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Faculty of Science
Department of Mathematics and Statistics

DISSERTATION

Eva PEKÁRKOVÁ **Brno 2009**

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ASYMPTOTIC PROPERTIES OF SECOND ORDER DIFFERENTIAL EQUATION WITH p-LAPLACIAN

Dissertation

Supervisor: Prof. RNDr. Miroslav Bartušek, DrSc. **Brno 2009**

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Introduction

In the last time the second ordinary differential equations with p-Laplacian and their applications are studied.

The goal of the dissertation is an investigation of sufficient conditions under the validity of that either every solution y of these differential equations is continuable or a solution with special Cauchy conditions is continuable, to generalize known results to a system of differential equations and to study a problem of the asymptotic behaviour of continuable solutions.

The thesis consists of five chapters. The first one includes the problem description, introduces and explains new results which are obtained. Chapters 2–5 involves the new results and they are composed by four author's articles, see [P1], [P2], [P3] and [P4], respectively.

History and summary of the results 1

1.1 History and basic concepts

In two last decades the existence / the nonexistence of noncontinuable solutions are investigated for differential equation

$$y^{(n)} = f(t, y, \dots, y^{(n-1)}) \tag{1.1}$$

and it special cases where $n \geq 2$, f is a continuous function on $\mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R} := (-\infty, \infty)$.

Definition 1.1. Let y be a solution of (1.1) defined on $[T, \tau) \subset \mathbb{R}_+$. Then y is called noncontinuable (singular of 2-nd kind) if $T < \infty$ and $\limsup_{t \to T_-} |y^{(n-1)}(t)| < \infty$. If $T = \infty$, y called continuable (global).

It is important to study the existence / nonexistence of noncontinuable solutions. They appear e.g. in water flow models in one space dimension (flood waves, a flow in sewerage systems); their existence very often mean that models failed and they have been much more precise, see e.g. [16].

Sometimes, the noncontinuability is very important in a definition of some problems. For example, the limit-circle/limit-point problem for (1.1) has an old history, see e.g. the monograph [7] and [8, 9, 10, 11, 12].

Definition 1.2. Let $\alpha \in \{-1, 1\}$ and $\alpha f(t, x_0, \dots, x_{n-1})$, $x_0 \geq 0$ on $\mathbb{R}_+ \times \mathbb{R}^n$. Equation (1.1) is said to be of the nonlinear limit-circle type if for any solution y defined on $\mathbb{R}_+$ and

$$\int_0^\infty y(t) f(t, y, \dots, y^{(n-1)}(t)) \, \mathrm{d}t < \infty$$

holds. Equation (1.1) is said to be of the nonlinear limit-point type if there exists a solution y of (1.1) defined on $\mathbb{R}_+$ such that

$$\int_0^\infty y(t) f(t, y, \dots, y^{(n-1)}(t)) \, \mathrm{d}t = \infty. \tag{1.2}$$

According to Definition 1.2 it is necessary to know if a solution y defined on $\mathbb{R}_+$ and satisfying (1.2) exists. The following example is very instructive.

Example 1.1. Consider the differential equation

$$y'' = t^\alpha |y|^\lambda \operatorname{sgn} y \tag{1.3}$$

with $\lambda > 1$ and $\alpha \in \mathbb{R}$.

(i) Then exists $\varepsilon > 0$ such that every solution y of (1.3) with Cauchy initial conditions $|y(0)| \le \varepsilon$, $|y'(0)| \le \varepsilon$ is continuable if and only if $\alpha < -\lambda - 1$ (see [13]). Hence, if $\alpha < -\lambda - 1$ then (1.3) is of the nonlinear limit-point type.

(ii) If $\alpha \ge -\lambda - 1$, then every solution of (1.3) satisfying $y(\tau)y'(\tau) > 0$ at some $\tau \in \mathbb{R}_+$ is noncontinuable (see [11, Lemma 5]). Moreover, if $\alpha \ge 0$, then (1.3) is of the nonlinear limit-circle type (see [11, Theorem 4]).

It is important to study the nonexistence of noncontinuable solutions from the mathematical point of view. Example 1.1(ii) shows that all nonoscillatory solutions of (1.3) are Kneser ones, i.e. $y(t)y'(t) < 0$ for large t holds. In this case it is a nonsense to investigate asymptotic properties of positive increasing solutions. As concern to problems in Example 1.1(ii) for (1.1), see e.g. [11, Theorem 4] $(n = 2)$, [12, Theorem 6] (n is even), [10, Theorem 6] $(n = 4)$.

The first results for the nonexistence of noncontinuable solutions of (1.1) (or its special cases) are given by Wintner, see [20] or [31]. Other results are obtained e.g. in [5, 12, 14, 22, 29, 30, 32, 35, 36], see references therein, too. Existence results can be found e.g. in [2, 5, 15, 19, 20, 32, 39].

In the last decade a lot of papers are devoted to the study of a differential equation with p-Laplacian (see e.g. [28])

$$(a(t)|y'|^{p-1}y')' + r(t)g(y) = 0 \tag{1.4}$$

where $p > 0$, $r \in C^0(\mathbb{R}_+)$, $g \in C^0(\mathbb{R})$ and $g(x)x \ge 0$ on $\mathbb{R}$.

In sublinear case, if $M > 0$ and

$$|g(x)| \le M|x|^p \quad \text{for large } |x|,$$

then every solution y of (1.4) is continuable, see [39, Theorem 1.1]. Furthermore, if $g(x) = |x|^\lambda \operatorname{sgn} x$, $\lambda > p$, $K_2 > 0$, $r > 0$ for large t and

$$\int_0^\infty a^{-\frac{1}{p}}(s)\,\mathrm{d}s = \infty, \quad r(t)\int_0^t a^{-\frac{1}{p}}(s)\,\mathrm{d}s \ge K_2 a^{-\frac{1}{p}}(t) \quad \text{for large } t,$$

then every solution y with $y(\tau)y'(\tau)$ at some $\tau \in \mathbb{R}_+$ is noncontinuable; i.e. y is Kneser solution. A similar result for (1.4) with forcing term is in [11, Lemma 5]. So it is convenient to investigate the more general equation

$$(a(t)|y'|^{p-1}y')' + b(t)g(y') + r(t)f(y) = e(t) \tag{1.5}$$

where $p > 0$, $a \in C^0(\mathbb{R}_+)$, $b \in C^0(\mathbb{R}_+)$, $r \in C^0(\mathbb{R}_+)$, $e \in C^0(\mathbb{R}_+)$, $f \in C^0(\mathbb{R})$, $g \in C^0(\mathbb{R})$ and $a > 0$ on $\mathbb{R}_+$.

The p-Laplace differential equation

$$\operatorname{div}(\|\nabla v\|)^{p-2}\nabla v) = h(\|x\|, v) \tag{1.6}$$

plays an important role in the theory of partial differential equations (see e.g. [41]), where ∇ is the gradient, $p > 0$ and $\|x\|$ is a norm of $x \in \mathbb{R}^n$ and $h(y, v)$ is a nonlinear function on $\mathbb{R} \times \mathbb{R}$. Radially symmetric solutions of the equation (1.6) depend on the scalar variable $r = \|x\|$ and they are solutions of the ordinary differential equation

$$r^{1-n}(r^{n-1}|v'|)' = h(r, v) \tag{1.7}$$

where $v' = \frac{\mathrm{d}v}{\mathrm{d}r}$ and $p > 1$. If $p \neq n$ then the change of variables $r = t^{\frac{p-1}{p-n}}$ transforms the equation (1.7) into the equation

$$(\Psi_p(u'))' = f(t, u)$$

where $\Psi_p(u') = |u'|^{p-2}u'$ is so called one-dimensional, or scalar p-Laplacian [41] and

$$f(t, u) = \left|\frac{p-1}{p-n}\right|^p t^{\frac{p-n}{p(1-n)}} h(t^{\frac{p-1}{p-n}}, u)\,.$$

In [44] the existence of periodic solutions of the system

$$(\Phi_p(u'))' + \frac{\mathrm{d}}{\mathrm{d}t}\nabla F(u) + \nabla G(u) = e(t)$$

is studied where

$$\Phi_p : \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi_p(u) = (|u_1|^{p-2}u_1, \ldots, |u_n|^{p-2}u_n)^T.$$

The operator $\Phi_p(u')$ is called multidimensional p-Laplacian. The study of radially symmetric solutions of the system of p-Laplace equations

$$\operatorname{div}(\|\nabla v_i\|^{p-2}\nabla v_i) = h_i(\|x\|, v_1, v_2, \ldots, v_n), \quad i = 1, 2, \ldots, n, \quad p > 1$$

leads to the system of ordinary differential equations

$$(|u_i'|^{p-2}u_i')' = f_i(t, u_1, u_2, \ldots, u_n), \quad i = 1, 2, \ldots, n, \quad p \neq n$$

where

$$f_i(t, u_1, u_2, \ldots, u_n) = \left|\frac{p-1}{p-n}\right|^p t^{\frac{p-n}{p(1-n)}} h_i(t^{\frac{p-1}{p-n}}, u_1, u_2, \ldots, u_n).$$

This system can be written in the form

$$(\Phi_p(u'))' = f(t, u)$$

where $f = (f_1, f_2, \ldots, f_n)^T$ and $\Phi_p(u')$ is the n-dimensional p-Laplacian. We will consider the operator Φ_{p+1} with $p > 0$ and for the simplicity we denote it as Φ_p, i. e.

$$\Phi_p(u) = (|u_1|^{p-1}u_1, |u_2|^{p-1}u_2, \ldots, |u_n|^{p-1}u_n).$$

So it is reasonable to shall study the initial value problem

$$(A(t)\Phi_p(y'))' + B(t)g(y') + R(t)f(y) = e(t), \tag{1.8}$$

$$y(0) = y_0, \quad y'(0) = y_1 \tag{1.9}$$

where $p > 0$, $y_0, y_1 \in \mathbb{R}^n$, $A(t)$, $B(t)$, $R(t)$ are continuous, matrix-valued functions on $\mathbb{R}_+$, $A(t)$ is regular for all $t \in \mathbb{R}_+$, $e : \mathbb{R}_+ \to \mathbb{R}^n$ and $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are continuous mappings.

The equation (1.8) with $n = 1$ has been studied in Chapter 2. Many papers are devoted to the study of the existence of periodic solutions of scalar differential equation with $p-$Laplacian and in some of them it is assumed that $A(0) = 0$. We study the system without this singularity. From the recently published papers and books see e.g. [33, 34, 41, 44]. The problems treated in Chapter 3 are close to those studied in [3, 4, 5, 6, 17, 18, 26, 27, 31, 39, 40, 41, 44].

We also study asymptotic properties of the second order differential equation with p-Laplacian

$$(|u'|^{p-1}u')' + f(t, u, u') = 0, \quad p \geq 1. \tag{1.10}$$

In the sequel, it is assumed that all solutions of the equation (1.10) are continuously extendable throughout the entire real axis. We shall prove sufficient conditions under which all global solutions are asymptotic to $at + b$, as $t \to +\infty$ where a, b are real numbers. The problem for ordinary second order differential equations without $p-$Laplacian has been studied by many authors, e.g. [23, 24, 25, 37, 38, 42, 43, 45, 46]. Our results are more close to these obtained in the papers [42, 43]. The main tool of the proofs are the Bihari's and Dannan's integral inequalities. We remark that sufficient conditions on the existence of continuable solutions for second order differential equations and second order functional-differential equations with $p-$Laplacian are proved in the papers [3, 4, 13]. Many references concerning differential equations with $p-$Laplacian can be found in the paper [41], where boundary value problems for such equations are treated.

Let

$$u(t_0) = u_0, \quad u'(t_0) = u_1 \tag{1.11}$$

where $u_0, u_1 \in \mathbb{R}$ be initial condition for solutions of (1.10).

1.2 The main results

The goal of the thesis is

- an investigation of sufficient conditions under the validity of that either every solution y of (1.5) or a solution with special Cauchy conditions is continuable,
- to generalize known results to a system of differential equations of the form (1.8),
- to study a problem of the asymptotic behaviour of continuable solutions.

In the second chapter, we study the existence of continuable solutions of a forced second order nonlinear differential equation of the form (1.5).

A special case of equation (1.5) is the unforced equation

$$(a(t)|y'|^{p-1}y')' + b(t)g(y') + r(t)f(y) = 0. \tag{1.12}$$

We will often use of the following assumptions

$$f(x)x \geq 0 \quad \text{on } \mathbb{R} \tag{1.13}$$

and

$$g(x)x \geq 0 \quad \text{on } \mathbb{R}_+. \tag{1.14}$$

Definition 1.3. A solution y of (1.5) is called proper if it is defined on $\mathbb{R}_+$ and $sup_{t\in[\tau,\infty)}|y(t)| > 0$ for every $\tau \in (0,\infty)$. A proper solution y is called nonoscillatory if $y \neq 0$ in a neighbourhood of ∞; it is called weakly oscillatory if it is nonoscillatory and y' has a sequence of zeros tending to ∞. A solution y of (1.5) is called singular of the 1-st kind if it is defined on $\mathbb{R}_+$, there exists $\tau \in (0,\infty)$ such that $y \equiv 0$ on $[\tau,\infty)$ and $sup_{T\leq t\leq \tau}|y(t)| > 0$ for every $T \in [0,\tau)$. It is called noncontinuable (singular of the 2-nd kind) if it is defined on $[0,\tau), \tau < \infty$ and $sup_{0\leq t<\tau}|y'(t)| = \infty$.

We define the function $R : \mathbb{R}_+ \to \mathbb{R}$ by $R(t) = a^{\frac{1}{p}}(t)r(t)$.

The following theorem gives a nonexistence result for noncontinuable solution.

Theorem 1.1. *Let $M > 0$ and $|g(x)| \leq |x|^p$ and $|f(x)| \leq |x|^p$ for $|x| \geq M$. Then there exist no noncontinuable solution y of* (1.5) *and all solutions of* (1.5) *are defined on $\mathbb{R}_+$.*

Remark 1.1. The result of Theorem 1.1 for Equation (1.5) with $p \leq 1$ and without the damping $(b \equiv 0)$ is a generalization of the well-known Wintner's Theorem, see e.g. [20, Theorem 11.5] or [31, Theorem 6.1].

The following result shows that noncontinuable solutions of (1.5) do not exist if $r > 0$ and R is smooth enough under weakened assumptions on f.

Theorem 1.2. *Let* (1.13), $R \in C^1(\mathbb{R}_+)$, $r > 0$ *on* $\mathbb{R}_+$ *and let either*

(i) $M \in (0, \infty)$ *exist such that* $|g(x)| \leq |x|^p$ *for* $|x| \geq M$
or

(ii) (1.14) *hold and* $b(t) \geq 0$ *on* $\mathbb{R}_+$.

Then all solutions of (1.5) *are defined on* $\mathbb{R}_+$.

Remark 1.2. Note that the condition $|g(x)| \leq |x|^p$ in (i) can not be improved.

Example 1.2. Let $\varepsilon \in (0,1)$. Then the function $y = \left(\frac{1}{1-t}\right)^{\frac{1-\varepsilon}{\varepsilon}}$ is a noncontinuable solution of the equation

$$y'' - |y'|^{\varepsilon} y' + C|y|^{\frac{1+\varepsilon}{1-\varepsilon}} \operatorname{sgn} y = 0$$

on $[0,1)$ with $C = \left(\frac{1-\varepsilon}{\varepsilon^2}\right)^{\varepsilon+1} - \frac{1-\varepsilon}{\varepsilon^2}$.

Remark 1.3.

(i) The result of Theorem 1.2 is obtained in [9] in case $b \equiv 0$ using a the similar method.

(ii) Note that Theorem 1.2 is not valid if $R \notin C^1(\mathbb{R}_+)$; see [3] or [21] for the case $g \equiv 0$.

Remark 1.4. Theorem 1.2 is not valid if $r < 0$ on an interval of a positive measure, see e.g. [20, Theorem 11.3] (for (1.4) and $p = 1$). The existence of noncontinuable solutions for (1.5) with $r > 0$ is an open problem.

The following lemma shows that $e(t)$ has to be trivial in a neighbourhood of ∞ if Equation (1.5) has a singular solution of the first kind.

Lemma 1.1. *Let* y *be a singular solution of the first kind of* (1.5). *Then* $e(t) \equiv 0$ *in a neighbourhood* ∞.

In what follows, we will only consider the equation (1.12).

Theorem 1.3. *Let* $M > 0$ *and*

$$|g(x)| \leq |x|^p \ \text{ and } \ |f(x)| \leq |x|^p \ \text{ for } |x| \leq M.$$

Then there exist no singular solution of the first kind of Equation (1.12).

Theorem 1.4. *Consider* (1.13), $R \in C^1(\mathbb{R}_+)$, $r > 0$ *on* $\mathbb{R}_+$ *and let either*

(i) $M \in (0, \infty)$ *exist such that* $|g(x)| \le |x|^p$ *for* $|x| \le M$
or

(ii) (1.14) *and* $b(t) \le 0$ *on* $\mathbb{R}_+$.

Then Equation (1.12) *has no singular solution of the first kind.*

Remark 1.5. Theorem 1.3 generalize results of [39, Theorem 1.2], obtained in case $b \equiv 0$. Results of [31, Theorem 9.4] with ($b \equiv 0, f(x) = |x|^p \operatorname{sgn} x$) and of [3, Theorem 1] ($b \equiv 0$) are special cases of Theorem 1.1 here.

Remark 1.6. Theorem 1.4 is not valid if $r < 0$ on an interval of positive measure; see e.g. [20, Theorem 11.1] (for (1.4) and $p = 1$). The existence of singular solutions of the first kind of (1.12) is an open problem.

Remark 1.7. If $R \notin C^1(\mathbb{R}_+)$, then the statement of Theorem 1.4 does not hold (see [3] for $g \equiv 0$ or [22]).

Note that condition (i) in Theorem 1.4 can not be improved.

Example 1.3. Let $\varepsilon \in (0, 1)$. Then function $y = (1-t)^{(1+\frac{1}{\varepsilon})}$ for $t \in [0, 1]$ and $y \equiv 0$ on $(1, \infty)$ is a singular solution of the first kind of the equation

$$y'' + \left(1 + \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\right)\left(1 + \frac{1}{\varepsilon}\right)^{\varepsilon - 1} |y'|^{1-\varepsilon} \operatorname{sgn} y' + |y|^{\frac{1-\varepsilon}{1+\varepsilon}} = 0.$$

Note that $p = 1$ in this case.

Theorems 1.1, 1.2, 1.3 and 1.4 give us sufficient conditions for all nontrivial solutions of (1.12) to be proper.

Corollary 1.1. *Let* $|g(x)| \le |x|^p$ *and* $|f(x)| \le |x|^p$ *for* $x \in \mathbb{R}$. *Then every nontrivial solution* y *of* (1.12) *is proper.*

Corollary 1.2. *Let* (1.13), $R \in C^1(\mathbb{R}_+)$, $r > 0$ *on* $\mathbb{R}_+$ *and* $|g(x)| \le |x|^p$ *on* $\mathbb{R}$ *hold. Then every nontrivial solution* y *of* (1.12) *is proper.*

Remark 1.8. The results of Corollary 1.1 and Corollary 1.2 are obtained in [3] for $b \equiv 0$.

In the last part of Chapter 2, simple asymptotic properties of solutions of (1.12) are studied. Mainly, sufficient conditions are given under which zeros of a nontrivial solutions are simple and zeros of a solution and its derivative separate from each other.

Corollary 1.3. *Let the assumptions either of Theorem 1.3 or of Theorem 1.4 hold. Then any nontrivial solution of* (1.12) *has no double zeros on* $\mathbb{R}_+$.

Corollary 1.4. *Let* $f(x)x > 0$ *for* $x \neq 0$ *and one of the following possibilities hold:*

(i) $r \neq 0$ *on* $\mathbb{R}_+$ *and*

$$|g(x)| \leq |x|^p \quad \textit{and} \quad |f(x)| \leq |x|^p \quad \textit{for} \quad x \in \mathbb{R};$$

(ii) $R \in C^1(\mathbb{R}_+)$, $r > 0$ *on* $\mathbb{R}_+$ *and*

$$|g(x)| \leq |x|^p \quad \textit{for} \quad |x| \in \mathbb{R};$$

(iii) $R \in C^1(\mathbb{R}_+)$, $b \leq 0$ *on* $\mathbb{R}_+$, $r > 0$ *on* $\mathbb{R}_+$, $g(x)x \geq 0$ *on* R_+ *and* $M > 0$ *exists such that*

$$|g(x)| \geq |x|^p \quad \textit{for} \quad |x| \geq M;$$

(iv) $R \in C^1(\mathbb{R}_+)$, $r > 0$ *on* $\mathbb{R}_+$, $b \geq 0$ *on* $\mathbb{R}_+$, $g(x)x \geq 0$ *on* $\mathbb{R}$ *and* M *exists such that*

$$|g(x)| \leq |x|^p \quad \textit{for} \quad |x| \leq M.$$

Then the zeros of y *and* y' *(if any) separate from each other, i.e. between two consecutive zeros of* y (y') *there is the only zero of* y (y').

Theorem 1.5. *Let* $g(0) = 0$, $r \neq 0$ *on* $\mathbb{R}_+$ *and* $f(x)x > 0$ *for* $x \neq 0$. *Then* (1.12) *has no weakly oscillatory solution and every nonoscillatory solution* y *of* (1.12) *has a limit as* $t \to \infty$.

The following examples show that some of the assumptions of Theorem 1.5 cannot be omitted.

Example 1.4. The function $y = 2 + \sin t$, $t \in \mathbb{R}_+$ is a weakly oscillatory solution of the equation

$$y'' - y' + \frac{\sin t + \cos t}{2 + \sin t} y = 0.$$

In this case $r \neq 0$, Theorem 1.5 is not valid.

Example 1.5. The function $y = 2 + \sin t$, $t \in \mathbb{R}_+$ is a weakly oscillatory solution of the equation

$$y'' - g(y') + 2y = 0 \;\; \textit{with} \;\; g(x) = \begin{cases} 4 + \sqrt{1 - x^2} & \text{for } |x| \leq 1; \\ 4 & \text{for } |x| > 1. \end{cases}$$

In this case $g(0) \neq 0$, Theorem 1.5 is not valid.

Remark 1.9. If $g \equiv 0$, the result of Theorem 1.5 is known, see e.g. [39, Lemma 5.1].

In Chapter 3, the problem of the existence of continuable solutions of the system (1.8), (1.9) is studied. In this chapter $\|x\|$, $x \in \mathbb{R}^n$ is the Euclidean norm.

Definition 1.4. A solution $y(t)$, $t \in [0, T)$ of the initial value problem (1.8), (1.9) is called nonextendable to the right if either $T < \infty$ and

$$\limsup_{t \to T^-} [\|y(t)\| + \|y'(t)\|] = \infty$$

or $T = \infty$, i.e. $y(t)$ is defined on $\mathbb{R}_+$. In the first (second) case $y(t)$ is called noncontinuable (continuable).

The main result is the following theorem.

Theorem 1.6. *Let $p > 0$, $A(t)$, $B(t)$, $R(t)$ be continuous matrix-valued functions on $\mathbb{R}_+$, $A(t)$ be regular for all $t \in \mathbb{R}_+$, $e : \mathbb{R}_+ \to \mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous mappings and $y_0, y_1 \in \mathbb{R}^n$. Let*

$$\int_0^\infty \|R(s)\| s^{m-1} \, \mathrm{d}s < \infty$$

and there exist constants $K_1, K_2 > 0$ such that

$$\|g(u)\| \le K_1 \|u\|^m, \quad \|f(v)\| \le K_2 \|v\|^m, \quad u, v \in \mathbb{R}^n. \tag{1.15}$$

Then the following assertions hold:

1. *If $1 < m \le p$, then any nonextendable to the right solution $y(t)$ of the initial value problem* (1.8), (1.9) *is continuable.*

2. *Let $m > p, m > 1$,*

$$A_\infty := \sup_{0 \le t < \infty} \|A(t)^{-1}\| < \infty, \quad R_0 = \int_0^\infty \|R(s)\| \, \mathrm{d}s,$$
$$E_\infty := \sup_{0 \le t < \infty} \| \int_0^t e(s) \, \mathrm{d}s \| < \infty, \quad Q(s) := \int_s^\infty \|R(\sigma)\| \sigma^{m-1} \, \mathrm{d}\sigma$$

and

$$n^{\frac{p}{2}} \frac{m-p}{p} D^{\frac{m-p}{p}} A_\infty \sup_{0 \le t < \infty} \int_0^t \Big(K_1 \|B(s)\| + 2^{m-1} K_2 Q(s) \Big) \, \mathrm{d}s < 1$$

for all $t \in [0, \infty)$ where

$$D = n^{\frac{p}{2}} A_\infty \Big(\|A(0) \Phi_p(y_1)\| + 2^{m-1} K_2 \|y_0\|^m R_0 + E_\infty \Big).$$

Then any nonextendable to the right solution $y(t)$ of the initial value problem (1.8), (1.9) *is continuable.*

In Chapter 2, sufficient conditions for all solution of (1.8) with $n = 1$ to be defined on $\mathbb{R}_+$ are given. The method of proofs is not applicable in the case $n > 1$. Our proof of Theorem 1.6 is completely different from those applied in Chapter 2. The main tool of our proof is the discrete and also continuous versions of the Jensen's inequality, Fubini theorem and a generalization of the Bihari theorem. The application of the Jensen's inequality is possible only under the assumption $m > 1$. Therefore we do not study the case $0 < m \le 1$. Note, that the case $m \le 1$ is studied in [13] where our method of the proof is used.

Let $y(t)$ be a solution of the initial value problem (1.8), (1.9) defined on an interval $[0,T)$, $0 < T \le \infty$. If we denote $u(t) = y'(t)$, then $y(t) = y_0 + \int_0^t u(s)\,\mathrm{d}s$ and the equation (1.8) can be rewritten as the following integro-differential equation for $u(t)$:

$$\Big(A(t)\Phi_p(u(t))\Big)' + B(t)g(u(t)) + R(t)f\Big(y_0 + \int_0^t u(s)\,\mathrm{d}s\Big) = e(t) \tag{1.16}$$

with

$$u(0) = y_1. \tag{1.17}$$

The following theorem is the main tool for the proof of Theorem 1.6 and the obtained estimates may be important for further investigations of solutions, too.

Theorem 1.7. *Let $p > 0$, $A(t)$, $B(t)$, $R(t)$ be continuous matrix-valued functions on $\mathbb{R}_+$, $A(t)$ regular for all $t \in \mathbb{R}_+$, $e : \mathbb{R}_+ \to \mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous mappings on $\mathbb{R}_+$, $y_0, y_1 \in \mathbb{R}^n$, $\int_0^\infty \|R(s)\| s^{m-1}\,\mathrm{d}s < \infty$ and $0 < T < \infty$. Let the condition* (1.15) *be satisfied and let $u : [0,T) \to \mathbb{R}^n$ be a solution of the equation* (1.16) *satisfying the condition* (1.17). *Let $R_0 := \int_0^\infty \|R(s)\|\,\mathrm{d}s$.*

Then the following assertions hold:

1. *If $m = p > 1$, then*

$$\|u(t)\| \le d_T\, \mathrm{e}^{\int_0^t F_T(s)\,\mathrm{d}s}, \quad 0 \le t \le T$$

where

$$\begin{aligned}
F_T(t) &:= n^{\frac{p}{2}} E_T\Big(K_1\|B(s)\| + 2^{m-1}K_2 Q(s)\Big),\\
Q(s) &= \int_s^\infty \|R(\sigma)\|\sigma^{m-1}\,\mathrm{d}\sigma,\\
E_T &:= \max_{0\le t\le T}\|E(t)\|, \quad E(t) := \int_0^t e(s)\,\mathrm{d}s,\\
d_T &= n^{\frac{p}{2}} A_T\Big(\|A(0)\Phi_p(y_1)\| + 2^{m-1}K_2\|y_0\|^m R_0 + E_T\Big),\\
A_T &= \max_{0\le t\le T}\|A(t)^{-1}\|.
\end{aligned}$$

2. *If* $1 < m < p$, *then*
$$\|u(t)\| \le \left(d_T^{\frac{p-m}{p}} + \frac{p-m}{p} d_T \int_0^t F_T(s)\,\mathrm{d}s\right)^{\frac{1}{p-m}}.$$

3. *Let* $m > p$, $m > 1$ *and*
$$A_\infty := \sup_{0 \le T < \infty} A_T < \infty, \quad E_\infty := \sup_{0 \le t \le \infty} \|E(t)\| < \infty,$$
$$n^{\frac{p}{2}} \frac{m-p}{p} D^{\frac{m-p}{p}} A_\infty \sup_{0 \le t < \infty} \int_0^t \Big(K_1 \|B(s)\| + 2^{m-1} K_2 Q(s)\Big)\,\mathrm{d}s < 1$$
where
$$D = n^{\frac{p}{2}} A_\infty \Big(\|A(0)\Phi_p(y_1)\| + 2^{m-1} K_2 \|y_0\|^m R_0 + E_\infty\Big).$$
Then
$$\|u(t)\| \le D^{\frac{1}{p}} \left(1 - n^{\frac{p}{2}} \frac{m-p}{p} A_\infty D^{\frac{m-p}{p}} \int_0^t \Big(K_1 \|B(s)\| + 2^{m-1} K_2 Q(s)\Big)\,\mathrm{d}s\right)^{-\frac{1}{m-p}}$$
where $0 \le t \le \infty$.

The fourth chapter studies the estimates from bellow of norms of a noncontinuable solution of (1.8) and its derivative. Estimates of solutions are important e. g. in proofs of the existence of such solutions, see e. g. [4], [5] for (1.1). For generalized Emden-Fowler equation of the form (1.1), some estimates are proved in [1].

We will derive estimates for a noncontinuable solution y on the fixed definition interval $[T, \tau) \subset \mathbb{R}_+$, $\tau < \infty$. Note, that the results of Theorem 1.6 are the basic tool of proofs of the following two theorems.

Theorem 1.8. *Let* y *be a noncontinuable solution of system* (1.8) *on* $[T, \tau) \subset \mathbb{R}_+$, $\tau - T \le 1$,
$$A_0 := \max_{T \le t \le \tau} \|A(t)\|^{-1}, \quad B_0 := \max_{T \le t \le \tau} \|B(t)\|,$$
$$R_0 := \max_{T \le t \le \tau} \|R(t)\|, \quad E_0 := \max_{T \le t \le \tau} \|e(t)\|$$
and let there exist positive constants K_1, K_2 *and* $m > p$ *such that*
$$\|g(u)\| \le K_1 \|u\|^m, \quad \|f(v)\| \le K_2 \|v\|^m, \quad u, v \in \mathbb{R}^n.$$

(i) *If* $p > 1$ *and* $M = \frac{2^{2m+1}(2m+3)}{(m+1)(m+2)}$, *then*
$$\|A(t)\Phi_p(y'(t))\| + 2^{m-1} K_2 \|y(t)\|^m R_0 + E_0(\tau - t) \ge C_1 (\tau - t)^{-\frac{p}{m-p}}$$
for $t \in [T, \tau)$ *where*
$$C_1 = n^{-\frac{pm}{2(m-p)}} A_0^{-\frac{p}{m-p}} \left(\frac{m-p}{p}\right)^{-\frac{p}{m-p}} \left[\frac{3}{2} K_1 B_0 + M K_2 R_0\right]^{-\frac{p}{m-p}}.$$

(ii) *If $p \leq 1$, then*

$$\|A(t)\Phi_p(y'(t))\| + 2^m K_1 B_0 \|y'(t)\|^m + 2^{2m+1} K_2 R_0 \|y(t)\|^m + \\ +E_0(\tau - t) \geq C_2(\tau - t)^{-\frac{p}{p-m}}$$

for $t \in [T, \tau)$ where

$$C_2 = 2^{-\frac{p(m+1)}{m-p}} A_0^{-\frac{m}{m-p}} \left(\frac{m-p}{p}\right)^{-\frac{p}{m-p}} \left[\frac{3}{2} K_1 B_0 + M K_2 R_0\right]^{-\frac{p}{m-p}}.$$

Now consider special case of equation (1.8)

$$(A(t)\Phi_p(y'))' + R(t) f(y) = 0. \tag{1.18}$$

In this case a better estimation can be proved.

Theorem 1.9. *Let $m > p$ and y be a noncontinuable solution of system* (1.18) *on interval $[T, \tau) \subset \mathbb{R}_+$. Let there exist constant $K_2 > 0$ such that*

$$\|f(v)\| \leq K_2 \|v\|^m, \quad v \in \mathbb{R}^n.$$

Let A_0, R_0 and M be given by Theorem 1.8. Then

$$\|A(t)\Phi_p(y'(t))\| + 2^{m+2} K_2 \|y(t)\|^m R_0 \geq C_1(\tau - t)^{-\frac{p(m+1)}{m-p}}$$

where

$$C_1 = n^{-\frac{pm}{2(m-p)}} A_0^{-\frac{p}{m-p}} \left(\frac{m-p}{p}\right)^{-\frac{p}{m-p}} \left[M K_2 R_0\right]^{-\frac{p}{m-p}} \quad \textit{in case} \quad p > 1$$

and

$$\|A(t)\Phi_p(y')\| + 2^{2m+1} K_2 \|y(t)\|^m R_0 \geq C_2(\tau - t)^{-\frac{p(m+1)}{m-p}}$$

with

$$C_2 = 2^{-\frac{p(m+1)}{m-p}} A_0^{-\frac{m}{m-p}} \left(\frac{m-p}{p}\right)^{-\frac{p}{m-p}} \left[M K_2 R_0\right]^{-\frac{p}{m-p}} \quad \textit{in case} \quad p \leq 1.$$

We can prove more results for a scalar differential equation

$$(a(t)\Phi_p(y'))' + r(t) f(y) = 0 \tag{1.19}$$

where $p > 0$, $a(t)$, $r(t)$ are continuous functions on $\mathbb{R}_+$, $a(t) > 0$ for $t \in \mathbb{R}_+$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous mapping and $\Phi_p(u) = |u|^{p-1}u$.

Definition 1.5. A noncontinuable solution y of (1.19) defined on $[0, \tau)$ is called oscillatory if there exists a sequence $\{t_k\}_{k=1}^{\infty}$, $t_k \in [0, \tau)$ of its zeros such that $\lim_{k\to\infty} t_k = \tau$; otherwise it is called nonoscillatory.

Theorem 1.10. *Let y be a noncontinuable oscillatory solution of equation* (1.19) *defined on $[T, \tau)$. Let there exist constant $K_2 > 0$ such that*

$$|f(v)| \le K_2 |v|^m, \quad v \in \mathbb{R}. \tag{1.20}$$

Let $\{t_k\}_1^\infty$ and $\{\tau_k\}_1^\infty$ be the increasing sequences of all local extremes of solution y and $y^{[1]} = a(t)\Phi_p(y')$ on $[T, \tau)$, respectively. Then constants C_1 and C_2 exist such that

$$|y(t_k)| \ge C_1(\tau - t_k)^{-\frac{p(m+1)}{m(m-p)}}$$

and in the case $r \ne 0$ on $\mathbb{R}_+$ and $f(x)x > 0$ for $x \ne 0$.

$$|y^{[1]}(\tau_k)| \ge C_2(\tau - \tau_k)^{-\frac{p(m+1)}{m-p}}$$

for $k \ge 1, 2, \ldots$.

Example 1.6. Consider (1.19) and (1.20) with $m = 2$, $p = 1$ and $a \equiv 1$. Then from Theorem 1.10 we obtain the following estimates

$$|y(t_k)| \ge C_1(\tau - t_k)^{-\frac{3}{2}}, \quad |y^{[1]}(\tau_k)| \ge C_2(\tau - \tau_k)^{-3}$$

where $M = \frac{56}{3}$, $C_1 = \frac{\sqrt{42}}{448K_2r_0}$ and $C_2 = \frac{3}{448K_2r_0}$.

Example 1.7. Consider (1.19) and (1.20) with $m = 3$, $p = 2$ and $a \equiv 1$. Then from Theorem 1.10 we obtain the following estimates

$$|y(t_k)| \ge C_1(\tau - t_k)^{-\frac{8}{3}}, \quad |y^{[1]}(\tau_k)| \ge C_2(\tau - \tau_k)^{-8}$$

where $M = \frac{288}{5}$, $C_1 = \frac{1}{32K_2r_0}\left(\frac{10}{9}\right)^{\frac{2}{3}}$ and $C_2 = \left(\frac{5}{144K_2r_0}\right)^2$.

Now, let us turn our attention to nonoscillatory solutions of (1.19).

Theorem 1.11. *Let $m > p$ and $M \ge 0$ hold such that*

$$|f(x)| \le |x|^m \quad \textit{for} \quad |x| \ge M.$$

If y be a nonoscillatory noncontinuable solution of (1.19) *defined on $[T, \tau)$, then constants C, C_0 and a left neighborhood J of τ exist such that*

$$|y'(t)| \ge C(\tau - t)^{-\frac{p(m+1)}{m(m-p)}}, \quad t \in J.$$

Let, moreover, $m < p + \sqrt{p^2 + p}$. Then

$$|y(t)| \ge C_0(\tau - t)^{m_1}, \quad \textit{with} \quad m_1 = \frac{m^2 - 2mp - p}{m(m-p)} < 0.$$

Our last application is devoted to the equation

$$y'' = r(t)|y|^m \operatorname{sgn} y \tag{1.21}$$

where $r \in C^0(\mathbb{R}_+)$ and $m > 1$.

Theorem 1.12. *Let $\tau \in (0, \infty)$, $T \in [0, \tau)$ and $r(t) > 0$ on $[t, \tau]$.*

(i) *Then* (1.21) *has a nonoscillatory noncontinuable solution which is defined in a left neighbourhood of τ.*

(ii) *Let y be a nonoscillatory noncontinuable solution of* (1.21) *defined on $[T, \tau)$. Then constants C, C_1, C_2 and a left neighbourhood I of τ exists such that*

$$|y(t)| \le C(\tau - t)^{-\frac{2(m+3)}{m-1}} \quad \textit{and} \quad |y'(t)| \ge C_1(\tau - t)^{-\frac{m+1}{m(m-1)}}, \quad t \in I.$$

If, moreover, $m < 1 + \sqrt{2}$, then

$$|y(t)| \le C_2(\tau - t)^{m_1} \quad \textit{with} \quad m_1 = \frac{m^2 - 2m - 1}{m(m-1)} < 0.$$

In the fifth chapter, we study asymptotic properties of the initial value problem (1.10), (1.11).

Definition 1.6. We say that a solution $u(t)$ of (1.10) possesses the property (L) if $u(t) = at + b + o(t)$ as $t \to \infty$, where a, b are real constants.

Theorem 1.13. *Let $p \ge 1$, $r > 0$ and $t_0 > 0$. Suppose that the following conditions are satisfied:*

(i) *$f(t, u, v)$ is a continuous function in $D = \{(t, u, v) : t \in [t_0, \infty), u, v \in \mathbb{R}\}$ where $t_0 > 0$;*

(ii) *there exist continuous functions $h, g : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t, u, v)| \le h(t) g\left(\left[\frac{|u|}{t}\right]^r\right) |v|^r,\ (t, u, v) \in D$$

where for $s > 0$, the function $g(s)$ is positive and nondecreasing,

$$\int_{t_0}^{\infty} h(s)\,\mathrm{d}s < \infty$$

and if we denote

$$G(x) = \int_{t_0}^{x} \frac{\mathrm{d}s}{s^{\frac{r}{p}} g(s^{\frac{r}{p}})},$$

then

$$G(\infty) = \int_{t_0}^{\infty} \frac{\mathrm{d}s}{s^{\frac{r}{p}} g(s^{\frac{r}{p}})} = \frac{p}{r} \int_{a}^{\infty} \frac{\tau^{\frac{p}{r} - 2}}{g(\tau)}\,\mathrm{d}\tau = \infty$$

where $a = (t_0)^{\frac{r}{p}}$.

Then any continuable solution $u(t)$ *of the equation* (1.10) *possesses the property* (L).

Example 1.8. Let $t_0 = 1,\ p \geq r > 0, p \geq 1$

$$f(t,u,u') = \eta(t) t^{1-\alpha} e^{-t} \left(\frac{u}{t}\right)^{p-r} \ln\left[2 + \left(\frac{|u|}{t}\right)^r\right] (u')^r,\ t \geq 1$$

where $0 < \alpha < 1$ and $\eta(t)$ is a continuous function on interval $[1,\infty)$ with $K = \sup_{t\geq 1} |\eta(t)| < \infty$. Then all conditions of Theorem 1.13 are satisfied for every continuable solution $u(t)$ of the initial value problem (1.10), (1.11) there exist numbers a, b such that $u(t) = at + b + o(t)$ as $t \to \infty$.

Theorem 1.14. *Let* $p \geq 1, r > 0$ *and* $t_0 > 0$. *Suppose the following conditions are satisfied:*

(i) *The function* $f(t,u,v)$ *is continuous in* $\mathrm{D} = \{(t,u,v) : t \in [t_0,\infty), u, v \in \mathbb{R}\}$*;*

(ii) *there exist continuous functions* $h_1, h_2, h_3, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|f(t,u,v)| \leq h_1(t) g_1\left(\left[\frac{|u|}{t}\right]^r\right) + h_2(t) g_2(|v|^r) + h_3(t),\ (t,u,v) \in D$$

for $s > 0$ *the functions* $g_1(s)$, $g_2(s)$ *are nondecreasing and if*

$$G(x) = \int_{t_0}^{x} \frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}}) + g_2(s^{\frac{r}{p}})},$$

then

$$G(\infty) = \int_{t_0}^{\infty} \frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}}) + g_2(s^{\frac{r}{p}})} = \frac{p}{r} \int_{a}^{\infty} \frac{\tau^{\frac{p}{r}-1}\,\mathrm{d}\tau}{g_1(\tau) + g_2(\tau)} = \infty$$

where $a = (t_0)^{\frac{r}{p}}$.

Then any continuable solution $u(t)$ *of the equation* (1.10) *possesses the property* (L).

Example 1.9. Let $t_0 = 1$, $p \geq r > 0$, $p \geq 1$

$$\begin{aligned} f(t,u,v) = {} & \eta_1(t) t^{1-\alpha_1} e^{-t} \left(\frac{u}{t}\right)^{p-r} \ln\left[2 + \left(\frac{u}{t}\right)^r\right] \\ & + \eta_2(t) t^{1-\alpha_2} e^{-t} v^{p-r} \ln(3 + v^r) + \eta_3(t) t^{1-\alpha_3} e^{-t} \end{aligned}$$

where $0 < \alpha_i < 1$ and $\eta_i(t)$ are continuous functions on interval $[1,\infty)$ with $K_i = \sup_{t\geq 1} |\eta_i(t)| < \infty$, $i = 1, 2, 3$.

Then all assumptions of Theorem 1.14 are satisfied and thus any continuable solution $u(t)$ of the equation (1.10) possesses the property (L).

Theorem 1.15. *Let* $t_0 > 0$. *Suppose that the following assumptions hold:*

(i) *There exist nonnegative continuous functions* $h_1, h_2, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|f(t,u,v)| \le h_1(t) g_1\left(\left[\frac{|u|}{t}\right]^r\right) + h_2(t) g_2(|v|^r);$$

(ii) *for* $s > 0$ *the functions* $g_1(s)$, $g_2(s)$ *are nondecreasing and*

$$g_1(\alpha u) \le \psi_1(\alpha) g_1(u), \qquad g_2(\alpha u) \le \psi_2(\alpha) g_2(u)$$

for $\alpha \ge 1$, $u \ge 0$, *where the functions* $\psi_1(\alpha)$, $\psi_2(\alpha)$ *are continuous for* $\alpha \ge 1$;

(iii) $\int_{t_0}^{\infty} h_i(s)\,\mathrm{d}s = H_i < \infty, i = 1, 2$. *Assume that there exists a constant* $K \ge 1$ *such that*

$$K^{-1}(\psi_1(K) + \psi_2(K)) 2^{p-1}(H_1 + H_2) \le \int_{t_0}^{+\infty} \frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}}) + g_2(s^{\frac{r}{p}})} = \frac{p}{r} \int_a^{+\infty} \frac{\tau^{\frac{p}{r}-1}\,\mathrm{d}\tau}{g_1(\tau) + g_2(\tau)}$$

where $a = (t_0)^{\frac{r}{p}}$.

Then any continuable solution $u(t)$ *of the equation* (1.10) *with initial data* $u(t_0) = u_0$, $u'(t_0) = u_1$ *such that* $(|u_0| + |u_1|)^p \le K$ *possesses the property* (L).

Example 1.10. Let $t_0 > 0$. Consider the equation (1.10) with $p \ge 1$, $\frac{p}{r} = 2$,

$$f(t,u,v) = h_1(t) u^2 + h_2(t) v^2$$

where $h_1(t) = \frac{\eta_1(t)}{t^2} t^{1-\alpha_1} e^{-t}$, $h_2(t) = \eta_2(t) t^{1-\alpha_2} e^{-t}$, $0 < \alpha_i \le 1, \eta_i(t), i = 1, 2$ are continuous functions on the interval $[0, \infty)$ with $K_i = \sup_{t \ge t_0} |\eta_i(t)| < \infty$. Then all assumptions of Theorem 1.15 are satisfied and therefore any continuable solution $u(t)$ of the equation (1.10) (independently on the initial values u_0, u_1) possesses the property (L).

Theorem 1.16. *Let* $t_0 > 0$. *Suppose that the asumptions* (i) *and* (iii) *of Theorem 1.15 hold, while* (ii) *is replaced by*

(ii′) *for* $s > 0$ *the functions* $g_1(s)$, $g_2(s)$ *are nonnegative, continuous and nondecreasing,* $g_1(0) = g_2(0) = 0$ *and satisfy a Lipschitz condition*

$$|g_1(u+v) - g_1(u)| \le \lambda_1 v, \quad |g_2(u+v) - g_2(u)| \le \lambda_2 v$$

where λ_1, λ_2 *are positive constants.*

Then any continuable solution $u(t)$ *of the equation* (1.10) *with initial data* $u(t_0) = u_0$, $u'(t_0) = u_1$ *such that* $|u_0|^p + |u_1|^p \le K$ *possesses property* (L).

Theorem 1.17. *Let $t_0 > 0$. Suppose that there exist continuous functions $h : \mathbb{R}_+ \to \mathbb{R}_+$, $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \le h(t) g_1\left(\left[\frac{|u|}{t}\right]^r\right) g_2(|v|^r)$$

where for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are nondecreasing;

$$\int_{t_0}^{\infty} h(s)\,\mathrm{d}s < \infty$$

and if we denote

$$G(x) = \int_{t_0}^{x} \frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}}) g_2(s^{\frac{r}{p}})},$$

then $G(+\infty) = \frac{p}{r}\int_a^\infty \frac{\tau^{\frac{p}{r}-1}}{g_1(\tau)g_2(\tau)}\,\mathrm{d}\tau = +\infty$ where $a = (t_0)^{\frac{r}{p}}$.

Then any continuable solution $u(t)$ of the equation (1.10) *possesses the property* (L).

Example 1.11. Let $t_0 = 1$, $p \ge r > 0$,

$$f(t,u,v) = \eta(t) t^{1-\alpha} e^{-t} \left[\left(\frac{u}{t}\right)^{p-r} \ln\left[2 + \left(\frac{u}{t}\right)^r\right]\right]^{\frac{3}{4}} \cdot \left[v^{p-r}\ln(2+v^r)\right]^{\frac{1}{4}}$$

where $\eta(t)$ is a continuous function on $[1,\infty)$ with $K = \sup_{t\in\langle 1,\infty)} \eta(t) < \infty$. Then all assumptions of Theorem 1.17 are satisfied and this means that any continuable solution of the equation (1.10) possesses the property (L).

Theorem 1.18. *Let $t_0 > 0$. Suppose that the following conditions hold:*

(i) *there exist nonnegative continuous functions $h, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \le h(t) g_1\left(\left[\frac{|u(t)|}{t}\right]^r\right) g_2(|v|^r);$$

(ii) *for $s > 0$ the functions $g_1(s), g_2(s)$ are nondecreasing and*

$$g_1(\alpha u) \le \psi_1(\alpha) g_1(u), \quad g_2(\alpha u) \le \psi_2(\alpha) g_2(u)$$

for $\alpha \ge 1, u \ge 0$, where the functions $\psi_1(\alpha), \psi_2(\alpha)$ are continuous for $\alpha \ge 1$;

(iii) $\int_{t_0}^\infty h(s)\,\mathrm{d}s = H < +\infty$. *Assume also that there exists a constant $K \ge 1$ such that*

$$K^{-1} H \psi_1(K)\psi_2(K) \le \int_1^\infty \frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}}) g_2(s^{\frac{r}{p}})} = \frac{p}{r}\int_a^\infty \frac{\tau^{\frac{p}{r}-1}\,\mathrm{d}\tau}{g_1(\tau)g_2(\tau)}$$

where $a = (t_0)^{\frac{r}{p}}$.

Then any continuable solution $u(t)$ *of the equation* (1.10) *with initial data* $u(t_0) = u_0$, $u'(t_0) = u_1$ *such that* $2^{p-1}(|u_0|^p + |u_1|^p) \leq K$ *possesses the property* (L).

Theorem 1.19. *Let* $t_0 > 0$. *Suppose that the assumptions* (i) *and* (iii) *of Theorem 1.18 hold, while* (ii) *is replaced by*

(ii′) *for* $s > 0$ *the functions* $g_1(s)$, $g_2(s)$ *are continuous and nondecreasing,* $g_1(0) = g_2(0) = 0$ *and satisfy a Lipschitz condition*

$$|g_1(u+v) - g_1(u)| \leq \lambda_1 v, \quad |g_2(u+v) - g_2(u)| \leq \lambda_2 v$$

where λ_1, λ_2 *are positive constants.*

Then any continuable solution $u(t)$ *of the equation* (1.10) *with initial data* $u(t_0) = u_0$, $u'(t_0) = u_1$ *such that* $|u_0|^p + |u_1|^p \leq K$ *possesses the property* (L).

Note, that the methods of proof in [P2] are used to other types of differential equations, see. e.g. [13, 15].

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On existence of proper solutions of quasilinear second order differential equations

2

2.1 Introduction

In this chapter, we study the existence of proper solutions of a forced second order nonlinear differential equation of the form

$$(a(t)|y'|^{p-1}y')' + b(t)g(y') + r(t)f(y) = e(t) \tag{2.1}$$

where $p > 0$, $a \in C^0(R_+)$, $b \in C^0(R_+)$, $r \in C^0(R_+)$, $e \in C^0(R_+)$, $f \in C^0(R)$, $g \in C^0(R)$, $R_+ = [0, \infty)$, $R = (-\infty, \infty)$ and $a > 0$ on R_+.

A special case of Equation (2.1) is the unforced equation

$$(a(t)|y'|^{p-1}y')' + b(t)g(y') + r(t)f(y) = 0. \tag{2.2}$$

We will often use of the following assumptions

$$f(x)x \geq 0 \quad \text{on } R \tag{2.3}$$

and

$$g(x)x \geq 0 \quad \text{on } R_+. \tag{2.4}$$

Definition 2.1. A solution y of (2.1) is called proper if it is defined on R_+ and $sup_{t\in[\tau,\infty)}|y(t)| > 0$ for every $\tau \in (0, \infty)$. It is called singular of the 1-st kind if it is defined on R_+, there exists $\tau \in (0, \infty)$ such that $y \equiv 0$ on $[\tau, \infty)$ and $sup_{T\leq t\leq\tau}|y(t)| > 0$ for every $T \in [0, \tau)$. It is called singular of the 2-nd kind if it is defined on $[0, \tau), \tau < \infty$ and $sup_{0\leq t<\tau}|y'(t)| = \infty$.

Note, that a singular solution y of the 2-nd kind is sometimes called noncontinuable.

Definition 2.2. A proper solution y of (2.1) is called oscillatory if there exists a sequence of its zeros tending to ∞. Otherwise, it is called nonoscillatory. A solution y of (2.1) is called weakly oscillatory if there exists a sequence of zeros of y' tending to ∞.

It is easy to see that (2.1) can be transformed into the system

$$y_1' = a(t)^{-\frac{1}{p}}|y_2|^{\frac{1}{p}} \operatorname{sgn} y_2,$$
$$y_2' = -b(t)g(a(t)^{-\frac{1}{p}}|y_2|^{\frac{1}{p}} \operatorname{sgn} y_2) - r(t)f(y_1) + e(t); \tag{2.5}$$

the relation between a solution y of (2.1) and a solution of (2.5) is $y_1(t) = y(t)$, $y_2(t) = a(t)|y'(t)|^{p-1}y'(t)$.

An important problem is the existence of solutions defined on R_+ or of proper solutions (for Equation (2.2)). Their asymptotic behaviour is studied by many authors (see e.g. monographs [7], [9] and [10], and the references therein). So, it is very important to know conditions under the validity of which all solutions of (2.1) are defined on R_+ or are proper. For a special type of the equation of (2.2), for the equation

$$(a(t)|y'|^{p-1}y')' + r(t)f(y) = 0, \tag{2.6}$$

sufficient conditions for all nontrivial solutions to be proper are given e.g. in [1], [8], [9] and [10]. It is known that for half-linear equations, i.e., if $f(x) = |x|^p \operatorname{sgn} x$, all nontrivial solutions of (2.4) are proper, see e.g. [6]. For the forced equation (2.1) with (2.3) holding, $a \in C^1(R_+)$, $a^{\frac{1}{p}} r \in AC^1_{loc}(R_+)$ and $b \equiv 0$, it is proved in [2] that all solutions are defined on R_+, i.e., the set of all singular solutions of the second kind is empty. On the other hand, in [4] and [5] examples are given for which Equation (2.6) has singular solutions of the first and second kinds (see [1], as well). Moreover, Lemma 4 in [3] gives sufficient conditions for the equation

$$(a(t)y')' + r(t)f(y) = 0$$

to have no proper solutions.

In the present paper, these problems are solved for (2.1). Sufficient conditions for the nonexistence of singular solutions of the first and second kinds are given, and so, sufficient conditions for all nontrivial solutions of (2.2) to be proper are given. In the last section, simple asymptotic properties of solutions of (2.2) are given.

Note that it is known that Equation (2.6) has no weakly oscillatory solutions (see e.g. [10]), but as we will see in Section 4, Equation (2.1) may have them.

It will be convenient to define the following constants:

$$\gamma = \frac{p+1}{p(\lambda+1)}, \quad \delta = \frac{p+1}{p}.$$

We define the function $R : R_+ \to R$ by $R(t) = a^{\frac{1}{p}}(t)r(t)$.

For any solution y of (2.1), we let

$$y^{[1]}(t) = a(t)|y'(t)|^{p-1}y'(t)$$

and if (2.3) and $r > 0$ on R_+ hold, let us define

$$\begin{aligned} V(t) &= \frac{a(t)}{r(t)}|y'(t)|^{p+1} + \gamma \int_0^{y(t)} f(s)\,\mathrm{d}s \\ &= \frac{|y^{[1]}(t)|^{\delta}}{R(t)} + \gamma \int_0^{y(t)} f(s)\,\mathrm{d}s \geq 0. \end{aligned} \tag{2.7}$$

For any continuous function $h : R_+ \to R$, we let $h_+(t) = \max\{h(t), 0\}$ and $h_-(t) = \max\{-h(t), 0\}$ so that $h(t) = h_+(t) - h_-(t)$.

2.2 Singular solutions of the second kind

In this section, the nonexistence of singular solutions of the second kind will be studied. The following theorem is a generalization of the well-known Wintner's Theorem to (2.1).

Theorem 2.1. *Let $M > 0$ and $|g(x)| \leq |x|^p$ and $|f(x)| \leq |x|^p$ for $|x| \geq M$. Then there exist no singular solution y of the second kind of* (2.1) *and all solutions of* (2.1) *are defined on R_+.*

Proof. Let, to the contrary, y be a singular solution of the second kind defined on $[0, \tau)$, $\tau < \infty$. Then,

$$\sup_{0 \leq t < \tau} |y'(t)| = \infty \qquad \text{and} \qquad \sup_{0 \leq t < \tau} |y^{[1]}(t)| = \infty. \tag{2.8}$$

The assumptions of the theorem yield

$$|f(x)| \leq M_1 + |x|^p \qquad \text{and} \qquad |g(x)| \leq M_2 + |x|^p \tag{2.9}$$

with $M_1 = \max_{|s| \leq M} |f(s)|$ and $M_2 = \max_{|s| \leq M} |g(s)|$. Let $t_0 \in [0, \tau)$ be such that

$$\tau - t_0 \leq 1, \quad \int_{t_0}^{\tau} a^{-1}(s)|b(s)|\,\mathrm{d}s \leq \frac{1}{2} \tag{2.10}$$

and

$$2^p \max_{0 \leq s \leq \tau} |r(s)| \Big(\int_{t_0}^{\tau} a^{-\frac{1}{p}}(s)\,\mathrm{d}s \Big)^p \leq \frac{1}{3}. \tag{2.11}$$

Using system (2.5), by an integration we obtain

$$|y_1(t)| \leq |y_1(t_0)| + \int_{t_0}^{t} a^{-\frac{1}{p}}(s)|y_2(s)|^{\frac{1}{p}}\,\mathrm{d}s \tag{2.12}$$

and

$$|y_2(t)| \le |y_2(t_0)| + \int_{t_0}^{t} \left(|b(s) g(a(s)^{-\frac{1}{p}} |y_2(s)|^{\frac{1}{p}} \operatorname{sgn} y_2(s))| + |r(s)||f(y_1(s))| + |e(s)| \right) \mathrm{d}s. \quad (2.13)$$

Hence, using (2.9), (2.10) and (2.12), we have for $t \in [t_0, \tau)$

$$\begin{aligned} |y_2(t)| &\le |y_2(t_0)| + \int_{t_0}^{t} |b(s)|[M_2 + a^{-1}(s)|y_2(s)|] \,\mathrm{d}s \\ &+ \int_{t_0}^{t} |r(s)|[M_1 + |y_1(s)|^p] \,\mathrm{d}s + \int_{t_0}^{t} |e(s)| \,\mathrm{d}s \\ &\le M_3 + \frac{1}{2} \max_{t_0 \le s \le t} |y_2(s)| \,\mathrm{d}s \\ &+ \int_{t_0}^{t} |r(s)|[|y_1(t_0)| + \int_{t_0}^{s} a^{-\frac{1}{p}}(\sigma)|y_2(\sigma)|^{\frac{1}{p}} \mathrm{d}\sigma]^p \,\mathrm{d}s \end{aligned} \quad (2.14)$$

with $M_3 = |y_2(t_0)| + M_2 \int_{t_0}^{\tau} |b(s)| \,\mathrm{d}s + M_1 \int_{t_0}^{\tau} |r(s)| \,\mathrm{d}s + \int_{t_0}^{\tau} |e(s)| \,\mathrm{d}s$.

Denote $v(t_0) = |y_2(t_0)|$ and $v(t) = \max_{t_0 \le s \le t} |y_2(s)|$, $t \in (t_0, \tau)$. Then (2.10), (2.12) and (2.14) yield

$$\begin{aligned} v(t) &\le M_3 + \frac{1}{2} v(t) + \int_{t_0}^{t} |r(s)|[|y_1(t_0)| + M_4 v(s)^{\frac{1}{p}}]^p \,\mathrm{d}s \\ &\le M_3 + \frac{1}{2} v(t) + 2^p M_5 \int_{t_0}^{t} [y_1^p(t_0) + M_4^p v(s)] \,\mathrm{d}s \\ &\le M_3 + \frac{1}{2} v(t) + 2^p M_5 y_1^p(t_0) + 2^p M_4^p M_5 v(t) \end{aligned}$$

with $M_4 = \int_{t_0}^{\tau} a^{-\frac{1}{p}}(\sigma) \mathrm{d}\sigma$, $M_5 = \max_{0 \le s \le \tau} |r(s)|$.

From this and from (2.11), we have

$$\frac{1}{6} v(t) \le M_3 + 2^p M_5 y_1^p(t_0), t \in [t_0, \tau).$$

But this inequality contradicts (2.8) and the definition of v. □

Remark 2.1. The results of Theorem 2.1 for Equation (2.1) with $p \le 1$ and without the damping $(b \equiv 0)$ is a generalization of the well-known Wintner's Theorem, see e.g. [9, Theorem $11.5.$] or [7, Theorem $6.1.$].

The following result shows that singular solutions of the second kind of (2.1) do not exist if $r > 0$ and R is smooth enough under weakened assumptions on f.

Theorem 2.2. *Let* (2.3), $R \in C^1(R_+)$, $r > 0$ *on* R_+ *and let either*

(i) $M \in (0, \infty)$ *exist such that* $|g(x)| \leq |x|^p$ *for* $|x| \geq M$
or

(ii) (2.4) *holds and* $b(t) \geq 0$ *on* R_+.

Then Equation (2.1) *has no singular solution of the second kind and all solutions of* (2.1) *are defined on* R_+.

Proof. Suppose y is a singular solution of the second kind defined on $I = [0, \tau)$. Then $sup_{t\in[0,\tau)}|y'(t)| = \infty$ and (2.7) yields

$$V'(t) = \left(\frac{1}{R(t)}\right)' |y^{[1]}(t)|^\delta + \frac{\delta}{r(t)} y'(t)(y^{[1]}(t))' + \delta f(y(t))y'(t)$$
$$= \left(\frac{1}{R(t)}\right)' |y^{[1]}(t)|^\delta + \frac{\delta}{r(t)} y'(t)[e(t) - b(t)g(y'(t))$$
$$- r(t)f(y(t))] + \delta f(t)y'(t)$$

or

$$V'(t) = \left(\frac{1}{R(t)}\right)' |y^{[1]}(t)|^\delta + \frac{\delta}{r(t)} y'(t)e(t) - \frac{\delta b(t)g(y'(t))y'(t)}{r(t)} \tag{2.15}$$

for $t \in I$. We will estimate the summands in (2.15). We have

$$\left(\frac{1}{R(t)}\right)' |y^{[1]}(t)|^\delta = \frac{-R'(t)}{R(t)} \frac{|y^{[1]}(t)|^\delta}{R(t)} \leq \frac{R'_-(t)}{R(t)} V(t) \tag{2.16}$$

on I.
From $|x| \leq |x|^s + 1$ for $s \geq 1$ and $x \in R$, we get

$$\left|\frac{\delta e(t)}{r(t)} y'(t)\right| = \left|\frac{\delta e(t) a^{\frac{1}{p}}(t) y'(t)}{R(t)}\right| \tag{2.17}$$
$$\leq \delta |e(t)| a^{\frac{1}{p}}(t) \frac{|y'(t)|^{p+1} + 1}{R(t)}$$
$$= \frac{\delta |e(t)| |y^{[1]}(t)|^\delta}{a(t)R(t)} + \frac{\delta |e(t)|}{r(t)} \leq \frac{\delta |e(t)| V(t)}{a(t)} + \frac{\delta |e(t)|}{r(t)}$$

on I. Furthermore, in case (ii), we have

$$-\frac{\delta b(t) g(y'(t)) y'(t)}{r(t)} \leq v(t) + \frac{\delta |b(t)| |y'(t)|^{p+1}}{r(t)}$$
$$= v(t) + \frac{\delta |b(t)| |y^{[1]}(t)|^\delta}{a(t)R(t)} \leq v(t) + \frac{\delta |b(t)| V(t)}{a(t)} \tag{2.18}$$

with

$$v(t)=\frac{\delta|b(t)|}{r(t)}\max_{|s|\leq M}|sg(s)|.$$

Due to the fact that $b\geq 0$, inequality (2.18) holds in case (i) with $v(t)\equiv 0$. From this and (2.15), (2.16) and (2.17), we obtain

$$V'(t)\leq\left[\frac{R'_{-}(t)}{R(t)}+\frac{\delta}{a(t)}[|e(t)|+|b(t)|]\right]V(t)+\frac{\delta|e(t)|}{r(t)}+v(t). \tag{2.19}$$

The integration of (2.19) on $[0,t]\in I$ yields

$$\begin{aligned} V(t)-V(0)\leq&\int_0^t\left[\frac{R'_{-}(s)}{R(s)}+\frac{\delta}{a(s)}[|e(s)|+|b(s)|]\right]V(s)\,\mathrm{d}s\\ &+\int_0^\tau\left[\frac{\delta|e(t)|}{r(t)}+v(t)\right]\,\mathrm{d}t.\end{aligned}$$

Hence, Gronwall's inequality yields

$$\begin{aligned} 0\leq V(t)\leq&\left[V(0)+\int_0^\tau\left[\frac{\delta|e(t)|}{r(t)}+v(t)\right]\,\mathrm{d}t\right]\\ &\times\exp\int_0^\tau\left[\frac{R'_{-}(t)}{R(t)}+\frac{\delta}{a(t)}[|e(t)|+|b(t)|]\right]\,\mathrm{d}t. \end{aligned}\tag{2.20}$$

Now $V(t)$ is bounded from above on I since I is a bounded interval, so (2.7) yields that $|y^{[1]}(t)|^\delta$ and $|y'(t)|$ are bounded above on I. But this inequality contradicts (2.8). □

Remark 2.2. It is clear from the proof of Theorem 2.2 (ii) that if $b\equiv 0$, then assumption (2.4) is not needed in case (ii).

Remark 2.3. Note that the condition $|g(x)|\leq|x|^p$ in (i) can not be improved upon even for Equation (2.2).

Example 2.1. Let $\varepsilon\in(0,1)$. Then the function $y=\left(\frac{1}{1-t}\right)^{\frac{1-\varepsilon}{\varepsilon}}$ is a singular solution of the second kind of the equation

$$y''-|y'|^\varepsilon y'+C|y|^{\frac{1+\varepsilon}{1-\varepsilon}}\operatorname{sgn}y=0$$

on $[0,1)$ with $C=\left(\frac{1-\varepsilon}{\varepsilon^2}\right)^{\varepsilon+1}-\frac{1-\varepsilon}{\varepsilon^2}$.

Remark 2.4.

(i) The result of Theorem 2.2 is obtained in [2] in case $b\equiv 0$ using a the similar method.

(ii) Note that Theorem 2.2 is not valid if $R \notin C^1(R_+)$; see [1] or [4] for the case $g \equiv 0$.

Remark 2.5. Theorem 2.2 is not valid if $r < 0$ on an interval of positive measure, see e.g. [9, Theorem $11.3.$] (for (2.6) and $p = 1$). The existence of singular solutions of the second kind for (2.1) with $r > 0$ is an open problem.

2.3 Singular solutions of the first kind

In this section, the nonexistence of singular solutions of the first kind mainly for (2.2) will be studied. The following lemma shows that $e(t)$ has to be trivial in a neighbourhood of ∞ if Equation (2.1) has a singular solution of the first kind.

Lemma 2.1. *Let y be a singular solution of the first kind of* (2.1)*. Then $e(t) \equiv 0$ in a neighbourhood ∞.*

Proof. Let y be a singular solution of (2.1) and τ the number from its domain of definition. Then $y \equiv 0$ on $[\tau, \infty)$ and Equation (2.1) yields $e(t) \equiv 0$ on $[\tau, \infty)$. □

In what follows, we will only consider Equation (2.2).

Theorem 2.3. *Let $M > 0$ and*

$$|g(x)| \leq |x|^p \ \textit{ and } \ |f(x)| \leq |x|^p \ \textit{ for } |x| \leq M. \tag{2.21}$$

Then there exist no singular solution of the first kind of Equation (2.2)*.*

Proof. Assume that y is a singular solution of the first kind and τ is the number from Definition 2.1. Using system (2.5), we have $y_1 \equiv y_2 \equiv 0$ on $[\tau, \infty)$. Let $0 \leq T < \tau$ be such that

$$|y_1(t)| \leq M, \quad |y_2(t)| \leq M \quad \text{on } [T, \tau] \tag{2.22}$$

and

$$\int_T^\tau a(s)|b(s)| \, \mathrm{d}s + \left(\int_T^\tau a^{-\frac{1}{p}}(s) \, \mathrm{d}s \right)^p \int_T^\tau |r(s)| \, \mathrm{d}s \leq \frac{1}{2}. \tag{2.23}$$

Define $I = [T, \tau]$ and

$$v_1(t) = \max_{t \leq s \ \leq \tau} |y_1(s)|, \quad t \in I \ , \tag{2.24}$$

$$v_2(t) = \max_{t \leq s \ \leq \tau} |y_2(s)|, \quad t \in I. \tag{2.25}$$

From the definition of τ, (2.22), (2.24) and (2.25), we have

$$0 < v_1(t) \le M, 0 < v_2(t) \le M \quad \text{on} \quad [T,\tau). \tag{2.26}$$

An integration of the first equality in (2.5) and (2.25) yield

$$\begin{aligned} |y_1(t)| &\le \int_t^\tau a^{-\frac{1}{p}}(s)|y_2(s)|^{\frac{1}{p}}\,\mathrm{d}s \le \int_t^\tau a^{-\frac{1}{p}}(s)|v_2(s)|^{\frac{1}{p}}\,\mathrm{d}s \\ &\le |v_2(t)|^{\frac{1}{p}} \int_t^\tau a^{-\frac{1}{p}}(s)\,\mathrm{d}s \end{aligned} \tag{2.27}$$

on I. If $M_1 = \int_T^\tau a^{-\frac{1}{p}}(s)\,\mathrm{d}s$, then

$$|y_1(t)| \le M_1 |v_2(t)|^{\frac{1}{p}} \tag{2.28}$$

and from (2.24) we obtain

$$v_1(t) \le M_1 |v_2(t)|^{\frac{1}{p}},\ t \in I. \tag{2.29}$$

Similarly, an integration of the second equality in (2.5) and (2.21) yield

$$\begin{aligned} |y_2(t)| &\le \int_t^\tau \left| b(s) g\big(a^{\frac{1}{p}}(s)|y_2(s)|^{\frac{1}{p}} \operatorname{sgn} y_2(s)\big)\right| \mathrm{d}s \\ &\quad + \int_t^\tau |r(s) f(y_1(s))|\,\mathrm{d}s \\ &\le \int_t^\tau |b(s)|(a^{\frac{1}{p}}(s)|v_2(s)|^{\frac{1}{p}})^p\,\mathrm{d}s + \int_t^\tau |r(s)|y_1(s)|^p\,\mathrm{d}s. \end{aligned} \tag{2.30}$$

Hence, from this, (2.21), (2.23) and (2.28)

$$\begin{aligned} |y_2(t)| &\le v_2(t)\left[\int_T^\tau a(s)|b(s)|\,\mathrm{d}s + v_1^p(t)\int_T^\tau |r(s)|\,\mathrm{d}s\right] \\ &\le v_2(t)\left[\int_T^\tau a(s)|b(s)|\,\mathrm{d}s + M_1^p\int_T^\tau |r(s)|\,\mathrm{d}s\right] \le \frac{v_2(t)}{2}. \end{aligned} \tag{2.31}$$

Hence $v_2(t) \le \frac{v_2(t)}{2}$ and so $v_2(t) \equiv 0$ on I. The contradiction with (2.26) proves the conclusion. ☐

Theorem 2.4. *Consider* (2.3), $R \in C^1(R_+)$, $r > 0$ *on* R_+ *and let either*

(i) $M \in (0,\infty)$ *exist such that* $|g(x)| \le |x|^p$ *for* $|x| \le M$
or

(ii) (2.4) *and* $b(t) \le 0$ *on* R_+.

Then Equation (2.2) *has no singular solution of the first kind.*

Proof. Let $y(t)$ be singular solution of the first kind of (2.2). Then there exists $\tau \in (0, \infty)$ such that $y(t) \equiv 0$ on $[\tau, \infty)$ and $\sup_{T \le s < \tau} |y(s)| > 0$ for $T \in [0, \tau)$. Then, similar to the proof of Theorem 2.2, (2.15) and the equality in (2.16) hold with $e \equiv 0$. From this we have

$$\begin{aligned} V'(t) &= \left(\frac{1}{R(t)}\right)' |y^{[1]}(t)|^{\delta} - \frac{\delta b(t) g(y'(t)) y'(t)}{r(t)} \\ &\ge -\frac{R'_+(t)}{R^2(t)} a^{\delta}(t) |y'(t)|^{p+1} - \frac{\delta a^{\frac{1}{p}}(t) b(t) g(y'(t)) y'(t)}{R(t)}. \end{aligned} \tag{2.32}$$

Let (i) be valid. Let $T \in [0, \tau)$ be such that $|y'(t)| \le M$ on $[T, \tau]$, and let $\varepsilon > 0$ be arbitrary. Then,

$$\begin{aligned} \frac{V'(t)}{V(t)+\varepsilon} &\ge -\frac{|y'(t)|^{p+1}}{R(t)[V(t)+\varepsilon]} \left(a^{\delta}(t) \frac{R'_+(t)}{R(t)} + \delta a^{\frac{1}{p}}(t) |b(t)| \right) \\ &\ge -\frac{V(t)}{V(t)+\varepsilon} \left(a^{\delta}(t) \frac{R'_+(t)}{R(t)} + \delta a^{\frac{1}{p}}(t) |b(t)| \right) \\ &\ge -\left(a^{\delta}(t) \frac{R'_+(t)}{R(t)} + \delta a^{\frac{1}{p}}(t) |b(t)| \right). \end{aligned} \tag{2.33}$$

An integration on the interval $[t, \tau] \subset [T, \tau]$ yields

$$\frac{\varepsilon}{V(t)+\varepsilon} = \frac{V(\tau)+\varepsilon}{V(t)+\varepsilon} \ge \exp\left\{ -\int_t^{\tau} \left[a^{\delta}(s) \frac{R'_+(s)}{R(s)} + \delta a^{\frac{1}{p}}(s) |b(s)| \right] \mathrm{d}s \right\}.$$

As $\varepsilon > 0$ is arbitrary, we have

$$0 \ge V(t) \exp\left\{ -\int_t^{\tau} \left[a^{\delta}(s) \frac{R'_+(s)}{R(s)} + \delta a^{\frac{1}{p}}(s) b(s) \right] \mathrm{d}s \right\}, \ t \in [T, \tau].$$

Hence, $V(t) \equiv 0$ on $[T, \tau]$ and (2.7) yield $y(t) = 0$ on $[T, \tau]$. The contradiction to $\sup_{t \in [T,\tau]} |y(t)| > 0$ proves that the conclusion holds in this case.
Let (ii) hold; then from (2.7) and (2.32) we have

$$\begin{aligned} \frac{V'(t)}{V(t)+\varepsilon} &\ge \left\{ -a^{\delta}(t) \frac{R'_+(t)}{R^2(t)} |y'(t)|^{p+1} - \frac{\delta a^{\frac{1}{p}}(t) b(t) g(y'(t)) y'(t)}{R(t)} \right\} \\ &\quad \times (V(t)+\varepsilon)^{-1} \\ &\ge -\frac{V(t)}{V(t)+\varepsilon} a^{\delta}(t) \frac{R'_+(t)}{R(t)} \ge -a^{\delta}(t) \frac{R'_+(t)}{R(t)} \end{aligned} \tag{2.34}$$

for $t \in [0, \tau]$. Hence, we have a similar situation to that in (2.33) and the proof is similar to case (i). ☐

Remark 2.6. Theorem 2.3 generalized results of [10, Theorem $1.2.$], obtained in case $b \equiv 0$. Results of [7, Theorem $9.4.$] with ($b \equiv 0, f(x) = |x|^p \operatorname{sgn} x$) and of [1, Theorem 1] ($b \equiv 0$) are special cases of Theorem 2.1 here.

Remark 2.7. Theorem 2.4 is not valid if $r < 0$ on an interval of positive measure; see e.g. [9, Theorem $11.1.$] (for (2.6) and $p = 1$). The existence of singular solutions of the first kind of (2.2) is an open problem.

Remark 2.8. If $R \notin C^1(R_+)$, then the statement of Theorem 2.4 does not hold (see [1] for $g \equiv 0$ or [5]).

Note, that condition (i) in Theorem 2.4 can not be improved.

Example 2.2. Let $\varepsilon \in (0, 1)$. Then function $y = (1 - t)^{(1+\frac{1}{\varepsilon})}$ for $t \in [0, 1]$ and $y \equiv 0$ on $(1, \infty)$ is a singular solution of the first kind of the equation

$$y'' + \left(1 + \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\right)\left(1 + \frac{1}{\varepsilon}\right)^{\varepsilon-1} |y'|^{1-\varepsilon} \operatorname{sgn} y' + |y|^{\frac{1-\varepsilon}{1+\varepsilon}} = 0.$$

Note that $p = 1$ in this case.

Theorems 2.1, 2.2, 2.3 and 2.4 gives us sufficient conditions for all nontrivial solutions of (2.2) to be proper.

Corollary 2.1. *Let $|g(x)| \leq |x|^p$ and $|f(x)| \leq |x|^p$ for $x \in R$. Then every nontrivial solution y of* (2.2) *is proper.*

Corollary 2.2. *Let* (2.3), *$R \in C^1(R_+)$, $r > 0$ on R_+ and $|g(x)| \leq |x|^p$ on R hold. Then every nontrivial solution y of* (2.2) *is proper.*

Remark 2.9. The results of Corollary 2.1 and Corollary 2.2 are obtained in [1] for $b \equiv 0$.

2.4 Further properties of solutions of (2.2)

In this section, simple asymptotic properties of solutions of (2.2) are studied. Mainly, sufficient conditions are given under which zeros of a nontrivial solutions are simple and zeros of a solution and its derivative separate from each other.

Corollary 2.3. *Let the assumptions either of Theorem 2.3 or of Theorem 2.4 hold. Then any nontrivial solution of* (2.2) *has no double zeros on R_+.*

Proof. Let y be a nontrivial solution of (2.2) defined on R_+ with a double zero at $\tau \in R_+$, i.e. $y(\tau) = y'(\tau) = 0$. Then it is clear that the function

$$\bar{y}(t) = y(t) \text{ on } [0, \tau], \quad \bar{y}(t) = 0 \text{ for } t > \tau$$

is also solution of (2.2). As $\bar{y}$ is a singular solution of the first kind, we obtain contradiction with either Theorem 2.3 or with Theorem 2.4. □

Lemma 2.2. *Let $g(0) = 0$, $r \neq 0$ on R_+, and $f(x)x > 0$ for $x \neq 0$. Let y be a nontrivial solution of* (2.2) *such that $y'(t_1) = y'(t_2) = 0$ with $0 \leq t_1 < t_2 < \infty$. Then there exists $t_3 \in [t_1, t_2]$ such that $y(t_3) = 0$.*

Proof. We may suppose without loss of generality that t_1 and t_2 are consecutive zeros of y'; if t_1 or t_2 is an accumulation point of zeros of y', the result holds. If we define $z(t) = y^{[1]}(t), t \in R_+$, then

$$z(t_1) = z(t_2) = 0 \quad \text{and} \quad z(t) \neq 0 \quad \text{on} \quad (t_1, t_2). \tag{2.35}$$

Suppose, contrarily, that $y(t) \neq 0$ on (t_1, t_2). Then either

$$y(t_1)y(t_2) > 0 \quad \text{on} \quad [t_1, t_2] \tag{2.36}$$

or

$$y(t_1)y(t_2) = 0 \tag{2.37}$$

holds. If (2.36) is valid, then (2.2) and the assumptions of the lemma yields

$$\text{sgn } z'(t_1) = \text{ sgn } z'(t_2) \neq 0$$

and the contradiction with (2.35) proves the statement in this case.

If (2.37) holds the conclusion is valid. □

Corollary 2.4. *Let $f(x)x > 0$ for $x \neq 0$ and one of the following possibilities hold:*

(i) $r \neq 0$ *on* R_+ *and*

$$|g(x)| \leq |x|^p \quad \textit{and} \quad |f(x)| \leq |x|^p \quad \textit{for} \quad x \in R;$$

(ii) $R \in C^1(R_+)$, $r > 0$ *on* R_+ *and*

$$|g(x)| \leq |x|^p \quad \textit{for} \quad |x| \in R;$$

(iii) $R \in C^1(R_+)$, $b \leq 0$ *on* R_+, $r > 0$ *on* R_+, $g(x)x \geq 0$ *on* R_+ *and* $M > 0$ *exists such that*

$$|g(x)| \geq |x|^p \quad \textit{for} \quad |x| \geq M;$$

(iv) *$R \in C^1(R_+)$, $r > 0$ on R_+, $b \geq 0$ on R_+, $g(x)x \geq 0$ on R and M exists such that*

$$|g(x)| \leq |x|^p \quad \textit{for} \quad |x| \leq M.$$

Then the zeros of y and y' (if any) separate from each other, i.e. between two consecutive zeros of y (y') there is the only zero of y (y').

Proof. Accounting to our assumptions, Corollary 2.3 holds and hence all zeros of any nontrivial solution y of (2.2) are simple, there exists no accumulation point of zeros of y on R_+, and there exists no interval $[\alpha, \beta] \in R_+, \alpha < \beta$ of zeros of y. Then, the statement follows from Lemma 2.2 and Rolle's Theorem. □

Theorem 2.5. *Let $g(0) = 0$, $r \neq 0$ on R_+ and $f(x)x > 0$ for $x \neq 0$. Then* (2.2) *has no weakly oscillatory solution and every nonoscillatory solution y of* (2.2) *has a limit as $t \to \infty$.*

Proof. Let y be a weakly oscillatory solution of (2.2). Then there exist t_0, t_1 and t_2 such that $0 \leq t_0 < t_1 < t_2$, $y(t) \neq 0$ on $[t_0, \infty)$ and $y'(t_1) = y'(t_2) = 0$. But this fact contradicts Lemma 2.2. □

The following examples show that some of the assumptions of Theorem 2.5 cannot be omitted.

Example 2.3. The function $y = 2 + \sin t,\ t \in R_+$ is a weakly oscillatory solution of the equation

$$y'' - y' + \frac{\sin t + \cos t}{2 + \sin t} y = 0.$$

In this case $r \neq 0$, Theorem 2.5 is not valid.

Example 2.4. The function $y = 2 + \sin t,\ t \in R_+$ is a weakly oscillatory solution of the equation

$$y'' - g(y') + 2y = 0 \ \ \textit{with} \ \ g(x) = \begin{cases} 4 + \sqrt{1 - x^2} & \text{for } |x| \leq 1; \\ 4 & \text{for } |x| > 1. \end{cases}$$

In this case $g(0) \neq 0$, Theorem 2.5 is not valid.

Remark 2.10. If $g \equiv 0$, the result of Theorem 2.5 is known, see e.g. [10, Lemma 5.1.] or a direct application of (2.5).

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Existence of global solutions for systems of second order differential equations with p-Laplacian

3

3.1 Introduction

The p-Laplace differential equation

$$\mathrm{div}(\|\nabla v\|)^{p-2}\nabla v) = h(\|x\|, v) \tag{3.1}$$

plays an important role in the theory of partial differential equations (see e.g. [21]) where ∇ is the gradient, $p > 0$ and $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^n$, $n > 1$ and $h(y, v)$ is a nonlinear function on $\mathbb{R} \times \mathbb{R}$. Radially symmetric solutions of the equation (3.1) depend on the scalar variable $r = \|x\|$ and they are solutions of the ordinary differential equation

$$r^{1-n}(r^{n-1}|v'|)' = h(r, v) \tag{3.2}$$

where $v' = \frac{\mathrm{d}v}{\mathrm{d}r}$ and $p > 1$. If $p \neq n$ then the change of variables $r = t^{\frac{p-1}{p-n}}$ transforms the equation (3.2) into the equation

$$(\Psi_p(u'))' = f(t, u) \tag{3.3}$$

where $\Psi_p(u') = |u'|^{p-2}u'$ is so called one-dimensional, or scalar p-Laplacian [21] and

$$f(t, u) = \left|\frac{p-1}{p-n}\right|^p t^{\frac{p-n}{p(1-n)}} h(t^{\frac{p-1}{p-n}}, u)\,.$$

In [22] the existence of periodic solutions of the system

$$(\Phi_p(u'))' + \frac{\mathrm{d}}{\mathrm{d}t}\nabla F(u) + \nabla G(u) = e(t) \tag{3.4}$$

is studied where

$$\Phi_p : \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi_p(u) = (|u_1|^{p-2}u_1, \ldots, |u_n|^{p-2}u_n)^T.$$

The operator $\Phi_p(u')$ is called multidimensional p-Laplacian. The study of radially symmetric solutions of the system of p-Laplace equations

$$\operatorname{div}(\|\nabla v_i\|^{p-2}\nabla v_i) = h_i(\|x\|, v_1, v_2, \ldots, v_n), \quad i = 1, 2, \ldots, n, \quad p > 1$$

leads to the system of ordinary differential equations

$$(|u_i'|^{p-2}u_i')' = f_i(t, u_1, u_2, \ldots, u_n), \quad i = 1, 2, \ldots, n, \quad p \neq n \tag{3.5}$$

where

$$f_i(t, u_1, u_2, \ldots, u_n) = \left|\frac{p-1}{p-n}\right|^p t^{\frac{p-n}{p(1-n)}} h_i(t^{\frac{p-1}{p-n}}, u_1, u_2, \ldots, u_n).$$

This system can be written in the form

$$(\Phi_p(u'))' = f(t, u) \tag{3.6}$$

where $f = (f_1, f_2, \ldots, f_n)^T$ and $\Phi_p(u')$ is the n-dimensional p-Laplacian. Throughout this paper we consider the operator Φ_{p+1} with $p > 0$ and for the simplicity we denote it as Φ_p, i. e.

$$\Phi_p(u) = (|u_1|^{p-1}u_1, |u_2|^{p-1}u_2, \ldots, |u_n|^{p-1}u_n).$$

We shall study the initial value problem

$$(A(t)\Phi_p(y'))' + B(t)g(y') + R(t)f(y) = e(t), \tag{3.7}$$

$$y(0) = y_0, \quad y'(0) = y_1 \tag{3.8}$$

where $p > 0$, $y_0, y_1 \in \mathbb{R}^n$, $A(t)$, $B(t)$, $R(t)$ are continuous, matrix-valued functions on $\mathbb{R}_+ := [0, \infty)$, $A(t)$ is regular for all $t \in \mathbb{R}_+$, $e : \mathbb{R}_+ \to \mathbb{R}^n$ and $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are continuous mappings. The equation (3.7) with $n = 1$ has been studied by many authors (see e. g. references in [21]). Many papers are devoted to the study of the existence of periodic solutions of scalar differential equation with $p-$Laplacian and in some of them it is assumed that $A(0) = 0$. We study the system without this singularity. From the recently published papers and books see e.g. [12, 13, 21, 22]. The problems treated in this paper are close to those studied in [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 18, 20, 21, 22]. The aim of the paper is to study the problem of the existence of global solutions to (3.7) in the sense of the following definition.

Definition 3.1. A solution $y(t)$, $t \in [0, T)$ of the initial value problem (3.7), (3.8) is called nonextendable to the right if either $T < \infty$ and

$$\limsup_{t\to T^-}[\|y(t)\| + \|y'(t)\|] = \infty$$

or $T = \infty$, i. e. $y(t)$ is defined on $\mathbb{R}_+ = [0, \infty)$. In the second case the solution $y(t)$ is called global.

The main result of this paper is the following theorem.

Theorem 3.1. *Let $p > 0$, $A(t)$, $B(t)$, $R(t)$ be continuous matrix-valued functions on $[0,\infty)$, $A(t)$ be regular for all $t \in \mathbb{R}_+$, $e : \mathbb{R}_+ \to \mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous mappings and $y_0, y_1 \in \mathbb{R}^n$. Let*

$$\int_0^\infty \|R(s)\| s^{m-1}\, \mathrm{d}s < \infty \tag{3.9}$$

and there exist constants $K_1, K_2 > 0$ such that

$$\|g(u)\| \le K_1 \|u\|^m, \quad \|f(v)\| \le K_2\|v\|^m, \quad u, v \in \mathbb{R}^n. \tag{3.10}$$

Then the following assertions hold:

1. *If $1 < m \le p$, then any nonextendable to the right solution $y(t)$ of the initial value problem* (3.7), (3.8) *is global.*

2. *Let $m > p, m > 1$,*

$$A_\infty := \sup_{0\le t<\infty} \|A(t)^{-1}\| < \infty, \quad R_0 = \int_0^\infty \|R(s)\|\, \mathrm{d}s,$$
$$E_\infty := \sup_{0\le t<\infty} \|\int_0^t e(s)\,\mathrm{d}s\| < \infty, \quad Q(s) := \int_s^\infty \|R(\sigma)\|\sigma^{m-1}\,\mathrm{d}\sigma$$

and

$$n^{\frac{p}{2}} \frac{m-p}{p} D^{\frac{m-p}{p}} A_\infty \sup_{0\le t<\infty} \int_0^t \Big(K_1\|B(s)\| + 2^{m-1}K_2 Q(s)\Big)\,\mathrm{d}s < 1$$

for all $t \in [0,\infty)$ where

$$D = n^{\frac{p}{2}} A_\infty \Big(\|A(0)\Phi_p(y_1)\| + 2^{m-1}K_2\|y_0\|^m R_0 + E_\infty\Big).$$

Then any nonextendable to the right solution $y(t)$ of the initial value problem (3.7), (3.8) *is global.*

In [5] a solution $u : [0,T) \to \mathbb{R}^n$ with $0 < T < \infty$ of the equation (3.7) with $n = 1$ is called singular of the second kind, if $\sup_{0<t<T} |y'(t)| = \infty$. By [5, Theorem 1] if $m = p > 0$ (we need to assume $m > 1$) and the condition (3.10) is fulfilled then there exists no singular solution of the second kind of (3.7) and all solutions of (3.7) are defined on $\mathbb{R}_+$, i. e. they are global. The proof of this result is based on the transformation $y_1(t) = y(t)$, $y_2(t) = A(t)|y'(t)|^{p-1}y'(t)$ transforming the scalar equation (3.7) into the form

$$\begin{aligned} y_1' &= A(t)^{-\frac{1}{p}}|y_2|^{\frac{1}{p}}\operatorname{sgn} y_2, \\ y_2' &= -B(t)g(A(t)^{-\frac{1}{p}}\operatorname{sgn} y_2) - R(t)f(y_1) + e(t). \end{aligned} \tag{3.11}$$

An estimate of the function $v(t) = \max_{0\leq s\leq t} |y_2(s)|$ proves the boundedness of $|y'(t)|$ on any bounded interval $[0, T)$. By [5, Theorem 2], if $n = 1$, $R \in C^1(\mathbb{R}_+, \mathbb{R})$, $R(t) > 0$, $f(x)x > 0$ for all $t \in \mathbb{R}_+$ and either $|g(x)| \leq |x|^p$ for $|x| \geq M$ for some $M \in (0, \infty)$ or $g(x)x \geq 0$ or $g(x) \geq 0$ for all $x \in \mathbb{R}_+$ then the equation (3.7) has no singular solution of the second kind and all its solutions are defined on $\mathbb{R}_+$, i. e. they are global. The method of proofs are based on the study of the boundedness from above of the scalar function $V(t) = \frac{A(t)}{R(t)}|y'(t)|^{p+1} + \frac{p+1}{p}\int_0^{y(t)} f(s)\,\mathrm{d}s$ on any bounded interval $[0, T)$. We remark that in [5] the case $n = 1, m = p > 0$ is studied. The method of proofs applied in [5] is not applicable in the case $n > 1$. Our proof of Theorem 3.1 is completely different from that applied in [5]. The main tool of our proof is the discrete and also continuous version of the Jensen's inequality, Fubini theorem and a generalization of the Bihari theorem (see Lemma), proved in this paper. The application of the Jensen's inequality is possible only under the assumption $m > 1$. Therefore we do not study the case $0 < m < 1$. This means that the problem is open for $n > 1$ and $0 < m < 1$. The natural problem is to formulate sufficient conditions for the existence of solutions which are not global, or solutions which are not of the second kind. This problem is not solved even for the scalar case and it seems to be not simple. By [5, Remark 5] the existence of singular solutions of the second kind of (3.7) is an open problem even in the scalar case. M. Bartušek proved (see [1, Theorem 4]) that if $n = 1$, $0 < p < m$, then there exists a positive function $R(t)$, $t \geq 0$ such that the scalar equation (3.7) with $A(t) \equiv 1$, $B(t) \equiv 0$, $e(t) \equiv 0$ and $f(y) = |y|^p$ has a singular solution of the second kind. The case $0 < p < m, n = 1$, studied by Bartušek, corresponds to the assertion 2 of our Theorem 3.1, however for the example given by Bartušek in [5] the assumptions of the assertion 2 are not satisfied. The function $R(t)$ is constructed using a continuous, piecewise polynomial function and the integral R_0 is not finite. Let us remark that for the case $p = 1$, i. e. for second order differential equations without p-Laplacian and also for higher order differential equations some sufficient conditions for the existence of singular solutions of the second kind are proved by Bartušek in the papers [2, 3, 4]. A result on the existence of singular solutions of the second kind for systems of nonlinear differential equations (without the p-Laplacian) are proved by Chanturia [7, Theorem 3] and also by Mirzov [18].

3.2 Proof of the main result

First we shall prove the following lemma.

Lemma 3.1. *Let $c > 0$, $m > 0$, $p > 0$, $t_0 \in \mathbb{R}$ be constants, $F(t)$ be a continuous, nonnegative function on $\mathbb{R}_+$ and $v(t)$ be a continuous, nonnegative function on $\mathbb{R}_+$ satisfying the inequality*

$$v(t)^p \le c + \int_{t_0}^{t} F(s)v(s)^m \,\mathrm{d}s, \quad t \ge t_0. \tag{3.12}$$

Then the following assertions hold:

1. *If $0 < m < p$, then*

$$v(t) \le \Big(c^{\frac{p-m}{p}} + \frac{p-m}{p}\int_{t_0}^{t} F(s)\,\mathrm{d}s\Big)^{\frac{1}{p-m}}, \quad t \ge t_0. \tag{3.13}$$

2. *If $m > p$, $m > 1$ and*

$$\frac{m-p}{p}c^{\frac{m-p}{p}} \sup_{t_0 \le t < \infty} \int_{t_0}^{t} F(s)\,\mathrm{d}s < 1,$$

then

$$v(t) \le \frac{c^{\frac{1}{p}}}{\Big(1 - \frac{m-p}{p}c^{\frac{m-p}{p}}\int_{t_0}^{t} F(s)\,\mathrm{d}s\Big)^{\frac{1}{m-p}}}, \quad t \ge t_0. \tag{3.14}$$

Proof. Let $G(t)$ be the right-hand side of the inequality (3.12). Then $v(t)^m \le G(t)^{\frac{m}{p}}$ which yields

$$\frac{F(t)v(t)^m}{G(t)^{\frac{m}{p}}} \le F(t),$$

i. e.

$$\frac{G'(t)}{G(t)^{\frac{m}{p}}} \le F(t).$$

Integrating this inequality from t_0 to t we obtain

$$\int_{t_0}^{t} \frac{G'(s)}{G(s)^{\frac{m}{p}}}\,\mathrm{d}s = \int_{G(t_0)}^{G(t)} \frac{\mathrm{d}\sigma}{\sigma^{\frac{m}{p}}} = \frac{p}{p-m}\Big(G(t)^{\frac{p-m}{p}} - G(t_0)^{\frac{p-m}{p}}\Big) \le \int_{t_0}^{t} F(s)\,\mathrm{d}s.$$

Since $G(t_0) = c$ we obtain

$$v(t) \le G(t)^{1/p} \le \Big(c^{\frac{p-m}{p}} + \frac{p-m}{p}\int_{t_0}^{t} F(s)\,\mathrm{d}s\Big)^{\frac{1}{p-m}}.$$

The assertions (3.1) and (3.2) follow from this inequality. □

Remark 3.1. If $p = 1$, $m > 0$ then this lemma is a consequence of the well known Bihari inequality (see [6]). Some results on integral inequalities with power nonlinearity on their left-hand sides can be found in the B. G. Pachpatte monograph [19]. The idea of the proof of this lemma is based on that used in the proofs of results on integral inequalities with singular kernels and power nonlinearities on their left-hand sides, published in the papers [16, 17].

Let $y(t)$ be a solution of the initial value problem (3.7), (3.8) defined on an interval $[0, T)$, $0 < T \leq \infty$. If we denote $u(t) = y'(t)$, then

$$y(t) = y_0 + \int_0^t u(s)\,\mathrm{d}s \tag{3.15}$$

and the equation (3.7) can be rewritten as the following integro-differential equation for $u(t)$:

$$\Big(A(t)\Phi_p(u(t))\Big)' + B(t)g(u(t)) + R(t)f\Big(y_0 + \int_0^t u(s)\,\mathrm{d}s\Big) = e(t) \tag{3.16}$$

with

$$u(0) = y_1. \tag{3.17}$$

Theorem 3.2. *Let $p > 0$, $A(t)$, $B(t)$, $R(t)$ be continuous matrix-valued functions on $\mathbb{R}_+$, $A(t)$ regular for all $t \in \mathbb{R}_+$, $e : \mathbb{R}_+ \to \mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous mappings on $\mathbb{R}_+$, $y_0, y_1 \in \mathbb{R}^n$, $\int_0^\infty \|R(s)\| s^{m-1}\,\mathrm{d}s < \infty$ and $0 < T < \infty$. Let the condition* (3.10) *be satisfied and let $u : [0, T) \to \mathbb{R}^n$ be a solution of the equation* (3.16) *satisfying the condition* (3.17). *Let $R_0 := \int_0^\infty \|R(s)\|\,\mathrm{d}s$.*

Then the following assertions hold:

1. *If $m = p > 1$, then*

$$\|u(t)\| \leq d_T\,\mathrm{e}^{\int_0^t F_T(s)\,\mathrm{d}s}, \quad 0 \leq t \leq T$$

where

$$\begin{aligned}
F_T(t) &:= n^{\frac{p}{2}} E_T\Big(K_1\|B(s)\| + 2^{m-1}K_2 Q(s)\Big),\\
Q(s) &= \int_s^\infty \|R(\sigma)\|\sigma^{m-1}\,\mathrm{d}\sigma,\\
E_T &:= \max_{0\leq t\leq T}\|E(t)\|, \quad E(t) := \int_0^t e(s)\,\mathrm{d}s,\\
d_T &= n^{\frac{p}{2}} A_T\Big(\|A(0)\Phi_p(y_1)\| + 2^{m-1}K_2\|y_0\|^m R_0 + E_T\Big),\\
A_T &= \max_{0\leq t\leq T}\|A(t)^{-1}\|.
\end{aligned}$$

2. *If* $1 < m < p$, *then*

$$\|u(t)\| \le \Big(d_T^{\frac{p-m}{p}} + \frac{p-m}{p} d_T \int_0^t F_T(s)\,\mathrm{d}s\Big)^{\frac{1}{p-m}}.$$

3. *Let* $m > p$, $m > 1$ *and*

$$A_\infty := \sup_{0\le T<\infty} A_T < \infty, \quad E_\infty := \sup_{0\le t\le\infty} \|E(t)\| < \infty,$$

$$n^{\frac{p}{2}} \frac{m-p}{p} D^{\frac{m-p}{p}} A_\infty \sup_{0\le t<\infty} \int_0^t \Big(K_1\|B(s)\| + 2^{m-1}K_2 Q(s)\Big)\,\mathrm{d}s < 1$$

where

$$D = n^{\frac{p}{2}} A_\infty \Big(\|A(0)\Phi_p(y_1)\| + 2^{m-1}K_2\|y_0\|^m R_0 + E_\infty\Big).$$

Then

$$\|u(t)\| \le D^{\frac{1}{p}} \Big(1 - n^{\frac{p}{2}} \frac{m-p}{p} A_\infty D^{\frac{m-p}{p}} \int_0^t \Big(K_1\|B(s)\| + 2^{m-1}K_2 Q(s)\Big)\,\mathrm{d}s\Big)^{-\frac{1}{m-p}}$$

where $0 \le t \le \infty$.

Proof. We shall give an explicit upper bound for the solution $u(t)$ of the equation (3.16), defined on the interval $[0, T)$, satisfying (3.17). From the equation (3.16) and the condition (3.17) it follows that

$$\begin{aligned}\Phi_p(u(t)) = A(t)^{-1}\Big\{&A(0)\Phi_p(y_1) - \int_0^t B(s)g(u(s))\,\mathrm{d}s \\ &- \int_0^t R(s) f\Big(y_0 + \int_0^s u(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s \; + E(t)\Big\}\end{aligned} \tag{3.18}$$

where $E(t) = \int_0^t e(s)\,\mathrm{d}s$. This inequality together with the conditions (3.10) yield

$$\begin{aligned}\|A(t)^{-1}\|^{-1}\|\Phi_p(u(t))\| \le{}& \|A(0)\Phi_p(y_1)\| + K_1 \int_0^t \|B(s)\|\|u(s)\|^m\,\mathrm{d}s \\ &+ K_2 \int_0^t \|R(s)\|\Big(\|y_0\| + \int_0^s \|u(\tau)\|d\tau\Big)^m\,\mathrm{d}s \; + \|E(t)\|.\end{aligned} \tag{3.19}$$

We shall use the integral version of the Jensen' s inequality

$$\Big(\int_0^t h(s)\,\mathrm{d}s\Big)^\kappa \le t^{\kappa-1} \int_0^t h(s)^\kappa\,\mathrm{d}s, \quad \kappa > 1,\ t \ge 0 \tag{3.20}$$

for $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ (for a more general integral Jensen's inequality, see e.g. [15, Chapter VIII, Theorem 2]). Also we shall use its discrete version

$$(A_1 + A_2 + \cdots + A_l)^\kappa \le l^{\kappa-1}(A_1^\kappa + A_2^\kappa + \cdots + A_l^\kappa) \tag{3.21}$$

for $A_1, A_2, \dots, A_l \geq 0$, $\kappa > 1$ (see [15, Chapter VIII, Corollary 4]).

Let $m > 1$. Then using the inequalities (3.20) and (3.21) we obtain the inequality

$$\Big(\|y_0\| + \int_0^s \|u(\tau)\|\,\mathrm{d}\tau\Big)^m \leq 2^{m-1}\Big(\|y_0\|^m + \Big(\int_0^s \|u(\tau)\|\,\mathrm{d}\tau\Big)^m\Big)$$
$$\leq 2^{m-1}\Big(\|y_0\|^m + s^{m-1}\int_0^s \|u(\tau)\|^m\,\mathrm{d}\tau\Big).$$

Putting this inequality into (3.19) we obtain

$$\begin{aligned}
&\|A(t)^{-1}\|^{-1}\|\Phi_p(u(t))\| \leq \|A(0)\Phi_p(y_1)\| + \\
&+ K_1\int_0^t \|B(s)\|\|u(s)\|^m\,\mathrm{d}s \ + 2^{m-1}K_2\|y_0\|^m\int_0^t \|R(s)\|\,\mathrm{d}s \\
&+ 2^{m-1}K_2\int_0^t \Big(\|R(s)\|s^{m-1}\int_0^s \|u(\tau)\|^m\,\mathrm{d}\tau\Big)\,\mathrm{d}s \ + \|E(t)\|.
\end{aligned} \tag{3.22}$$

Now we shall apply the following consequence of the Fubini theorem (see e. g. [23, Theorem 3.10 and Exercise 3.27]): If $h : [a,b] \times [a,b] \to \mathbb{R}$ is an integrable function, then

$$\int_a^b \int_a^y h(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_x^b h(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

If $h(\tau, s) = \|R(s)\|s^{m-1}\|u(\tau)\|^m$, $a = 0$, $b = t$, $y = s$, $x = \tau$, then

$$\int_0^t \int_0^s h(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s = \int_0^t \int_\tau^t h(\tau,s)\,\mathrm{d}s\,\mathrm{d}\tau,$$

i. e.

$$\int_0^t \int_0^s \|R(s)\|s^{m-1}\|u(\tau)\|^m\,\mathrm{d}\tau\,\mathrm{d}s = \int_0^t \Big(\int_\tau^t \|R(s)\|s^{m-1}\,\mathrm{d}s\Big)\|u(\tau)\|^m\,\mathrm{d}\tau.$$

This yields

$$\int_0^t \|R(s)\|s^{m-1}\int_0^s \|u(\tau)\|^m\,\mathrm{d}\tau\,\mathrm{d}s \ \leq \int_0^t Q(\tau)\|u(\tau)\|^m\,\mathrm{d}\tau \tag{3.23}$$

where

$$Q(\tau) := \int_\tau^\infty \|R(s)\|s^{m-1}\,\mathrm{d}s$$

for $\tau \geq 0$.

Let $0 < T < \infty$ and $t \in [0, T)$. From the inequalities (3.22) and (3.23) it follows that

$$\|A(t)^{-1}\|^{-1}\|\Phi_p(u(t))\| \leq c_T + \int_0^t F_0(s)\|u(s)\|^m\,\mathrm{d}s \tag{3.24}$$

where

$$c_T = \|A(0)\Phi_p(y_1)\| + 2^{m-1}K_2\|y_0\|^m R_0 + E_T, \tag{3.25}$$
$$F_0(s) = K_1\|B(s)\| + 2^{m-1}K_2 Q(s), \tag{3.26}$$
$$E_T = \max_{0\le t\le T} \|E(t)\|. \tag{3.27}$$

If $k \in \{1, 2, \dots, n\}$, then

$$\begin{aligned}|u_k(t)|^p \le \|\Phi_p(u(t))\| &= \left(|u_1(t)|^{2p} + |u_2(t)|^{2p} + \cdots + |u_n(t)|^{2p}\right)^{\frac{1}{2}}\\ &\le A_T c_T + \int_0^t A_T F_0(s)\|u(s)\|^m \,\mathrm{d}s;\end{aligned}$$

i. e.

$$|u_k(t)|^p \le c_{0T} + \int_0^t F_{0T}(s)\|u(s)\|^m \,\mathrm{d}s \tag{3.28}$$

where

$$A_T := \max_{0\le t\le T} \|A(t)^{-1}\| \quad \text{if } T < \infty, \tag{3.29}$$
$$c_{0T} = A_T c_T, \quad F_{0T}(t) = A_T F_0(t). \tag{3.30}$$

This yields

$$\|u(t)\| \le n^{\frac{p}{2}}\left(c_{0T} + \int_0^t F_{0T}(s)\|u(s)\|^m \,\mathrm{d}s\right)^{\frac{1}{p}}$$

and therefore we have obtained the inequality

$$\|u(t)\|^p \le d_T + \int_0^t F_T(s)\|u(s)\|^m \,\mathrm{d}s \tag{3.31}$$

where

$$d_T = n^{\frac{p}{2}} c_{0T}, F_T(t) = n^{\frac{p}{2}} F_{0T}(t). \tag{3.32}$$

Now applying Lemma 3.1 (case $m = p$ follows from the Gronwall's lemma) to the inequality (3.31) we obtain the assertion 1. and assertion 2. In the proof of the assertion 3. we use the assumptions $A_\infty := \sup_{0\le t<\infty} \|A(t)^{-1}\| < \infty$ and $E_\infty := \sup_{0\le t\le\infty} \|E(t)\| < \infty$. From the inequality (3.31) we obtain the inequality,

$$\|u(t)\|^p \le D + \int_0^t G(s)\|u(s)\|^m \,\mathrm{d}s \tag{3.33}$$

where D is defined in Theorem 3.1,

$$G(s) := n^{\frac{p}{2}} A_\infty(K_1\|B(s)\| + 2^{m-1}K_2 Q(s))$$

and $Q(s) = \int_s^\infty \|R(\sigma)\sigma^{m-1}\| \,\mathrm{d}\sigma$. Now if we put in Lemma 3.1 $t_0 = 0$, $v(t) = \|u(t)\|$, $c = D$ and $F(t) = G(t)$, then we obtain the inequality from the assertion 3. $\square$

Proof of Theorem 3.1. Let $y : [0, T) \to \mathbb{R}^n$ be a nonextendable to the right solution of the initial value problem (3.16), (3.17) with $T < \infty$. Then

$$y(t) = y_0 + \int_0^t u(s)\,\mathrm{d}s$$

where $u(t)$ is a solution of the equation (3.16) satisfying the condition (3.17). From Theorem 3.2 it follows that $M = \sup_{0\le t\le T} \|u(t)\| < \infty$ and since (3.15) yields $\|y(t)\| \le \|y_0\| + t\sup_{0\le s\le T} \|u(s)\|$ we obtain $\lim_{t\to T^-} \|y(s)\| < \infty$. This is a contradiction with nonextendability of $y(t)$. $\square$

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Estimations of noncontinuable solutions

4

4.1 Introduction

Consider the differential equation

$$(A(t)\Phi_p(y'))' + B(t)g(y') + R(t)f(y) = e(t), \tag{4.1}$$

where $p > 0$ and $A(t)$, $B(t)$, $R(t)$ are continuous, matrix-valued function on $\mathbb{R}_+ := [0, \infty)$, $A(t)$ is regular for all $t \in \mathbb{R}_+$, $e : \mathbb{R}^n \to \mathbb{R}_+$ and $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are continuous mappings, $\Phi_p(u) = (|u_1|^{p-1}u_1, \ldots, |u_n|^{p-1}u_n)$ for $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$. Let $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$. Then $\|u\| = \max_{1\le i\le n} |u_i|$.

Definition 4.1. A solution y of (4.1) defined on $t \in [0, T)$ is called noncontinuable if $T < \infty$ and $\limsup_{t\to T^-} \|y'(t)\| = \infty$. The solution y is called continuable if $T = \infty$.

Note, that noncontinuable solutions are called singular of the second kind, too, see e.g. [2], [8], [13].

Definition 4.2. A noncontinuable solution y is called oscillatory if there exists an increasing sequence $\{t_k\}_{k=1}^{\infty}$ of zeros of y such that $\lim_{k\to\infty} t_k = \tau$; otherwise y is called nonoscillatory.

In the two last decades the existence and properties of noncontinuable solutions of special types of (4.1) are investigated. In the scalar case, see e.g. [2, 3, 5, 6, 9, 11, 12, 13, 15] and references therein. In particular, noncontinuable solutions do not exist if f is sublinear in neighbourhoods of $\pm\infty$, i. e. if

$$|g(x)| \le |x|^p \quad \text{and} \quad |f(x)| \le |x|^p \quad \text{for large } |x| \tag{4.2}$$

with R is positive. Hence, noncontinuable solutions may exist mainly in the superlinear case, i. e. if $|f(x)| \ge |x|^m$ with $m > p$.

As concern to system (4.1), see papers [7], [14] where sufficient conditions are given for (4.1) to have continuable solutions.

The scalar equation (4.1) may be applied in problems of radially symmetric solutions of the p-Laplace differential equation, see e.g. [14]; noncontinuable solutions appear e.g. in water flow problems (flood waves, a flow in sewerage systems), see e.g. [10].

The present paper studies the estimations from bellow of norms of a noncontinuable solution of (4.1) and its derivative. Estimations of solutions are important e.g. in proofs of the existence of such solutions, see e.g. [3], [8] for

$$y^{(n)} = f(t, y, \ldots, y^{(n-1)}) \tag{4.3}$$

with $n \geq 2$ and $f \in C^0(\mathbb{R}_+, \mathbb{R}^n)$. For generalized Emden-Fowler equation of the form (4.3), some estimation are proved in [1].

In the paper [14] there is studied differential equation (4.1) with initial conditions

$$y(0) = y_0, \quad y'(0) = y_1 \tag{4.4}$$

where $y_0, y_1 \in \mathbb{R}^n$.

We will use the following version of Theorem $1.2.$ from [14] and of Theorem $1.2.$ from [7].

Theorem A. Let $m > p$ and there exist positive constants K_1, K_2 such that

$$\|g(u)\| \leq K_1 \|u\|^m, \quad \|f(v)\| \leq K_2 \|v\|^m, \quad u, v \in \mathbb{R}^n. \tag{4.5}$$

and $\int_0^\infty \|R(\infty)\| s^m \, \mathrm{d}s < \infty$. Denote

$$A_\infty := \sup_{0 \leq t < \infty} \|A(t)\|^{-1} < \infty, \quad E_\infty := \sup_{0 \leq t < \infty} \| \int_0^t e(s) \, \mathrm{d}s \| < \infty,$$
$$R_\infty := \int_0^\infty \|R(s)\| \, \mathrm{d}s, \quad B_\infty = \int_0^\infty \|B(t)\| \, \mathrm{d}t.$$

(i) Let $m > 1$ and let

$$n^{\frac{p}{2}} \frac{m-p}{p} A_\infty D_1^{\frac{m-p}{p}} \int_0^\infty \Big(K_1 \|B(s)\| + 2^{m-1} K_2 s^m \|R(s)\| \Big) \, \mathrm{d}s < 1$$

for all $t \in \mathbb{R}_+$ where

$$D_1 = n^{\frac{p}{2}} A_\infty \Big\{ \|A(0)\Phi_p(y_1)\| + 2^{m-1} K_2 \|y_0\|^m R_\infty + E_\infty \Big\}.$$

(ii) Let $m \leq 1$ and let

$$2^{m+1} \frac{m-p}{p} A_\infty D_2^{\frac{m-p}{p}} \int_0^\infty \Big(K_1 \|B(s)\| + K_2 s^m \|R(s)\| \Big) \, \mathrm{d}s < 1$$

for all $t \in \mathbb{R}_+$ where

$$D_2 = A_\infty \Big\{ \|A(0)\Phi_p(y_1)\| + 2^m K_1 \|y_1\|^m B_\infty + 2^{2m+1} K_2 R_\infty \|y_0\|^m + E_\infty \Big\}.$$

Then any nonextendable to the right solution $y(t)$ of the initial value problem (4.1), (4.4) is continuable.

4.2 Main results

We will derive estimates for a noncontinuable solution y on the fixed definition interval $[T,\tau) \subset \mathbb{R}_+$, $\tau < \infty$.

Theorem 4.1. *Let y be a noncontinuable solution of system* (4.1) *on $[T,\tau) \subset \mathbb{R}_+$, $\tau - T \leq 1$,*

$$A_0 := \max_{T\leq t\leq \tau} \|A(t)\|^{-1}, \quad B_0 := \max_{T\leq t\leq \tau} \|B(t)\|,$$
$$R_0 := \max_{T\leq t\leq \tau} \|R(t)\|, \quad E_0 := \max_{T\leq t\leq \tau} \|e(t)\|$$

and let there exist positive constants K_1, K_2 and $m > p$ such that

$$\|g(u)\| \leq K_1\|u\|^m, \quad \|f(v)\| \leq K_2\|v\|^m, \quad u,v \in \mathbb{R}^n. \tag{4.6}$$

(i) *If $p > 1$ and $M = \frac{2^{2m+1}(2m+3)}{(m+1)(m+2)}$, then*

$$\|A(t)\Phi_p(y'(t))\| + 2^{m-1}K_2\|y(t)\|^m R_0 + E_0(\tau - t) \geq C_1(\tau - t)^{-\frac{p}{m-p}} \tag{4.7}$$

for $t \in [T,\tau)$ where

$$C_1 = n^{-\frac{pm}{2(m-p)}} A_0^{-\frac{p}{m-p}} \left(\frac{m-p}{p}\right)^{-\frac{p}{m-p}} \left[\frac{3}{2}K_1B_0 + MK_2R_0\right]^{-\frac{p}{m-p}}.$$

(ii) *If $p \leq 1$, then*

$$\begin{aligned}\|A(t)\Phi_p(y'(t))\| + 2^m K_1B_0\|y'(t)\|^m + 2^{2m+1}K_2R_0\|y(t)\|^m + \\ +E_0(\tau - t) \geq C_2(\tau - t)^{-\frac{p}{p-m}}\end{aligned} \tag{4.8}$$

for $t \in [T,\tau)$ where

$$C_2 = 2^{-\frac{p(m+1)}{m-p}} A_0^{-\frac{m}{m-p}} \left(\frac{m-p}{p}\right)^{-\frac{p}{m-p}} \left[\frac{3}{2}K_1B_0 + MK_2R_0\right]^{-\frac{p}{m-p}}.$$

Proof. Let y be a singular solution of system (4.1) on interval $[T,\tau)$. We take t to be fixed in interval $[T,\tau)$ and for simplicity sign

$$D = n^{-\frac{pm}{2(m-p)}} A_0^{-\frac{p}{m-p}} \left(\frac{m-p}{p}\right)^{-\frac{p}{m-p}}. \tag{4.9}$$

Assume, contrarily, that

$$\begin{aligned}&\|A(t)\Phi_p(y'(t))\| + 2^{m-1}K_2\|y(t)\|^m R_0 + E_0(\tau - t) \\ &< D\left[\frac{3}{2}K_1B_0 + MK_2R_0\right]^{-\frac{p}{m-p}} (\tau - t)^{-\frac{p}{m-p}}.\end{aligned} \tag{4.10}$$

Together with the Cauchy problem

$$(A(x)\Phi_p(y'))' + B(x)g(y') + R(x)f(y) = e(x), \quad x \in [t,\tau) \tag{4.11}$$

and

$$y(t) = y_0, \quad y'(t) = y_1 \tag{4.12}$$

we construct a helpful system

$$(\bar{A}(s)\Phi_p(z'))' + \bar{B}(s)g(z') + \bar{R}(s)f(z) = \bar{e}(s), \tag{4.13}$$

$$z(0) = z_0, \quad z'(0) = z_1 \tag{4.14}$$

where $s \in \mathbb{R}_+$, z_0, $z_1 \in \mathbb{R}^n$, $\bar{A}(s)$, $\bar{B}(s)$, $\bar{R}(s)$ are continuous, matrix-valued function on $\mathbb{R}_+$ given by

$$\bar{A}(s) = \begin{cases} A(s+t) & \text{if } 0 \le s < \tau - t, \\ A(\tau) & \text{if } \tau - t \le s < \infty, \end{cases} \tag{4.15}$$

$$\bar{B}(s) = \begin{cases} B(s+t) & \text{if } 0 \le s < \tau - t, \\ -\frac{B(\tau-t)}{\tau-t}s + 2B(\tau-t) & \text{if } \tau - t \le s < 2(\tau-t), \\ 0 & \text{if } 2(\tau-t) \le s < \infty, \end{cases} \tag{4.16}$$

$$\bar{R}(s) = \begin{cases} R(s+t) & \text{if } 0 \le s < \tau - t, \\ -\frac{R(\tau-t)}{\tau-t}s + 2R(\tau-t) & \text{if } \tau - t \le s < 2(\tau-t), \\ 0 & \text{if } 2(\tau-t) \le s < \infty, \end{cases} \tag{4.17}$$

$$\bar{e}(s) = \begin{cases} e(s) & \text{if } 0 \le s < \tau - t, \\ -\frac{e(\tau-t)}{\tau-t}s + 2e(\tau-t) & \text{if } \tau - t \le s < 2(\tau-t), \\ 0 & \text{if } 2(\tau-t) \le s < \infty. \end{cases} \tag{4.18}$$

We can see that $\bar{A}(s)$ is regular for all $s \in \mathbb{R}_+$.

Hence, systems (4.11) on $[t,\tau)$ and (4.13) on interval $[0, \tau - t)$ are equivalent with $s = x - t$. Let $z_0 = y(t)$ and $z_1 = y'(t)$. Then the definitions of functions $\bar{A}, \bar{B}, \bar{R}, \bar{e}$ give that

$$z(s) = y(s+t), s \in [0, \tau - t) \quad \text{is the noncontinuable solution} \tag{4.19}$$

of system (4.13), (4.14) on $[0, \tau - t)$. By the application of Theorem A(i) on system (4.13), (4.14) we will see that every z satisfying

$$\begin{aligned} &\|\bar{A}(0)\Phi_p(z_1)\| + 2^{m-1}K_2\|z_0\|^m R_0 + \int_0^\infty \|\bar{e}(s)\| \, \mathrm{d}s \\ &< D\left[\int_0^\infty \Big(K_1\|\bar{B}(w)\| + 2^{m-1}K_2\|\bar{R}(w)\|w^m\Big) \, \mathrm{d}w\right]^{-\frac{p}{m-p}} \end{aligned} \tag{4.20}$$

is continuable. Note, that according to (4.15)–(4.20) all assumptions of Theorem A are valid. Furthermore, we will show that (4.10) yields (4.20).

We calculate integrals and estimate the right side of the (4.20)

$$
\begin{aligned}
G &:= D\left[\int_0^{\infty}\left(K_1\|\bar{B}(w)\| + 2^{m-1}K_2\|\bar{R}(w)\|w^m\right)\mathrm{d}w\right]^{-\frac{p}{m-p}} \\
&\geq D\left[\int_0^{2(\tau-t)}\left(K_1\|\bar{B}(w)\| + 2^{m-1}K_2\|\bar{R}(w)\|w^m\right)\mathrm{d}w\right]^{-\frac{p}{m-p}} \\
&\geq D\Bigg[K_1 \max_{0\leq s\ \leq \tau-t}\|B(s+t)\|(\tau-t) \\
&\quad + K_1\int_{\tau-t}^{2(\tau-t)} \| -\frac{B(\tau-t)}{\tau-t}w + 2B(\tau-t)\ \|\ \mathrm{d}w \\
&\quad + 2^{m-1}K_2 \max_{0\leq s\ \leq (\tau-t)}\|R(s+t)\|\frac{(\tau-t)^{m+1}}{m+1}\,\mathrm{d}w \\
&\quad + 2^{m-1}K_2\int_{\tau-t}^{2(\tau-t)} \| -\frac{R(\tau-t)}{\tau-t}w + 2R(\tau-t)\ \|\ w^m\,\mathrm{d}w\Bigg]^{-\frac{p}{m-p}}, \\
G &\geq D\Bigg[K_1 \max_{T\leq t\leq \tau}\|B(t)\|(\tau-t) + \frac{1}{2}K_1\|B(\tau-t)\|(\tau-t) \\
&\quad + M_1K_2\max_{T\leq t\leq\tau}\|R(t)\|(\tau-t)^{m+1} + M_2K_2\|R(\tau-t)\|(\tau-t)^{m+1}\Bigg]^{-\frac{p}{m-p}}
\end{aligned}
$$

where

$$
M_1 = \frac{2^{m-1}}{m+1} \quad \text{and} \quad M_2 = 2^{m-1}\frac{2^{m+2}(2m+3)-3m-5}{(m+1)(m+2)}.
$$

Hence,

$$
G > D\left[\frac{3}{2}K_1B_0(\tau-t) + MK_2R_0(\tau-t)^{m+1}\right]^{-\frac{p}{m-p}} \tag{4.21}
$$

as $M > M_1 + M_2$.

As we assume that $\tau - t \leq 1$, (4.10) and (4.21) imply

$$
\begin{aligned}
G &> D\left[\frac{3}{2}K_1B_0 + MK_2R_0\right]^{-\frac{p}{m-p}}(\tau-t)^{-\frac{p}{m-p}} = C_1(\tau-t)^{-\frac{p}{m-p}} \qquad (4.22) \\
&\geq \|A(t)\Phi_p(y'(t))\| + 2^{m-1}K_2\|y(t)\|^mR_0 + E_0(\tau-t) \\
&= \|\bar{A}(0)\Phi_p(z_1)\| + 2^{m-1}K_2\|z_0\|^mR_0 + \int_0^{\infty}\|\bar{e}(s)\|\,\mathrm{d}s
\end{aligned}
$$

where $C_1 = D\Big[\frac{3}{2}K_1B_0 + MK_2R_0\Big]^{-\frac{p}{m-p}}$. Hence (4.20) holds and the solution z of (4.13) satisfying the initial condition $z(0) = y_0$ and $z'(0) = y_1$ is continuable. The contradiction with (4.19) proves the statement.

If $m \le 1$ then the proof is similar, we have to use Theorem A(ii) instead of Theorem A(i). □

Now consider special case of equation (4.1)

$$(A(t)\Phi_p(y'))' + R(t)f(y) = 0 \tag{4.23}$$

for all $t \in \mathbb{R}_+$. In this case a better estimation can be proved.

Theorem 4.2. *Let $m > p$ and y be a noncontinuable solution of system* (4.23) *on interval $[T, \tau) \subset \mathbb{R}_+$. Let there exist constant $K_2 > 0$ such that*

$$\|f(v)\| \le K_2\|v\|^m, \quad v \in \mathbb{R}^n. \tag{4.24}$$

Let A_0, R_0 and M be given by Theorem 4.1. Then

$$\|A(t)\Phi_p(y'(t))\| + 2^{m+2}K_2\|y(t)\|^m R_0 \ge C_1(\tau - t)^{-\frac{p(m+1)}{m-p}} \tag{4.25}$$

where

$$C_1 = n^{-\frac{pm}{2(m-p)}} A_0^{-\frac{p}{m-p}} \left(\frac{m-p}{p}\right)^{-\frac{p}{m-p}} \big[MK_2R_0\big]^{-\frac{p}{m-p}} \quad \textit{in case} \quad p > 1$$

and

$$\|A(t)\Phi_p(y')\| + 2^{2m+1}K_2\|y(t)\|^m R_0 \ge C_2(\tau - t)^{-\frac{p(m+1)}{m-p}}$$

with

$$C_2 = 2^{-\frac{p(m+1)}{m-p}} A_0^{-\frac{m}{m-p}} \left(\frac{m-p}{p}\right)^{-\frac{p}{m-p}} \big[MK_2R_0\big]^{-\frac{p}{m-p}} \quad \textit{in case} \quad p \le 1.$$

Proof. Proof is similar the one of the Theorem 4.1 for $B(t) \equiv 0$ and $e(t) \equiv 0$. Let $p > 1$. We do not use assumption $\tau - t \le 1$, we are able to improve an exponent of the estimation (4.7). Equation (4.22) has changed to

$$\begin{aligned} G &\ge C_1(\tau - t)^{-\frac{p(m+1)}{m-p}} \\ &\ge \|A(t)\Phi_p(y'(t))\| + 2^{m-1}K_2\|y(t)\|^m R_0 \\ &\ge \|\bar{A}(0)\Phi_p(z'(0))\| + 2^{m-1}K_2\|z(0)\|^m R_0 \end{aligned}$$

where $C_1 = D[MK_2R_0]^{-\frac{p}{(m-p)}}$. If $p \le 1$, the proof is similar. □

4.3 Applications

In this section we study a scalar differential equation

$$(a(t)\Phi_p(y'))' + r(t)f(y) = 0 \tag{4.26}$$

where $p > 0$, $a(t)$, $r(t)$ are continuous functions on R_+, $a(t) > 0$ for $t \in \mathbb{R}_+$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous mapping and $\Phi_p(u) = |u|^{p-1}u$.

Theorem 4.3. *Let y be a noncontinuable oscillatory solution of equation* (4.26) *defined on $[T, \tau)$. Let there exist constants $K_2 > 0$ such that*

$$|f(v)| \leq K_2|v|^m, \quad v \in \mathbb{R}. \tag{4.27}$$

Let $\{t_k\}_1^\infty$ and $\{\tau_k\}_1^\infty$ be increasing sequences of all local extremes of solution y and $y^{[1]} = a(t)\Phi_p(y')$ on $[T, \tau)$, respectively. Then constants C_1 and C_2 exists such that

$$|y(t_k)| \geq C_1(\tau - t_k)^{-\frac{p(m+1)}{m(m-p)}} \tag{4.28}$$

and in the case $r \neq 0$ on $\mathbb{R}_+$ and $f(x)x > 0$ for $x \neq 0$.

$$|y^{[1]}(\tau_k)| \geq C_2(\tau - \tau_k)^{-\frac{p(m+1)}{m-p}} \tag{4.29}$$

for $k \geq 1, 2, \ldots$.

Proof. Let $m > p$ and y be a oscillatory noncontinuable solution of equation (4.26) defined on $[T, \tau)$. An application of Theorem 4.2 to (4.26) gives

$$|y^{[1]}(t)| + 2^{2m+1}K_2|y(t)|^m r_0 \geq C(\tau - t)^{-\frac{p(m+1)}{m-p}} \tag{4.30}$$

where C is a suitable constant and $r_0 = \max_{T \leq t \leq \tau} |r(t)|$. Note, that according to (4.26), x ($x^{[1]}$) has a local extreme at $t_0 \in (T, \tau)$ if and only if $x^{[1]}(t_0)$ ($x(t_0) = 0$). From this an accumulation point of zeros of x ($x^{[1]}$) does not exist in $[T, \tau)$. Otherwise, it holds $y(\tau) = 0$ and $y'(\tau) = 0$. That is in contradiction with (4.30). If $\{t_k\}_1^\infty$ is sequence of all extremes of solution y, then $y'(t_k) = 0$, i.e. $y^{[1]}(t_k) = 0$. We obtain the following estimate for $y(t_k)$ from (4.30)

$$|y(t_k)| \geq C_1(\tau - t_k)^{-\frac{p(m+1)}{m(m-p)}}$$

where $C_1 = C^{\frac{1}{m}}(2^{2m+1}K_2 r_0)^{-\frac{1}{m}}$ and (4.28) is valid. If $\{\tau_k\}_1^\infty$ is the sequence of all extremes of $y^{[1]}(\tau_k)$, then $y(\tau_k) = 0$. We obtain the following estimate for $y^{[1]}(\tau_k)$ from (4.30)

$$|y^{[1]}(\tau_k)| \geq C_2(\tau - \tau_k)^{-\frac{p(m+1)}{m-p}}$$

where $C_2 = C$ and (4.29) is valid. ∎

Example 4.1. Consider (4.26) and (4.27) with $m = 2$, $p = 1$ and $a \equiv 1$. Then from Theorem 4.3 we obtain the following estimates

$$|y(t_k)| \geq C_1(\tau - t_k)^{-\frac{3}{2}}, \quad |y^{[1]}(\tau_k)| \geq C_2(\tau - \tau_k)^{-3}$$

where $M = \frac{56}{3}$, $C_1 = \frac{\sqrt{42}}{448K_2r_0}$ and $C_2 = \frac{3}{448K_2r_0}$.

Example 4.2. Consider (4.26) and (4.27) with $m = 3$, $p = 2$ and $a \equiv 1$. Then from Theorem 4.3 we obtain the following estimates

$$|y(t_k)| \geq C_1(\tau - t_k)^{-\frac{8}{3}}, \quad |y^{[1]}(\tau_k)| \geq C_2(\tau - \tau_k)^{-8}$$

where $M = \frac{288}{5}$, $C_1 = \frac{1}{32K_2r_0}\left(\frac{10}{9}\right)^{\frac{2}{3}}$ and $C_2 = \left(\frac{5}{144K_2r_0}\right)^2$.

The following lemma is a special case of Lemma 11.2. in [13].

Lemma 4.1. *Let $y \in C^2[a,b)$, $\delta \in (0, \frac{1}{2})$ and $y'(t)y(t) > 0$, $y''(t)y(t) \geq 0$ on $[a,b)$. Then*

$$\big(y'(t)y(t)\big)^{-\frac{1}{1-2\delta}} \geq \omega \int_t^b |y''(s)|^\delta |y(s)|^{3\delta-2}\,\mathrm{d}s, \quad t \in [a,b) \tag{4.31}$$

where $\omega = [(1-2\delta)\delta^\delta(1-\delta)^{1-\delta}]^{-1}$.

Now, let us turn our attention to nonoscillatory solutions of (4.26).

Theorem 4.4. *Let $m > p$ and $M \geq 0$ hold such that*

$$|f(x)| \leq |x|^m \quad \textit{for} \quad |x| \geq M. \tag{4.32}$$

If y be a nonoscillatory noncontinuable solution of (4.26) *defined on $[T,\tau)$, then constants C, C_0 and a left neighborhood J of τ exist such that*

$$|y'(t)| \geq C(\tau - t)^{-\frac{p(m+1)}{m(m-p)}}, \quad t \in J. \tag{4.33}$$

Let, moreover, $m < p + \sqrt{p^2 + p}$. Then

$$|y(t)| \geq C_0(\tau - t)^{m_1} \quad \textit{with} \quad m_1 = \frac{m^2 - 2mp - p}{m(m-p)} < 0. \tag{4.34}$$

Proof. Let y be a nonoscillatory noncontinuable solutions of (4.26) defined on interval $[T,\tau)$. Then $t_0 \in [T,\tau)$ exists such that $y(t)y^{[1]}(t) > 0$ for $t \in [t_0,\tau)$. Let

$$y(t) > 0 \text{ and } y'(t) > 0 \quad \text{for } t \in J := [t_0,\tau);$$

the opposite case $x < 0$ and $x' < 0$ can be studied similarly. As y is noncontinuable, $\lim_{t\to\tau^-} y'(t) = \infty$. Moreover, $\lim_{t\to\infty} y(t) = \infty$ as, otherwise, $y^{[1]}$ and y are bounded on the finite interval J. Hence, $t_1 \in J$ exists such that $y'(t) \geq 1$ for $[t_1, \tau)$, $y(t) \geq M$ for $t \geq t_1$ and

$$y(t) = y(t_0) + \int_{t_0}^{t} y'(s)\,\mathrm{d}s \leq y(t_0) + \tau y'(t) \leq 2\tau y'(t), \quad t \in [t_1, \tau). \tag{4.35}$$

Note, that due to $y \geq M$ it is enough to suppose (4.32) in stead of (4.24) for an application of Theorem 4.2. Hence, Theorem 4.2 applied to (4.26), (4.35) and $y' \geq 1$ imply

$$\begin{aligned} C_1(\tau - t)^{-\frac{p(m+1)}{m-p}} &\leq a(t)(y'(t))^p + C_2 y^m(t) \\ &\leq a(t)(y'(t))^p + C_2(2\tau)^m (y'(t))^m \\ &\leq C_3(y'(t))^m \end{aligned}$$

or

$$y'(t) \geq C_4(\tau - t)^{-\frac{p(m+1)}{m(m-p)}} \quad \text{on } [t_1, \tau)$$

where C_1, C_2, C_3 and C_4 are positive constants which do not depend on y. Moreover, the integration of (4.33) implies

$$\begin{aligned} y(t) = y(t_0) + \int_{t_0}^{t} y'(s)\,\mathrm{d}s &\geq C \int_{t_0}^{t} (\tau - s)^{-\frac{p(m+1)}{m(m-p)}}\,\mathrm{d}s \\ &\geq \frac{C}{|m_1|}[(\tau - t)^{m_1} - (\tau - t_0)^{m_1}] \geq \frac{C}{2|m_1|}(\tau - t)^{m_1} \end{aligned}$$

for t lying in a left neighbourhood I_1 of τ. Hence, (4.33) and (4.34) are valid. □

Our last application is devoted to the equation

$$y'' = r(t)|y|^m \operatorname{sgn} y \tag{4.36}$$

where $r \in C^0(\mathbb{R}_+)$, $m > 1$.

Theorem 4.5. *Let $\tau \in (0, \infty)$, $T \in [0, \tau)$ and $r(t) > 0$ on $[t, \tau]$.*

(i) *Then* (4.36) *has nonoscillatory noncontinuable solution which is defined in a left neighbourhood of τ.*

(ii) *Let y be a nonoscillatory noncontinuable solution of* (4.36) *defined on $[T, \tau)$. Then constants C, C_1, C_2 and a left neighbourhood I of τ exist such that*

$$|y(t)| \leq C(\tau - t)^{-\frac{2(m+3)}{m-1}} \quad \textit{and} \quad |y'(t)| \geq C_1(\tau - t)^{-\frac{m+1}{m(m-1)}}, \quad t \in I.$$

If, moreover, $m < 1 + \sqrt{2}$, then

$$|y(t)| \leq C_2(\tau - t)^{m_1} \quad \textit{with} \quad m_1 = \frac{m^2 - 2m - 1}{m(m-1)} < 0.$$

Proof. (i) The conclusion follows from Theorem 2 in [4].
(ii) Let y be a noncontinuable solution of (4.36) defined on $[T, \tau)$. According to Theorem 4.4 and its proof we have $\lim_{t\to\tau^-} |y(t)| = \infty$ and (4.33) holds. Hence, suppose that $t_0 \in [T, \tau)$is such that

$$y(t) \geq 1 \text{ and } y'(t) > 0 \quad \text{on } [t_0, \tau).$$

Furthermore, $t_1 \in [t_0, \tau)$ exists such that

$$y(t) = y(t_0) + \int_{t_0}^{t} y'(s)\,\mathrm{d}s \leq y(t_0) + y'(t)(\tau - t_0) \leq C_3 y'(t)$$

for $t \in [t_1, \tau)$ with $C_3 = 2(\tau - t_0)$. Now, we estimate y from bellow. Applying Lemma 4.1 with $[a, b) = [t_1, \tau)$ and $\delta = \frac{2}{m+3} \in (0, \frac{1}{2})$. We have $\delta m + 3\delta - 2 = 0$ and

$$\begin{aligned} C_3^{\frac{m+3}{m-1}} y^{-\frac{2(m+3)}{m-1}}(t)m &\geq (y'(t)y(t))^{-\frac{1}{1-2\delta}} \geq \omega \int_t^{\tau} (y''(s))^{\delta} (y(s))^{3\delta-2}\,\mathrm{d}s \\ &\geq C_4 \int_t^{\tau} y^{\delta m + 3\delta - 2}(s)\,\mathrm{d}s = C_4(\tau - t) \quad \text{on } [t_1, \tau) \end{aligned}$$

where $C_4 = \omega \min_{t_0 \leq \sigma \leq \tau} |r(\sigma)|$. From this

$$y(t) \leq C(\tau - t)^{-\frac{m-1}{2(m+3)}} \quad \text{on } [t_1, \tau)$$

with a suitable positive C. The rest of statement follows from Theorem 4.5. $\square$

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Large behavior of second order differential equation with p-Laplacian

5

5.1 Introduction

In this paper, we study asymptotic properties of the second order differential equation with p-Laplacian

$$(|u'|^{p-1}u')' + f(t,u,u') = 0, \quad p \geq 1. \tag{5.1}$$

In the sequel, it is assumed that all solutions of the equation (5.1) are continuously extendable throughout the entire real axis. We refer to such solutions as to global solutions. We shall prove sufficient conditions under which all global solutions are asymptotic to $at + b$, as $t \to +\infty$, where a, b are real numbers. The problem for ordinary second order differential equations without $p-$Laplacian has been studied by many authors, e. g. by D. S. Cohen [6], A. Constantin [7], F. M. Dannan [8], T. Kusano and W. F. Trench [9, 10], Y. V. Rogovchenko [13], S. P. Rogovchenko, Y. V. Rogovchenko [14], J. Tong [15] and W. F. Trench [16]. Our results are more close to these obtained in the papers [13, 14]. The main tool of the proofs are the Bihari's and Dannan's integral inequalities. We remark that sufficient conditions on the existence of global solutions for second order differential equations and second order functional-differential equations with $p-$Laplacian are proved in the papers [1, 2, 3, 4, 11]. Many references concerning differential equations with $p-$Laplacian can be found in the paper by I. Rachunková, S. Staněk and M. Tvrdý [12], where boundary value problems for such equations are treated.

Let

$$u(t_0) = u_0, \quad u'(t_0) = u_1, \tag{5.2}$$

where $u_0, u_1 \in \mathbb{R}$ be initial condition for solutions of (5.1).

Definition 5.1. We say that a solution $u(t)$ of (5.1) possesses the property (L) if $u(t) = at + b + o(t)$ as $t \to \infty$, where a, b are real constants.

5.2 Main results

Theorem 5.1. *Let $p \geq 1$, $r > 0$ and $t_0 > 0$. Suppose that the following conditions are satisfied:*

(i) *$f(t, u, v)$ is a continuous function in $D = \{(t, u, v) : t \in [t_0, \infty), u, v \in \mathbb{R}\}$ where $t_0 > 0$.*

(ii) *There exist continuous functions $h, g : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}_+$ such that*

$$|f(t, u, v)| \leq h(t)g\left(\left[\frac{|u|}{t}\right]^r\right)|v|^r,\ (t, u, v) \in D$$

where for $s > 0$, the function $g(s)$ is positive and nondecreasing,

$$\int_{t_0}^{\infty} h(s)\,\mathrm{d}s < \infty$$

and if we denote

$$G(x) = \int_{t_0}^{x} \frac{\mathrm{d}s}{s^{\frac{r}{p}} g(s^{\frac{r}{p}})},$$

then

$$G(\infty) = \int_{t_0}^{\infty} \frac{\mathrm{d}s}{s^{\frac{r}{p}} g(s^{\frac{r}{p}})} = \frac{p}{r} \int_{a}^{\infty} \frac{\tau^{\frac{p}{r}-2}}{g(\tau)}\,\mathrm{d}\tau = \infty$$

where $a = (t_0)^{\frac{r}{p}}$.

Then any global solution $u(t)$ of the equation (5.1) *possesses the property* (L).

Proof. Without loss of generality we may assume $t_0 = 1$. Let $u(t)$ be a solution of (5.1), (5.2). Then

$$|u'(t)|^p \leq c_2 + \int_1^t |f(s, u(s), u'(s))|\,\mathrm{d}s \tag{5.3}$$

where $c_2 = |u_1|^p$. Let $w(t)$ be the right-hand side of inequality (5.3). Then

$$|u'(t)| \leq w(t)^{\frac{1}{p}}$$

and

$$|u(t)| \leq c_1 + \int_1^t w(s)^{\frac{1}{p}}\,\mathrm{d}s \leq c_1 + (t-1)w(t)^{\frac{1}{p}} \leq t[c_1 + w(t)^{\frac{1}{p}}] \tag{5.4}$$

where $c_1 = |u_0|$, i. e.

$$u(t) \leq t[c_1 + w(t)^{\frac{1}{p}}], \quad t \geq 1.$$

Applying the inequality $(A+B)^p \le 2^{p-1}(A^p+B^p)$, $A, B \ge 0$ and the assumption (ii) of Theorem 5.1 we obtain from (5.4):

$$\begin{aligned}\left(\frac{|u(t)|}{t}\right)^p &\le 2^{p-1}c_1^p + 2^{p-1}w(t) \qquad (5.5)\\ &\le 2^{p-1}c_1^p + 2^{p-1}\left(c_2 + \int_1^t h(s)g\left(\left[\frac{|u(s)|}{s}\right]^r\right)|u'(s)|^r\,\mathrm{d}s\right).\end{aligned}$$

Let

$$d = 2^{p-1}(c_1^p + c_2), H(t) = 2^{p-1}h(t). \qquad (5.6)$$

Then

$$\left(\frac{|u(t)|}{t}\right)^p \le d + \int_1^t H(s)g\left(\left[\frac{|u(s)|}{s}\right]^r\right)|u'(s)|^r\,\mathrm{d}s := z(t), \qquad (5.7)$$

i. e.

$$\left(\frac{|u(t)|}{t}\right)^r \le z(t)^{\frac{r}{p}}.$$

From the assumption (ii) of Theorem 5.1 and the inequality (5.3) it follows

$$|u'(t)|^p \le u_1^p + \int_1^t h(s)g\left(\left[\frac{|u(s)|}{s}\right]^r\right)|u'(s)|^r\,\mathrm{d}s \le z(t),$$

i. e. we have

$$|u'(t)|^p \le z(t).$$

Since $g(s)$ is nondecreasing, the inequality (5.5) yields

$$g\left(\left[\frac{|u(t)|}{t}\right]^r\right) \le g(z(t)^{\frac{r}{p}})$$

and so we conclude for $t \ge 1$

$$z(t) \le d + \int_1^t H(s)g(z(t)^{\frac{r}{p}})z(t)^{\frac{r}{p}}\,\mathrm{d}s.$$

From the assumption (ii) of Theorem 5.1 it follows that the inverse G^{-1} of G is defined on the interval $(G(+0), \infty)$. Applying the Bihari theorem (see [5]) we obtain

$$z(t) \le G^{-1}\left(G(d) + 2^{p-1}\int_1^\infty h(s)\,\mathrm{d}s\right) := K < \infty.$$

Therefore the inequality (5.6) yields

$$|u'(t)| \le L := K^{\frac{1}{p}}$$

and from (5.5) we have

$$\frac{|u(t)|}{t} \le L.$$

Since

$$\int_1^t |f(s,u(s),u'(s))|\,\mathrm{d}s \le \int_1^t h(s)g\bigg(\Big[\frac{|u(s)|}{s}\Big]^r\bigg)|u'(s)|^r\,\mathrm{d}s \le z(t) \le K$$

for $t \ge 1$, the integral $\int_1^\infty |f(s,u(s),u'(s))|\,\mathrm{d}s$ exists. From (5.7) it follows that there exists $a \in \mathbb{R}$ such that

$$\lim_{t\to\infty} u'(t) = a.$$

By the l'Hospital rule, we can conclude that

$$\lim_{t\to\infty}\frac{u(t)}{t} = \lim_{t\to\infty}\frac{u_1 + \int_1^t u'(\tau)\mathrm{d}\tau}{t} = \lim_{t\to\infty} u'(t) = a.$$

Therefore there exist $b \in \mathbb{R}$ such that $u(t) = at + b + o(t)$. □

Example 5.1. Let $t_0 = 1,\ p \ge r > 0, p \ge 1$

$$f(t,u,u') = \eta(t)t^{1-\alpha}e^{-t}\Big(\frac{u}{t}\Big)^{p-r}\ln\bigg[2+\Big(\frac{|u|}{t}\Big)^r\bigg](u')^r,\ t \ge 1$$

where $0 < \alpha < 1$ and $\eta(t)$ is a continuous function on interval $[1,\infty)$ with $K = \sup_{t\ge1}|\eta(t)| < \infty$.

The function $f(t,u,u')$ can be written in the form

$$f(t,u,u') = h(t)g\bigg(\Big[\frac{u}{t}\Big]^r\bigg)(u')^r$$

where $h(t) = \eta(t)t^{1-\alpha}e^{-t},\ g(u) = u^{\frac{p}{r}-1}\ln(2+|u|)$. Obviously $g(u)$ is positive, continuous and nondecreasing function, $\int_1^\infty |h(s)|ds < K\Gamma(\alpha) = K\int_0^\infty s^{1-\alpha}e^{-s}\,\mathrm{d}s$ and

$$\int_1^\infty \frac{\tau^{\frac{p}{r}-1}\mathrm{d}\tau}{\tau g(\tau)} = \int_1^\infty \frac{\mathrm{d}\tau}{\tau\ln(2+\tau)} > \int_1^\infty \frac{\mathrm{d}\tau}{(2+\tau)\ln(2+\tau)} = \infty.$$

Thus we have proved that all conditions of Theorem 5.1 are satisfied. This means that for every solution $u(t)$ of the initial value problem (5.1), (5.2) there exist numbers a, b such that $u(t) = at + b + o(t)$ as $t \to \infty$.

Theorem 5.2. *Let $p \ge 1, r > 0$ and $t_0 > 0$. Suppose the following conditions are satisfied:*

(i) *The function $f(t,u,v)$ is continuous in $\mathrm{D} = \{(t,u,v) : t \in [t_0,\infty), u, v \in \mathbb{R}\}$.*

(ii) *There exist continuous functions $h_1, h_2, h_3, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \le h_1(t)g_1\bigg(\Big[\frac{|u|}{t}\Big]^r\bigg) + h_2(t)g_2(|v|^r) + h_3(t),\ (t,u,v) \in D$$

for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are nondecreasing and if

$$G(x) = \int_{t_0}^{x} \frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}}) + g_2(s^{\frac{r}{p}})},$$

then

$$G(\infty) = \int_{t_0}^{\infty} \frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}}) + g_2(s^{\frac{r}{p}})} = \frac{p}{r}\int_{a}^{\infty} \frac{\tau^{\frac{p}{r}-1}\,\mathrm{d}\tau}{g_1(\tau) + g_2(\tau)} = \infty$$

where $a = (t_0)^{\frac{r}{p}}$.

Then any global solution $u(t)$ of the equation (5.1) *possesses the property* (L).

Proof. Without loss of generality we may assume $t_0 = 1$. By the standard existence results, it follows from the continuity of the function f that equation (5.1) has solution $u(t)$ corresponding to the initial data $u(1) = u_0$, $u'(1) = u_1$. Two times of integration (5.1) from 1 to t, yields for $t \geq 1$

$$|u'(t)|^p \leq |u_1|^p + \int_1^t |f(s, u(s), u'(s))|\,\mathrm{d}s, \tag{5.8}$$

$$|u(t)| \leq u_0 + (t-1)\Big[u_1^p + \int_1^t |f(s, u(s), u'(s))|\,\mathrm{d}s\Big]^{\frac{1}{p}}. \tag{5.9}$$

It follows from (5.8) and (5.9) that for $t \geq 1$

$$|u'(t)| \leq w(t)^{\frac{1}{p}},$$

$$|u(t)| \leq t\big(c_1 + w(t)^{\frac{1}{p}}\big)$$

where $c_1 = |u_0|$, $c_2 = |u_1|^p$, $w(t) = c_2 + \int_1^t |f(s, u(s), u'(s))|\,\mathrm{d}s$. Using the assumption (ii) of Theorem 5.2 we obtain for $t \geq 1$

$$|u'(t)| \leq \Big[c_2 + \int_1^t h_1(s) g_1\Big(\Big[\frac{|u(s)|}{s}\Big]^r\Big)\,\mathrm{d}s + \int_1^t h_2(s) g_2(|u'(s)|^r)\,\mathrm{d}s + \int_1^t h_3(s)\,\mathrm{d}s\Big]^{\frac{1}{p}},$$

$$\frac{|u(t)|}{t} \leq c_1 + \Big[c_2 + \int_1^t h_1(s) g_1\big(\Big[\frac{|u(s)|}{s}\Big]^r\big)\,\mathrm{d}s + \int_1^t h_2(s) g_2(|u'(s)|^r)\,\mathrm{d}s + \int_1^t h_3(s)\,\mathrm{d}s\Big]^{\frac{1}{p}}.$$

Applying the inequality $(A+B)^p \leq 2^{p-1}(A^p + B^p)$, where $A, B \geq 0$, we obtain

$$\begin{aligned}\Big(\frac{|u(t)|}{t}\Big)^p \leq d &+ \int_1^t H_1(s) g_1\Big(\Big[\frac{|u(s)|}{s}\Big]^r\Big)\,\mathrm{d}s + \int_1^t H_2(s) g_2(|u'(s)|^r)\,\mathrm{d}s \\ &+ \int_1^t H_3(s)\,\mathrm{d}s,\end{aligned} \tag{5.10}$$

where $d = 2^{p-1}(c_1^p + c_2)$, $H_i(t) = 2^{p-1}h_i(t)$, $i = 1, 2, 3$.

Denote by $z(t)$ the right-hand side inequality (5.10)

$$|u'(t)|^r \leq z(t)^{\frac{r}{p}},$$
$$\left(\frac{|u(t)|}{t}\right)^r \leq z(t)^{\frac{r}{p}}. \tag{5.11}$$

Since the function $g_1(s)$ and $g_2(s)$ are nondecreasing for $s > 0$, we obtain

$$g_1\Big(|u'(t)|^r\Big) \leq g_1\Big(z(t)^{\frac{r}{p}}\Big),$$
$$g_1\Big(\Big[\frac{|u(t)|}{t}\Big]^r\Big) \leq g_2\Big(z(t)^{\frac{r}{p}}\Big).$$

Thus, for $t \geq 1$

$$z(t) \leq d + \int_1^t H_1(s) g_1(z(s)^{\frac{r}{p}})\,\mathrm{d}s + \int_1^t H_2(s) g_2(z(s)^{\frac{r}{p}})\,\mathrm{d}s + \int_1^t H_3(s)\,\mathrm{d}s.$$

Furthermore, due to evident inequality

$$H_1(s)g_1(z(s)^{\frac{r}{p}}) + H_2(s)g_2(z(s)^{\frac{r}{p}}) \leq (H_1(s)+H_2(s))(g_1(z(s)^{\frac{r}{p}}) + g_2(z(s)^{\frac{r}{p}})) \tag{5.12}$$

we have by (5.12)

$$z(t) \leq d + \bar{H}_3 + \int_1^t (H_1(s)+H_2(s))(g_1(z(s)^{\frac{r}{p}}) + g_2(z(s)^{\frac{r}{p}}))\,\mathrm{d}s$$

where $\bar{H}_3 = \int_1^t H_3(s)\,\mathrm{d}s$. I.e.

$$z(t) \leq d + 2^{p-1}\bar{h}_3 + 2^{p-1}\int_1^t (h_1(s)+h_2(s))(g_1(z(s)^{\frac{r}{p}}) + g_2(z(s)^{\frac{r}{p}}))\,\mathrm{d}s. \tag{5.13}$$

Applying Bihari's inequality (see [5]) to (5.13), we obtain for $t \geq 1$

$$z(t) \leq G^{-1}\Big(G(d + 2^{p-1}\bar{h}_3) + 2^{p-1}\int_1^t (h_1(s)+h_2(s))\,\mathrm{d}s\Big)$$

where

$$G(x) = \int_1^x \frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}}) + g_2(s^{\frac{r}{p}})}$$

and $G^{-1}(x)$ is the inverse function for $G(x)$ defined for $x \in (G(+0), \infty)$. Note that $G(+0) < 0$, and $G^{-1}(x)$ is increasing.

Now, let

$$K = G(d + 2^{p-1}\bar{h}_3) + 2^{p-1}(\bar{h}_1 + \bar{h}_2) < \infty$$

where $\bar{h}_i = \int_1^t h_i(s)\,\mathrm{d}s$, $i = 1, 2, 3$. Since $G^{-1}(x)$ is increasing, we have

$$z(t) \leq G^{-1}(K) < \infty,$$

so (5.11) yields

$$\frac{|u(t)|}{t} \le \left(G^{-1}(K)\right)^{\frac{1}{p}} \quad \text{and} \quad |u'(t)| \le \left(G^{-1}(K)\right)^{\frac{1}{p}}.$$

Using the assumption (ii) of the Theorem 5.2, we have

$$\begin{aligned}\int_1^t |f(s,u(s),u'(s))|\,\mathrm{d}s &\le h_1(t)g_1\left(\left[\frac{|u|}{t}\right]^r\right) + h_2(t)g_2(|u'(t)|^r) + h_3(t)\\ &\le z(t) \le \left(G^{-1}(K)\right)^{\frac{1}{p}}\end{aligned}$$

where $t \ge 1$, the integral $\int_1^t |f(s,u(s),u'(s))|\,\mathrm{d}s$ converges and there exists an $a \in \mathbb{R}$ such that

$$\lim_{t\to\infty} u'(t) = a.$$

□

Example 5.2. Let $t_0 = 1$, $p \ge r > 0$, $p \ge 1$

$$\begin{aligned}f(t,u,v) &= \eta_1(t)t^{1-\alpha_1}e^{-t}\left(\frac{u}{t}\right)^{p-r}\ln\left[2+\left(\frac{u}{t}\right)^r\right]\\ &\quad + \eta_2(t)t^{1-\alpha_2}e^{-t}v^{p-r}\ln(3+v^r) + \eta_3(t)t^{1-\alpha_3}e^{-t}\end{aligned}$$

where $0 < \alpha_i < 1$ and $\eta_i(t)$ are continuous functions on interval $[1,\infty)$ with $K_i = \sup_{t\ge 1}|\eta_i(t)| < \infty$, $i = 1,2,3$. Then $f(t,u,u')$ can be written as

$$f(t,u,v) = h_1(t)g_1\left(\left[\frac{u}{t}\right]^r\right) + h_2(t)g_2(v^r) + h_3(t)$$

where $h_i(t) = \eta_i(t)t^{1-\alpha_i}e^{-t}$, $i = 1,2,3$, $g_1(u) = u^{\frac{p}{r}}\ln(2+u)$, $g_2(u) = u^{\frac{p}{r}}\ln(2+u)$. Then

$$|f(t,u,v)| \le |h_1(t)|g_1\left(\left[\frac{u}{t}\right]^r\right) + |h_2(t)|g_2(|v|^r) + |h_3(t)|$$

where $(t,u,v) \in D = \{(t,u,v) : t \in [1,\infty), u,v \in \mathbb{R}\}$, $|h_i(t)| \le K_i\Gamma(\alpha_i)$, $i = 1,2,3$ and obviously we have

$$\begin{aligned}G(\infty) &= \int_1^\infty \frac{\tau^{\frac{p}{r}-1}\,\mathrm{d}\tau}{g_1(\tau)+g_2(\tau)} = \int_1^\infty \frac{\tau^{\frac{p}{r}-1}\,\mathrm{d}\tau}{\tau^{\frac{p}{r}}[\ln(2+\tau)+\ln(3+\tau)]}\\ &\ge \frac{1}{2}\int_1^\infty \frac{\mathrm{d}\tau}{(3+\tau)\ln(3+\tau)} = \infty.\end{aligned}$$

This means that all assumptions of Theorem 5.2 are satisfied and thus any global solution $u(t)$ of the equation (5.1) possesses the property (L).

Theorem 5.3. *Let $t_0 > 0$. Suppose that the following assumptions hold:*

(i) *there exist nonnegative continuous function $h_1, h_2, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \leq h_1(t) g_1\left(\left[\frac{|u|}{t}\right]^r\right) + h_2(t) g_2(|v|^r);$$

(ii) *for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are nondecreasing and*

$$g_1(\alpha u) \leq \psi_1(\alpha) g_1(u), \qquad g_2(\alpha u) \leq \psi_2(\alpha) g_2(u)$$

for $\alpha \geq 1$, $u \geq 0$, where the functions $\psi_1(\alpha)$, $\psi_2(\alpha)$ are continuous for $\alpha \geq 1$;

(iii) $\int_{t_0}^{\infty} h_i(s)\,\mathrm{d}s = H_i < \infty, i = 1, 2$. *Assume that there exists a constant $K \geq 1$ such that*

$$\begin{aligned} K^{-1}(\psi_1(K) + \psi_2(K)) 2^{p-1}(H_1 + H_2) &\leq \int_{t_0}^{+\infty} \frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}}) + g_2(s^{\frac{r}{p}})} \\ &= \frac{p}{r} \int_a^{+\infty} \frac{\tau^{\frac{p}{r}-1}\,\mathrm{d}\tau}{g_1(\tau) + g_2(\tau)} \end{aligned}$$

where $a = (t_0)^{\frac{r}{p}}$.

Then any global solution $u(t)$ of the equation (5.1) *with initial data $u(t_0) = u_0$, $u'(t_0) = u_1$ such that $(|u_0| + |u_1|)^p \leq K$ possesses the property* (L).

Proof. Without loss of generality we may assume $t_0 = 1$. Arguing in the same way as in Theorem 5.1, we obtain by assumption (i) of Theorem 5.3

$$|u'(t)| \leq \left[|u_1|^p + \int_1^t h_1(s) g_1\left(\left[\frac{u(s)}{s}\right]^r\right) \mathrm{d}s + \int_1^t h_2(s) g_2(|u'(s)|^r)\,\mathrm{d}s \right]^{\frac{1}{p}} \tag{5.14}$$

$$\frac{|u(t)|}{t} \leq |u_0| + \left[|u_1|^p + \int_1^t h_1(s) g_1\left(\left[\frac{u(s)}{s}\right]^r\right) \mathrm{d}s + \int_1^t h_2(s) g_2(|u'(s)|^r)\,\mathrm{d}s \right]^{\frac{1}{p}}$$

where $t \geq 1$.

$$\left(\frac{|u(t)|}{t}\right)^p \leq K + 2^{p-1} \left[\int_1^t h_1(s) g_1\left(\left[\frac{u(s)}{s}\right]^r\right) \mathrm{d}s + \int_1^t h_2(s) g_2(|u'(s)|^r)\,\mathrm{d}s \right] \tag{5.15}$$

where $K = 2^{p-1}(|u_0|^p + |u_1|^p) \geq (|u_0| + |u_1|)^p$. Denoting by $z(t)$ the right-hand side of inequality (5.15) we have by (5.14) and (5.15)

$$|u'(t)|^r \leq z(t)^{\frac{r}{p}}, \tag{5.16}$$
$$\left(\frac{|u(t)|}{t}\right)^r \leq z(t)^{\frac{r}{p}}.$$

Since the functions $g_1(s)$, $g_2(s)$ are nondecreasing for $s > 0$, (5.16) yields for $t \geq 1$

$$z(t) \leq K \ + 2^{p-1}\bigg(\int_1^t h_1(s) g_1\big(z(s)^{\frac{r}{p}}\big)\,\mathrm{d}s + \int_1^t h_2(s) g_2(z(s)^{\frac{r}{p}}) \bigg)\,\mathrm{d}s. \tag{5.17}$$

By assumption (ii) of Theorem 5.3, the functions $g_1(u)$, $g_2(u)$ belong to the class $\mathbb{H}$. Furthermore, if $g_1(u)$ and $g_2(u)$ belong to the class $\mathbb{H}$ with corresponding multiplier functions $\psi_1(\alpha)$, $\psi_2(\alpha)$, respectively, then the sum $g_1(u) + g_2(u)$ also belongs to the class $\mathbb{H}$ with corresponding multiplier function $(\psi_1(\alpha) + \psi_2(\alpha))$. Applying Bihari's Theorem (see [5]) to (5.17), we have for $t \geq 1$

$$z(t) \leq KW^{-1}(K^{-1}(\psi_1(K) + \psi_2(K)))2^{p-1} \int_1^t (h_1(s) + h_2(s))\,\mathrm{d}s \tag{5.18}$$

where

$$W(u) = \int_1^u \frac{\mathrm{d}s}{g_1\big(s^{\frac{r}{p}}\big) + g_2\big(s^{\frac{r}{p}}\big)}$$

and $W^{-1}(u)$ is inverse function for $W(u)$. Inequality (5.18) holds for all $t \geq 1$ because

$$(K^{-1}(\psi_1(K) + \psi_2(K))2^{p-1}(H_1 + H_2) = L < \infty.$$

Since $W^{-1}(u)$ is increasing, we get

$$z(t) \leq KW^{-1}(L) < \infty,$$

so it follows from (5.16), (5.17) that

$$\frac{|u(t)|}{t} \leq \big(KW^{-1}(L)\big)^{\frac{1}{p}} \quad \text{and} \quad |u'(t)| \leq \big(KW^{-1}(L)\big)^{\frac{1}{p}}.$$

The rest of the proof is similar to that of Theorem 5.2 and thus it is omitted. □

Example 5.3. Let $t_0 > 0$. Consider the equation (5.1) with $p \geq 1$, $\frac{p}{r} = 2$,

$$f(t, u, v) = h_1(t)u^2 + h_2(t)v^2$$

where $h_1(t) = \frac{\eta_1(t)}{t^2} t^{1-\alpha_1} e^{-t}$, $h_2(t) = \eta_2(t) t^{1-\alpha_2} e^{-t}$, $0 < \alpha_i \leq 1, \eta_i(t), i = 1, 2$ are continuous functions on the interval $[0, \infty)$ with $K_i = \sup_{t \geq t_0} |\eta_i(t)| < \infty$. Then we can write

$$f(t, u, v) = \eta_1(t) t^{1-\alpha_1} e^{-t} \left(\frac{u}{t}\right)^2 + \eta_2(t) t^{1-\alpha_2} e^{-t} v^2$$

and

$$|f(t,u,u')| \le K_1\Gamma(\alpha_1)g_1(u) + K_2\Gamma(\alpha_2)g_2(u')$$

where $g_1(u) = u^2$, $g_2(u') = (u')^2$. The functions g_1, g_2 satisfy the condition (ii) of Theorem 5.3 with $\psi_1(\alpha) = \psi_2(\alpha) = \alpha^2$ and

$$\int_{t_0}^{\infty} \frac{\tau^{\frac{p}{r}-1}\,\mathrm{d}\tau}{g_1(\tau)+g_2(\tau)} = \int_{t_0}^{\infty} \frac{\mathrm{d}\tau}{\tau} = \infty.$$

Thus all assumptions of Theorem 5.3 are satisfied and therefore any global solution $u(t)$ of the equation (5.1) (independently on the initial values u_0, u_1) possesses the property (L).

Theorem 5.4. *Let $t_0 > 0$. Suppose that the asumptions* (i) *and* (iii) *of Theorem 5.3 hold, while* (ii) *is replaced by*

(ii') *for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are nonnegative, continuous and nondecreasing, $g_1(0) = g_2(0) = 0$ and satisfy a Lipschitz condition*

$$|g_1(u+v) - g_1(u)| \le \lambda_1 v, \quad |g_2(u+v) - g_2(u)| \le \lambda_2 v$$

where λ_1, λ_2 are positive constants.

Then any global solution $u(t)$ of the equation (5.1) *with initial data $u(t_0) = u_0$, $u'(t_0) = u_1$ such that $|u_0|^p + |u_1|^p \le K$ possesses property* (L).

Proof. Applying [8, Corollary 2] to (5.17), we have for $t \ge 1$

$$\begin{aligned} z(t) \le K\ &+ 2^{p-1}\int_{t_0}^{t} (h_1(s)+h_2(s))(g_1(K)+g_2(K)) \\ &\times \exp\left(2^{p-1}\int_{t_0}^{t}(\lambda_1+\lambda_2)(h_1(\tau)+h_2(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &\le K + 2^{p-1}(H_1+H_2)(g_1(K)+g_2(K))\exp\left(2^{p-1}(\lambda_1+\lambda_2)(H_1+H_2)\right) \\ &< +\infty. \end{aligned}$$

The proof can be completed with the same argument as in Theorem 5.2. □

Theorem 5.5. *Let $t_0 > 0$. Suppose that there exist continuous functions $h : \mathbb{R}_+ \to \mathbb{R}_+$, $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \le h(t)g_1\left(\left[\frac{|u|}{t}\right]^r\right)g_2(|v|^r)$$

where for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are nondecreasing;

$$\int_{t_0}^{\infty} h(s)\,\mathrm{d}s < \infty$$

and if we denote

$$G(x)=\int_{t_0}^{x}\frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}})g_2(s^{\frac{r}{p}})},$$

then $G(+\infty)=\frac{p}{r}\int_a^\infty\frac{\tau^{\frac{p}{r}-1}}{g_1(\tau)g_2(\tau)}\,\mathrm{d}\tau=+\infty$ *where* $a=(t_0)^{\frac{r}{p}}$.

Then any global solution $u(t)$ *of the equation* (5.1) *possesses the property* (L).

Proof. Without loss of generality we may assume $t_0=1$. Arguing as in the proof of Theorem 5.2, we obtain for $t\geq 1$

$$|u'(t)|\leq\left[|u_1|^p+\int_1^t h(s)g_1\left(\left[\frac{u(s)}{s}\right]^r\right)g_2(|u'(s)|^r)\,\mathrm{d}s\right]^{\frac{1}{p}}$$

$$\frac{|u(t)|}{t}\leq|u_0|+\left[|u_1|^p+\int_1^t h(s)g_1\left(\left[\frac{u(s)}{s}\right]^r\right)g_2(|u'(s)|^r)\,\mathrm{d}s\right]^{\frac{1}{p}}$$

$$\left(\frac{|u(t)|}{t}\right)^p\leq C+2^{p-1}\int_1^t h(s)g_1\left(\left[\frac{u(s)}{s}\right]^r\right)g_2(|u'(s)|^r)\,\mathrm{d}s \tag{5.19}$$

where $C=2^{p-1}(|u_0|^p+|u_1|^p)\geq(|u_0|+|u_1|)^p$. Denoting by $z(t)$ the right-hand side of inequality (5.19) and using the assumptions of the Theorem 5.5, we have for $t\geq 1$

$$z(t)\leq 1+C+2^{p-1}\int_1^t h(s)g_1(z^{\frac{r}{p}})g_2(z^{\frac{r}{p}})\,\mathrm{d}s. \tag{5.20}$$

Applying Bihari's inequality (see [5]) to (5.20), we obtain for $t\geq 1$

$$z(t)\leq G^{-1}\left(G(1+C)+2^{p-1}\int_1^t h(s)\,\mathrm{d}s\right)\leq G^{-1}(K)$$

where

$$G(w)=\int_1^w\frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}})g_2(s^{\frac{r}{p}})}$$

and $G^{-1}(w)$ is the inverse function for $G(w)$. The function $G^{-1}(w)$ is defined for $w\in(G(+0),\infty)$, where $G(+0)<0$, it is increasing and

$$K=G(1+C)+2^{p-1}\int_1^\infty h(s)\,\mathrm{d}s<\infty.$$

The rest of proof is similar that of Theorem 5.2 and thus is omitted. $\square$

Example 5.4. Let $t_0=1$, $p\geq r>0$,

$$f(t,u,v)=\eta(t)t^{1-\alpha}e^{-t}\left[\left(\frac{u}{t}\right)^{p-r}\ln\left[2+\left(\frac{u}{t}\right)^r\right]\right]^{\frac{3}{4}}\cdot\left[v^{p-r}\ln(2+v^r)\right]^{\frac{1}{4}}$$

where $\eta(t)$ is a continuous function on $[1,\infty)$ with $K = \sup_{t\in\langle 1,\infty)} \eta(t) < \infty$. Let

$$g_1(u) = \left[u^{\frac{p}{r}-1}\ln(2+u)\right]^{\frac{3}{4}},\ g_2(v) = \left[v^{\frac{p}{r}-1}\ln(2+v)\right]^{\frac{1}{4}},\ h(t) = \eta(t)t^{1-\alpha}e^{-t}.$$

Then

$$f(t,u,v) = h(t)g_1\left(\left[\frac{u}{t}\right]^r\right)g_2(v^r)$$

and

$$\begin{aligned} G(+\infty) &= \frac{p}{r}\int_1^\infty \frac{\tau^{\frac{p}{r}-1}}{g_1(\tau)g_2(\tau)}\,\mathrm{d}\tau = \frac{p}{r}\int_1^\infty \frac{\mathrm{d}\tau}{\tau\ln(2+\tau)} \\ &> \frac{p}{r}\int_1^\infty \frac{\mathrm{d}\tau}{(2+\tau)\ln(2+\tau)} = +\infty. \end{aligned}$$

Obviously $|f(t,u,v)|$ can be estimated as in Theorem 5.5. Thus all assumptions of Theorem 5.5 are satisfied and this means that any global solution of the equation (5.1) possesses the property (L).

Theorem 5.6. *Let $t_0 > 0$. Suppose that the following conditions hold:*

(i) *there exist nonnegative continuous functions $h, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \le h(t)g_1\left(\left[\frac{|u(t)|}{t}\right]^r\right)g_2(|v|^r)$$

(ii) *for $s > 0$ the functions $g_1(s), g_2(s)$ are nondecreasing and*

$$g_1(\alpha u) \le \psi_1(\alpha)g_1(u), \quad g_2(\alpha u) \le \psi_2(\alpha)g_2(u)$$

for $\alpha \ge 1, u \ge 0$, where the functions $\psi_1(\alpha), \psi_2(\alpha)$ are continuous for $\alpha \ge 1$;

(iii) $\int_{t_0}^\infty h(s)\,\mathrm{d}s = H < +\infty$.

Assume also that there exists a constant $K \ge 1$ such that

$$K^{-1}H\psi_1(K)\psi_2(K) \le \int_1^\infty \frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}})g_2(s^{\frac{r}{p}})} = \frac{p}{r}\int_a^\infty \frac{\tau^{\frac{p}{r}-1}\,\mathrm{d}\tau}{g_1(\tau)g_2(\tau)} \tag{5.21}$$

where $a = (t_0)^{\frac{r}{p}}$.

Then any global solution $u(t)$ of the equation (5.1) *with initial data $u(t_0) = u_0$, $u'(t_0) = u_1$ such that $2^{p-1}(|u_0|^p + |u_1|^p) \le K$ possesses the property* (L).

Proof. Without loss of generality we may assume $t_0 = 1$. With the same argument as in Theorem 5.2, we have for $t \geq 1$

$$|u'(t)| \leq \left[|u_1|^p + \int_1^t h(s) g_1\left(\left[\frac{|u(s)|}{s} \right]^r \right) g_2(|u'(s)|^r)\, \mathrm{d}s \right]^{\frac{1}{p}},$$

$$\frac{|u(t)|}{t} \leq |u_0| + \left[|u_1|^p + \int_1^t h(s) g_1\left(\left[\frac{|u(s)|}{s} \right]^r \right) g_2(|u'(s)|^r)\, \mathrm{d}s \right]^{\frac{1}{p}}.$$

Applying the inequality $(A+B)^p \leq 2^{p-1}(A^p + B^p)$, $A, B \geq 0$ we obtain

$$\left(\frac{|u(t)|}{t} \right)^p \leq 2^{p-1}(|u_0|^p + |u_1|^p) + 2^{p-1} \left[\int_1^t g_1\left(\left[\frac{|u(s)|}{s} \right]^r \right) g_2(|u'(s)|^r)\, \mathrm{d}s \right]. \quad (5.22)$$

Denoting by $z(t)$ the right-hand side of inequality (5.22) we obtain for $t \geq 1$

$$z(t) \leq K + \int_1^t H(s) g_1(z(s)^{\frac{r}{p}}) g_2(z(s)^{\frac{r}{p}})\, \mathrm{d}s \quad (5.23)$$

where $K = 2^{p-1}(|u_0|^p + |u_1|^p)$ and $H(t) = 2^{p-1} h(t)$. Assumption (ii) implies that the functions $g_1(u)$, $g_2(u)$ belong to the class $\mathbb{H}$. Furthermore, it follows from [6, Lemma 1], that if $g_1(u)$ and $g_2(u)$ belong to the class $\mathbb{H}$ with the corresponding multiplier functions $\psi_1(\alpha)$ and $\psi_2(\alpha)$ respectively, then the product $g_1(u)g_2(u)$ also belongs to $\mathbb{H}$ and the corresponding multiplier function is $\psi_1(\alpha)\psi_2(\alpha)$. Thus, applying [8, Theorem 1] to (5.23), we have for $t \geq 1$

$$z(t) \leq K W^{-1}\left(K^{-1} \psi_1(K) \psi_2(K) \int_1^t H(s)\, \mathrm{d}s \right) \quad (5.24)$$

where

$$W(u) = \int_1^u \frac{\mathrm{d}s}{g_1(s^{\frac{r}{p}}) g_2(s^{\frac{r}{p}})}$$

and $W^{-1}(u)$ is the inverse function for $W(u)$. Evidently, inequality (5.24) holds for all $t \geq 1$ since by (5.21)

$$K^{-1} \psi_1(K) \psi_2(K) \int_1^t H(s)\, \mathrm{d}s \in \mathrm{Dom}(W^{-1})$$

for all $t \geq 1$. The rest of the proof is analogous to that of Theorem 5.2 and is omitted. □

Theorem 5.7. *Let $t_0 > 0$. Suppose that the assumptions* (i) *and* (iii) *of Theorem 5.6 hold, while* (ii) *is replaced by* (ii') *for* $s > 0$ *the functions* $g_1(s)$, $g_2(s)$ *are continuous and nondecreasing,* $g_1(0) = g_2(0) = 0$ *and satisfy a Lipschitz condition*

$$|g_1(u+v) - g_1(u)| \leq \lambda_1 v, \quad |g_2(u+v) - g_2(u)| \leq \lambda_2 v$$

where λ_1, λ_2 *are positive constants.*

Then any global solution $u(t)$ *of the equation* (5.1) *with initial data* $u(t_0) = u_0$, $u'(t_0) = u_1$ *such that* $|u_0|^p + |u_1|^p \leq K$ *possesses the property* (L).

Proof. Without loss of generality we may assume $t_0 = 1$. Applying [8, Corollary 2] to (5.23), we have for $t \geq 1$

$$\begin{aligned} z(t) &\leq K \ + g_1(K)g_2(K) \int_1^t H(s) \exp\left(\lambda_1\lambda_2 \int_1^t H(\tau)\mathrm{d}\tau\right) \mathrm{d}s \\ &\leq K \ + \bar{H} g_1(K)g_2(K) \exp\left(\lambda_1\lambda_2 \bar{H}\right) < +\infty. \end{aligned}$$

The proof can be completed with the same argument as in Theorem 5.2. □

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Conclusions

The main benefit of thesis is derivation the sufficient conditions for existence continuable solutions and the investigation one type of asymptotic behavior for special type of second order differential equation with p-Laplacian. Specially, in the Chapter 3 is used the original method in the proof Theorem 3.1.

The next research could study different kind of asymptotic formulas. Furthermore, it is possible to study the same problem applied to other differential equations, e.g. n-th differential equation, differential equation with delay which appears in many applications (e.g. biology, chemistry).