (-\infty, \infty)$. Then there exists a solution $z_0(t)$ of (3.2) such that*

$$\lim_{t\to\infty} \min_{\theta(t)\leq s\leq t} |z_0(s)| = 0. \quad (4.64)$$

*Proof.* The proof is same as the proof of Theorem 4.3 except that we use Theorem 4.4 instead of Theorem 4.1 and Theorem 4.5 instead of Theorem 4.2. □

*Remark* 4.8. Theorem 4.6 is a generalization of results published in the papers [20] and [27].

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = t - r_0$, where $0 \leq r_0 \leq r$, we obtain a generalization of Theorem 3 of [20]. Notice that in the case $r_0 = 0$ (i. e. $\theta_1(t) \equiv t$) the condition (4.62) can be omitted and (4.64) is of the form $\lim\limits_{t\to\infty} |z(t)| = 0$.

If we take $A_1(t) = A(t)$, $A_k \equiv 0$ for $k = 2, \ldots, m$, $B_1(t) = B(t)$, $B_k \equiv 0$ for $k = 2, \ldots, m$, $\theta_1(t) = \theta(t)$, we get Theorem 3 from [27].

### 4.2.3 Corollaries and examples

From Theorem 4.4 we easily obtain several corollaries which can be proven similarly as the corollaries in the Subsection 4.1.3.

**Corollary 4.4.** *Let the assumptions of Theorem 4.4 be fulfilled with $R_0 > 0$. If*

$$\liminf_{t\to\infty}\left[\int_{t_1}^{t}\tilde{\Lambda}_0(s)\,ds - \ln(|\tilde{\gamma}(t)| + |\tilde{c}(t)|)\right] = \varsigma > \nu, \tag{4.65}$$

*then to any $\varepsilon$, $0 < \varepsilon < R_0 e^{\varsigma-\nu}$, there is a $t_2 \ge t_1$ such that*

$$|z(t)| > \varepsilon \tag{4.66}$$

*for all $t \ge t_2$ for which $z(t)$ is defined.*

**Corollary 4.5.** *Let the assumptions of Theorem 4.4 be fulfilled with $R_0 > 0$. If*

$$\lim_{t\to\infty}\left[\int_{t_1}^{t}\tilde{\Lambda}_0(s)\,ds - \ln(|\tilde{\gamma}(t)| + |\tilde{c}(t)|)\right] = \infty, \tag{4.67}$$

*then for any $\varepsilon > 0$ there exists $t_2 \ge t_1$ such that (4.66) holds for all $t \ge t_2$ for which $z(t)$ is defined.*

The efficiency of Theorem 4.4 is demonstrated in the following example which shows that it is worth to consider the case (4.37) as well.

**Example 4.3.** Consider the equation (3.2) where $a(t) \equiv 4 + 3i$, $b(t) \equiv 2$, $A_k(t) \equiv 0$, $B_k(t) \equiv 0$ for $k = 1, \dots, m$, $\theta_k(t) = t + \frac{1}{2k}(\cos kt - 1)$, $g(t, z, w_1, \dots, w_m) = 3z + \sum\limits_{k=1}^{m} \frac{1}{2m}\,\mathrm{e}^{-t}\, w_k$. Obviously $t - \frac{1}{k} \le \theta_k(t) \le t$ and $\frac{1}{2} \le \theta_k'(t) \le \frac{3}{2}$. Suppose $t_0 = 1$ and $T \ge 2$. Then $\gamma(t) = |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2} \equiv 5 + \sqrt{21}$, $c(t) = \bar{a}(t)b(t)/|a(t)| \equiv \frac{8-6i}{5}$, $\tilde{\gamma} \equiv 3 + \sqrt{5}$, $\tilde{c} \equiv -2i$. Further,

$$\begin{aligned}
|\gamma(t)g(t, z, w_1, \dots, w_m) + c(t)\bar{g}(t, z, w_1, \dots, w_m)| &\le 3|\gamma(t)z + c(t)\bar{z}| \\
&\quad + \sum_{k=1}^{m} \frac{1}{2m}\,\mathrm{e}^{-t}\,|\gamma(\theta_k(t))w_k + c(\theta_k(t))\bar{w}_k|, \\
|\tilde{\gamma}(t)g(t, z, w_1, \dots, w_m) + \tilde{c}(t)\bar{g}(t, z, w_1, \dots, w_m)| &\le 3|\tilde{\gamma}(t)z + \tilde{c}(t)\bar{z}| \\
&\quad + \sum_{k=1}^{m} \frac{1}{2m}\,\mathrm{e}^{-t}\,|\tilde{\gamma}(\theta_k(t))w_k + \tilde{c}(\theta_k(t))\bar{w}_k|.
\end{aligned}$$

Following Theorem 4.1, we obtain $\varkappa_0(t) \equiv 3$, $\kappa_{0k}(t) = \frac{1}{2m}\,\mathrm{e}^{-t}$, $\vartheta(t) \equiv 0$, $\alpha(t) \equiv \frac{3}{5}$, $\Lambda_0(t) \le \Theta_0(t) = -\frac{3}{5} + m\beta_0(t) \le -\frac{3}{5} - m\lambda_{0k}(t)(\theta_k'(t))^{-1} \le -\frac{3}{5} < 0$ and we see that neither Theorem 4.1 nor Corollary 4.2 is applicable, because the relations (4.12) and (4.36) cannot be fulfilled. On the other hand, taking $\tilde{\varkappa}_0(t) \equiv 3$, $\tilde{\kappa}_{0k}(t) = \frac{1}{2m}\,\mathrm{e}^{-t}$, $\tau_0 = T$, $R_0 = 0$, $\tilde{\vartheta}(t) \equiv 0$, $\tilde{\beta}_0(t) = -\frac{1}{m}\,\mathrm{e}^{-t}$, $\tilde{\Lambda}_0(t) = \tilde{\Theta}_0(t) = 1 - \mathrm{e}^{-t}$ $(> 0)$ in Theorem 4.4, we have $\theta_k'(t)\tilde{\beta}_0(t) \le -\tilde{\lambda}_{0k}(t)$, $\tilde{\beta}_0'(t) \ge \tilde{\Theta}_0(t)\tilde{\beta}_0(t)$ for $t \in [T, \infty)$ and Theorem 4.4 and Corollary 4.5 are applicable to the considered equation.

The following corollary is a consequence of Theorem 4.5.

**Corollary 4.6.** *Let the assumptions of Theorem 4.5 be satisfied. If*

$$\limsup_{t\to\infty}\left[\frac{1}{|\tilde{\gamma}(t)|-|\tilde{c}(t)|}\exp\left(\int_T^t \xi(s)\,ds\right)\right]<\infty,$$

*then there is a bounded solution $z_0(t)$ of (3.2). If*

$$\lim_{t\to\infty}\left[\frac{1}{|\tilde{\gamma}(t)|-|\tilde{c}(t)|}\exp\left(\int_T^t \xi(s)\,ds\right)\right]=0,$$

*then there is a solution $z_0(t)$ of (3.2) such that*

$$\lim_{t\to\infty} z_0(t)=0.$$

# Chapter 5

# Conclusion

This thesis is focused on the study of asymptotic behaviour of solutions of real two-dimensional differential delayed systems. The goal is to generalize previous results to the case of finite number of nonconstant delays. We use the combination of two methods, the method of complexification and the method of Lyapunov-Krasovskii functional. In some cases we utilize a version of Ważewski topological principle. This approach allows us to treat the two-dimensional system as one equation and thus to simplify the computations and the finding of examples.

In the beginning we went through preliminary part containing historical review of studying of the problem. Then we made a recapitulation of used methods and previous results. Chapter 3 we devoted to the case corresponding to the situation when the singular point 0 of the autonomous system (1.3) is stable. Acquired results on the stability and asymptotic stability are more general than those published in previous papers and we supplied several corollaries and explanatory examples. In Chapter 4 we focused on the case corresponding to the situation when the singular point 0 of the autonomous system (1.3) is unstable. We obtained improved criteria for instability properties of the solutions as well as conditions for the existence of bounded solutions. Corollaries and examples are appended to this part as well.

The methods and techniques utilized in this thesis can be used in further applications. For instance, the semistable case corresponding to the situation when the singular point 0 of the autonomous system (1.3) is a saddle point can be investigated. We can try to find the conditions for the existence of periodic solutions. One of the open problems is the application to the study of asymptotic behaviour of solutions of other types of equations with delay, for example neutral equations. Another open problem is to find similar easy applicable results for equations with advanced argument.

The thesis is a generalization of previous papers and works. The results are new and unpublished except for the results of the Section 4.1 which are to be published in [26].

# Chapter 6

# Appendix

This chapter contains several older results which are used in proofs.

Theorem (1.1) from Chapter IX. of [46]:

**Theorem 6.1.** *Given a finite function of a real variable $F$, each of the following sets is at most countable:*

*(i) the set of points at which the function $F$ assumes a strict maximum or minimum;*

*(ii) the set of the points $x$ at which*

$$\limsup_{t\to x} F(t) > \limsup_{t\to x+} F(t) \quad \text{or} \quad \liminf_{t\to x} F(t) < \liminf_{t\to x+} F(t);$$

*(iii) the set of the points $x$ at which*

$$D^+F(x) < D_-F(x) \quad \text{or} \quad D^-F(x) < D_+F(x).$$

Theorem "Věta 90" from [17], p. 189:

**Theorem 6.2.** *Let $F$ be a finite function on $(a, b)$. Let $N_1$ be the set of such $x \in (a, b)$ for which $D^+F(x) < D_-F(x)$; let $N_2$ be the set of such $x \in (a, b)$ for which $D^-F(x) < D_+F(x)$. Then $N_1 \cup N_2$ is at most countable.*

*Remark* 6.1. Here the symbol $D^+F(x)$ represents the right upper derivative of $F$. The other symbols have analogous meanings.

Theorem 2.1 from [45] and preliminaries:

**Definition 6.1.** Let $A \subset X$ be any two sets of topological space. $A$ is called a retract of $X$, if there exists a continuous mapping $f \colon X \mapsto A$ such that $f(x) = x$ for all $x \in A$. $f$ is called a retraction of $X$ onto $A$.

**Definition 6.2.** Let $\Lambda$ be a convergence space, let $\Omega \subset \mathbb{R} \times \Lambda$ be open in $\mathbb{R} \times \Lambda$, and let $x$ be a mapping, associating with every $(\sigma, \lambda) \in \Omega$ a function $x(\sigma, \lambda) \colon D_{\sigma,\lambda} \to \mathbb{R}^n$ where $D_{\sigma,\lambda}$ is an interval in $\mathbb{R}$. Assume (1), through (3):

(1) $\sigma \in D_{\sigma,\lambda}$;

(2) if $(\sigma_n, \lambda_n) \in \Omega$, $(\sigma, \lambda) \in \Omega$, $t_n$, $t \in \mathbb{R}$, $t \in \mathrm{int} D_{\sigma,\lambda}$, and if $\sigma_n \to \sigma$, $\lambda_n \to \lambda$, $t_n \to t$ as $n \to \infty$, then there is an $n_0$ such that for all $n \geq n_0$ it holds that $t_n \in D_{\sigma_n,\lambda_n}$;

(3) if $(\sigma_n, \lambda_n) \in \Omega$, $(\sigma, \lambda) \in \Omega$, $\sigma_n \to \sigma$, $\lambda_n \to \lambda$ and $t_n \in D_{\sigma_n,\lambda_n}$, $t \in D_{\sigma,\lambda}$, $t_n \to t$, then $x(\sigma_n, \lambda_n)(t_n) \to x(\sigma, \lambda)(t)$.

If all the above conditions are satisfied, then $(\Lambda, \Omega, x)$ is called a system of curves in $\mathbb{R}^n$.

Let $r \geq 0$, and $C = C([-r, 0], \mathbb{R}^n)$. If $t \in \mathbb{R}$ and $x\colon [-r + t, t] \to \mathbb{R}^n$ is a continuous mapping, then $x_t$ is the element of $C$ defined as $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$.

Suppose $\Omega \subset \mathbb{R} \times C$ is an open set and $F\colon \Omega \to \mathbb{R}^n$ is a mapping. Consider the equation

$$x' = F(t, x_t). \tag{6.1}$$

A mapping $x\colon [-r + \sigma, \sigma + A] \to \mathbb{R}^n$, $A > 0$ is called a solution of (6.1) on $[\sigma, \sigma + A)$, if $x(t) \in AC_{\mathrm{loc}}([\sigma, \sigma + A), \mathbb{R}^n)$ and $x'(t)$ exists and is equal to $F(t, x_t)$ a. e. on $[\sigma, \sigma + A)$.

**Definition 6.3.** Let $\{l_i\}$, $\{m_j\}$, $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, q\}$ be two classes of real-valued functions defined on $\mathbb{R} \times \mathbb{R}^n$. One of the classes may be empty. We assume that every $l_i$ and every $m_j$ is an element of $Lip_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \cap AC_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. The set $\omega = \{(t, x) \times \mathbb{R} \times \mathbb{R}^n |\ l_i(t, x) < 0,\ m_j(t, x) < 0, i \in \{1, \ldots, p\}, j \in \{1, \ldots, q\}\}$ is called a (time-dependent) polyfacial set generated by $\{l_i\} \cup \{m_j\}$.

**Definition 6.4.** Let $\omega$ be a (time-dependent) polyfacial set. $\omega$ is called regular with respect to (6.1), if $(\alpha)$, $(\beta)$ and $(\gamma)$ below hold:

$(\alpha)$ If $(t, \phi) \in \mathbb{R} \times C$ is such that $(t + \theta, \phi(\theta)) \in \omega$ for all $\theta \in [-r, 0)$, then $(t, \phi) \in \Omega$.

$(\beta)$ If $i \in \{1, \ldots, p\}$ is arbitrary and if $(t, y) \in \partial\omega$ and $\phi \in C$ are such that $l_i(t, y) = 0$, $\phi(0) = y$, and $\phi(\theta)) \in \omega$ for all $\theta \in [-r, 0)$, then there exists a neighbourhood $V(t, \phi) \subset \Omega$ of $(t, \phi)$ and a null-set $N(t, \phi) \subset \mathbb{R}$ such that for every $(s, \psi) \in V(t, \phi) \setminus (N(t, \phi) \times C)$:

$$\limsup_{h \to 0^+} (1/h)(l_i(s + h, \psi(0) + h \cdot F(s, \psi)) - l_i(s, \psi(0))) > 0.$$

$(\gamma)$ If $j \in \{1, \ldots, q\}$ is arbitrary and if $(t, y) \in \partial\omega$ and $\phi \in C$ are such that $m_j(t, y) = 0$, $\phi(0) = y$, and $\phi(\theta)) \in \omega$ for all $\theta \in [-r, 0)$, then there exists a neighbourhood $V(t, \phi) \subset \Omega$ of $(t, \phi)$ and a null-set $N(t, \phi) \subset \mathbb{R}$ such that for every $(s, \psi) \in V(t, \phi) \setminus (N(t, \phi) \times C)$:

$$\liminf_{h \to 0^+} (1/h)(m_j(s + h, \psi(0) + h \cdot F(s, \psi)) - m_j(s, \psi(0))) < 0.$$

**Theorem 6.3.** *Let $F\colon \Omega \to \mathbb{R}^n$ satisfy the Carathéodory condition on $\omega$ and let the equation* (6.1) *satisfy the uniqueness property of solutions. Suppose $\omega$ is a regular polyfacial set with respect to* (6.1)*, and let $W$ be defined as follows:*

$$W = \{(t, y) \in \partial\omega \mid m_j(t, y) < 0 \text{ for all } j \in \{1, \dots, q\}\}.$$

*Assume that there is a subset $Z \subset \omega \cup W$ satisfying (i) or (ii) below:*

*(i) $Z \cap W$ is a retract of $W$, but $Z \cap W$ is not a retract of $Z$.*

*(ii) $Z = \omega \cup W$ and $W$ is not a strong deformation retract of $Z$.*

*Finally, let $p\colon B = \overline{Z} \cap (Z \cup W) \to C$ be a continuous mapping such that if $z = (t, y) \in B$, then $(t, p(z)) \in \Omega$ and such that (iii) and (iv) below hold:*

*(iii) If $A = \{z = (\sigma, y) \in Z \cap \omega \mid$ There is a $t > \sigma$ such that $(t, x(\sigma, p(z))(t)) \notin \omega\}$, then $(\sigma + \theta, p(z)(\theta)) \in \omega$ for $\theta \in [-r, 0]$ and $z \in A$.*

*(iv) If $z = (\sigma, y) \in W \cap B$, then $p(z)(0) = y$ and $(\sigma + \theta, p(z)(\theta)) \in \omega$ for $\theta \in [-r, 0)$.*

*Under all the above hypotheses, there is a $z_0 = (\sigma_0, y_0) \in Z \cap \omega$ such that for all $t \geq \sigma_0$, if $x(\sigma_0, p(z_0))(t)$ is defined, then $(t, x(\sigma_0, p(z_0))(t)) \in \omega$.*

*Remark* 6.2. In the thesis, $\Omega = J \times \mathcal{C}$, $F$ is defined below $(3.2')$ on the page 49, $\omega = \Omega^0$, $W = \Omega_{\widetilde{U}}$, $l_1(t, y) = \widetilde{U}(t, z, \bar{z})$, $m_1(t, y) = \hat{U}(t)$, $Z = \big\{(t_2, z) \in \Omega^0 \cup \Omega_{\widetilde{U}},\ \text{where } t_2 > \tau + r \text{ is fixed}\big\}$, $\sigma = t_2$, $\sigma_0 = t_2$.

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