MASARYK UNIVERSITY
Faculty of Science

PHASES AND OSCILLATION THEORY OF SECOND ORDER DIFFERENCE EQUATIONS

Ph.D. Thesis

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I hereby declare that prof. RNDr. Z. Došlá, DSc. and prof. RNDr. O. Došlý, DrSc. share in the authorship of this dissertation.

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Název práce: Fáze a oscilační teorie diferenčních rovnic druhého řádu
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Abstrakt:

Uvažujme 2×2 symplektický diferenční systém tvaru

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \qquad k \in \mathbf{Z}, \tag{S}$$

kde (a_k), (b_k), (c_k) a (d_k) jsou posloupnosti reálných čísel s vlastností $a_k d_k - b_k c_k = 1$ a jeho speciální případ, Sturm-Liouvilleovu diferenční rovnici tvaru

$$\Delta(r_k \Delta x_k) + q_k x_{k+1} = 0, \qquad k \in \mathbf{Z}, \tag{S-L}$$

kde (r_k), (q_k) jsou reálné posloupnosti a $r_k \neq 0$. V této práci je zavedena první a druhá fáze normalizované báze symplektického systému (S). Je studován vztah mezi těmito fázemi a jejich souvislost s oscilatorickými vlastnostmi systému (S).

Pomocí teorie fází a Riccatiho rovnice je zkoumána konjugovanost rovnice (S-L). K tomuto účelu byl zaveden pojem fáze libovolné báze rovnice (S-L) a následně formulovány její vlastnosti. Krátce je studována problematika konstrukce rovnice (S-L), jejímiž rešeními jsou dvě posloupnosti tvořící bázi a mající předepsaný počet zobecněných nulových bodů.

Dále je ukázán vztah mezi symplektickým systémem (S), rovnicí (S-L) v rekurentním tvaru

$$r_{k+1} x_{k+2} + \beta_k x_{k+1} + r_k x_k = 0, \qquad r_k \neq 0, \tag{R}$$

a tridiagonální symetrickou maticí

$$\mathcal{L} = - \begin{pmatrix} \beta_0 & r_1 & 0 & \dots & 0 & 0 & 0 \\ r_1 & \beta_1 & r_2 & & 0 & 0 & 0 \\ 0 & r_2 & \beta_2 & & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & & & & \\ 0 & \dots & & 0 & r_{N-2} & \beta_{N-2} & r_{N-1} \\ 0 & \dots & & & 0 & r_{N-1} & \beta_{N-1} \end{pmatrix}, \tag{$\mathcal{L}$}$$

kde (r_k) and (β_k) jsou vhodné posloupnosti reálných čísel. Je studována transformace rekurentní rovnice (R) na tzv. trigonometrickou rekurentní rovnici a transformace matice $(\mathcal{L})$ na tzv. trigonometrickou matici. Jsou ukázány vztahy mezi všemi zmíněnými pojmy.

Klíčová slova: Sturm-Liouvilleova diferenční rovnice, symplektický diferenční systém, fáze, zobecněný nulový bod, konjugovanost, trigonometrický symplektický systém, trigonometrická rekurentní rovnice, trigonometrická matice

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Abstract:

Consider 2×2 symplectic difference system of the form

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \qquad k \in \mathbf{Z}, \tag{S}$$

where (a_k), (b_k), (c_k) and (d_k) are real-valued sequences such that $a_k d_k - b_k c_k = 1$. Consider a special case of (S), i.e., Sturm-Liouville difference equation

$$\Delta(r_k \Delta x_k) + q_k x_{k+1} = 0, \qquad k \in \mathbf{Z}, \tag{S-L}$$

with real-valued sequences (r_k), (q_k) and $r_k \neq 0$. First and second phases of a normalized basis of (S) are introduced. The relation between both phases and its connection to the oscillatory properties of system (S) is shown.

By means of a phase theory and the Riccati equation the conjugacy of the equation (S-L) is studied. To this goal it is introduced a concept of a phase of any basis of (S-L) and its properties. A construction of the equation (S-L) is described which has two linearly independent solutions with a prescribed number of generalized zeros.

Further, a relation is given between a symplectic system (S), a recurrence form of equation (S-L)

$$r_{k+1} x_{k+2} + \beta_k x_{k+1} + r_k x_k = 0, \qquad r_k \neq 0, \tag{R}$$

and a tridiagonal symmetric matrix

$$\mathcal{L} = - \begin{pmatrix} \beta_0 & r_1 & 0 & \dots & 0 & 0 & 0 \\ r_1 & \beta_1 & r_2 & & 0 & 0 & 0 \\ 0 & r_2 & \beta_2 & & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & & & & \\ 0 & \dots & & 0 & r_{N-2} & \beta_{N-2} & r_{N-1} \\ 0 & \dots & & & 0 & r_{N-1} & \beta_{N-1} \end{pmatrix}, \tag{$\mathcal{L}$}$$

with the suitable real-valued sequences (r_k) and (β_k). A transformation of the recurrence relation (R) into the trigonometric recurrence relation and a transformation of the matrix $(\mathcal{L})$ into the trigonometric matrix is studied. The relations between the above concepts are given as well.

Keywords: Sturm-Liouville difference equation, symplectic difference system, phase, generalized zero, conjugacy, trigonometric symplectic system, trigonometric recurrence relation, trigonometric matrix

Contents

Chapter 1

Introduction

The aim of this text is to study oscillatory properties of the second order linear difference equations and their connection to phases. The basis of this text consists of results which were received and published in previous articles [9, 15, 28]. We put these results together to make a compact text. Contrary to the papers it turned out more convenient to explain some of the proofs to make the text plainer.

We consider the *Sturm-Liouville difference equation*

$$\Delta(r_k \Delta x_k) + q_k x_{k+1} = 0, \qquad r_k \neq 0, \qquad k \in \mathbb{Z}, \tag{1.1}$$

where $\Delta x_k := x_{k+1} - x_k$ and (r_k), (q_k) are real-valued sequences. This equation can be written as a recurrence relation

$$r_{k+1} x_{k+2} + \beta_k x_{k+1} + r_k x_k = 0, \qquad r_k \neq 0, \qquad k \in \mathbb{Z}, \tag{R}$$

where $\beta_k = q_k - r_k - r_{k+1}$.

We also consider the more general form of (1.1), the 2×2 *symplectic difference system*

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \qquad k \in \mathbb{Z}, \tag{S}$$

where (a_k), (b_k), (c_k), (d_k) are real-valued sequences such that $a_k d_k - c_k b_k = 1$.

The motivation of our research consists mostly of the objects and theories surveyed in monographs [4, 7]. Especially, we were motivated by a theory of transformation of the second order linear differential equations introduced by O. Borůvka in [7]. The results of his theory were extended in several directions (see e.g. [11] by O. Došlý, [24] by F. Neuman, [29] by S. Staněk) and also to the discrete case by Z. Došlá and D. Škrabáková in their article [10]. Here the *first phases* of the normalized basis of (1.1) and (S) were defined and the fundamental relations to their oscillatory properties were shown.

We started our investigation just by the open problems which were stated in [10]. Some of the acquired results were published in article [28]. We formulated the definition of the *second phase* of (S) and its relationship to oscillatory properties of (S) and to the *reciprocal system* to (S) there.

The main subject of our study consists of the application of the discrete phase theory, namely of the conjugacy of the second order linear difference equations. We were inspirated by the conjugacy criterion in [16, Theorem 1] by O. Došlý and P. Řehák. We have shown a slightly different proof of the generalized formulation of this criterion using phases. To this goal we had to generalize a definition of the first phase of a normalized basis of (1.1) and so we defined the first phase of an arbitrary basis of (1.1). We took up again by some other conjugacy criteria, namely we extended criteria formulated in [3] to their disrete counterparts. These results were published in article [9].

The text is organized as follows. In Chapter 2 we recall the basic concepts of oscillation theory of Sturm-Liouville difference equations and 2×2 symplectic difference systems. In Chapter 3 we establish their first and second phases. Further, in Chapter 4 we introduce a generalized definition of phases of Sturm-Liouville difference equations and we show a construction of these equations with prescribed properties. The main results are given in Chapter 5, where we introduce the conjugacy criteria of Sturm-Liouville difference equations, which is an important application of the phase theory. Finally, in Chapter 6 we point out some algebraic aspects of symplectic systems. Among other things, we show the connection between (R), (S), linear system $\mathcal{L}x = 0$, where

$$\mathcal{L} = -\begin{pmatrix} \beta_0 & r_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ r_1 & \beta_1 & r_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & r_2 & \beta_2 & r_3 & & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots & \\ & & & & & & & \\ 0 & 0 & & 0 & r_{N-3} & \beta_{N-3} & r_{N-2} & 0 \\ 0 & \dots & & & & r_{N-2} & \beta_{N-2} & r_{N-1} \\ 0 & \dots & & & & 0 & r_{N-1} & \beta_{N-1} \end{pmatrix},$$

$x = (x_k)_{k=1}^N$, and their "trigonometric" counterparts. Open problems are given at the end of Chapter 3, 4 and 5.

Chapter 2

Basic Concepts of Oscillation Theory

The main purpose of this chapter is to present a brief survey of the basic terms and results of the discrete second order oscillation theory.

2.1 Sturm-Liouville differential equation

First of all we recall the elementary well-known concepts of oscillation theory for the Sturm-Liouville differential equation.

By the *Sturm-Liouville differential equation* we understand the equation

$$(r(t)x'(t))' + q(t)x(t) = 0, \qquad r(t) > 0 \tag{2.1}$$

with continuous functions r and q in a given interval $I = (a, b) \subseteq \mathbb{R}$, $-\infty \leq a < b \leq \infty$ and $a, b \in \mathbb{R}$. The most important oscillatory properties of (2.1) are described by the so-called *Reid roundabout theorem.* It shows the connection among such concepts like *disconjugacy* of (2.1) on I, positive definiteness of the *quadratic functional*

$$\mathcal{F}(x; a, b) = \int_a^b \left[r(t)x'^2(t) - q(t)x^2(t)\right] \, dt$$

over the set of admissible functions x, and solvability of the *Riccati equation*

$$w' + q(t) + \frac{w^2}{r(t)} = 0$$

on I. A consequence of this theorem yields the *Sturm separation theorem* and the *Sturm comparsion theorem.* For more information conserning oscillatory properties of (2.1) we refer to e.g. [7, 19, 24, 26, 27, 30, 31].

Let us recall an important property of equation (2.1); that is, its *self-reciprocity.* If x is a solution of (2.1) and $q(t) \neq 0$ on I, then $y = r(t)x'$ is a solution of the *reciprocal* equation to (2.1)

$$\left(\frac{1}{q(t)}y'\right)' + \frac{1}{r(t)}y = 0. \tag{2.2}$$

An equation that coincides with its reciprocal equation is called *self-reciprocal.*

Observe, that the equation (2.1) can be transformed via the transformation

$$x = h(t)y \tag{2.3}$$

into the equation

$$\left(\frac{1}{p(t)}y'\right) + p(t)y = 0, \qquad p(t) = \frac{1}{r(t)h(t)^2}, \tag{2.4}$$

where $h(t) = \sqrt{x_1^2(t) + x_2^2(t)}$ and x_1, x_2 are two linearly independent solutions of (2.1) with the Wronskian w

$$w \equiv r(t)(x_1(t)x_2'(t) - x_2(t)x_1'(t))$$

such that $w \equiv 1$. Note, that equation (2.4) is self-reciprocal.

The importance of the transformation (2.3) consists in the following: We are able to transform equation (2.1) to its "simpler" form (2.4) which is explicitly solvable. Note that the functions

$$y_1(t) = \cos\left(\int^t p(s)\ ds\right) \qquad \text{and} \qquad y_2(t) = \sin\left(\int^t p(s)\ ds\right)$$

are two linearly independent solutions of (2.4). Then in view of (2.3), a basis (x_1, x_2) of (2.1) takes the form

$$x_1(t) = h(t)\cos\left(\int^t p(s)\ ds\right) \qquad \text{and} \qquad x_2(t) = h(t)\sin\left(\int^t p(s)\ ds\right).$$

2.2 Sturm-Liouville difference equation

In the following we recall results which were obtained as the discrete counterparts of oscillatory concepts of equation (2.1) and play an important role in our research.

A discrete analogue of equation (2.1) is the so-called *Sturm-Liouville difference equation* (1.1)

$$\Delta(r_k \Delta x_k) + q_k x_{k+1} = 0, \qquad r_k \neq 0,$$

where $k \in \mathbb{Z}$, r and q are sequences of real numbers and $\Delta x_k := x_{k+1} - x_k$.

Definition 1. A pair of two linearly independent solutions $x^{[1]}$ and $x^{[2]}$ of (1.1) with the Casoratian ω, where

$$\omega \equiv r_k(x_k^{[1]}\Delta x_k^{[2]} - x_k^{[2]}\Delta x_k^{[1]}) = r_k(x_k^{[1]}x_{k+1}^{[2]} - x_k^{[2]}x_{k+1}^{[1]}), \tag{2.5}$$

is called a *basis* of the equation (1.1) and we denote it $(x^{[1]}, x^{[2]})$. If $\omega \equiv 1$, we talk about a *normalized basis.* The relation (2.5) is called the *Wronskian identity.*

Here, let us state the definition of a generalized zero, which is a discrete analogue of a zero of a solution of (2.1). The reason to formulate a concept of a generalized zero was to keep the validity of Sturmian theory for second order linear difference equations (for details see e.g. [8, 30]).

Definition 2. We say that an interval $(M, M+1]$, $M \in \mathbb{Z}$, contains a *generalized zero* of a solution x of (1.1) if

$$x_M \neq 0 \qquad \text{and} \qquad r_M x_M x_{M+1} \leq 0, \tag{2.6}$$

see e.g. [1, 4]. Equivalently, we say that a solution x of (1.1) has a *generalized zero* at M, $M \in \mathbb{Z}$, if (2.6) holds.

Remark 1. If $r_k > 0$ for every k, Definition 2 agrees with that given by P. Hartman in [18]: We say that an integer M is a "node" for a solution $x \not\equiv 0$ of (1.1) if either $x_M = 0$ or $x_{M-1}x_M < 0$.

Let us illustrate Definition 2 by the following difference equation. If $x_1 = x_2 = 1$, it determines the so-called *Fibonacci numbers.*

Example 1. *Consider the equation*

$$\Delta((-1)^k \Delta x_k) + (-1)^k x_{k+1} = 0, \qquad k \in \mathbb{N}, \tag{2.7}$$

which can be written as the recurrence relation

$$x_{k+2} = x_{k+1} + x_k, \qquad k \in \mathbb{N}. \tag{2.8}$$

The characteristic equation of (2.8) *is of the form*

$$\lambda^2 - \lambda - 1 = 0$$

with two roots λ_1 *and* λ_2, *where*

$$\lambda_1 = \frac{1-\sqrt{5}}{2} < 0, \qquad \lambda_2 = \frac{1+\sqrt{5}}{2} > 0.$$

Hence, a basis of (2.8) *is given by the solutions*

$$x_k^{[1]} = \left(\frac{1-\sqrt{5}}{2}\right)^k, \qquad x_k^{[2]} = \left(\frac{1+\sqrt{5}}{2}\right)^k.$$

In view of the fact that $r_k = (-1)^k$ *we obtain*

$$x_k^{[1]} \neq 0, \qquad r_k x_k^{[1]} x_{k+1}^{[1]} \begin{cases} < 0, & k = 2m \\ > 0, & k = 2m-1, \ m \in \mathbb{N} \end{cases}$$

and

$$x_k^{[2]} \neq 0, \qquad r_k x_k^{[2]} x_{k+1}^{[2]} \begin{cases} > 0, & k = 2m \\ < 0, & k = 2m-1, \ m \in \mathbb{N}. \end{cases}$$

It means that the solution $x^{[1]}$ possesses a generalized zero at each even integer and $x^{[2]}$ at each odd integer.

Notice that by the definiton of a zero of a solution of (1.1) *in Hartman's sense $x^{[1]}$ has a "node" at every integer and $x^{[2]}$ has no "nodes".*

As a consequence of the previous definition, we introduce the following important concepts:

Definition 3.

- A solution x of (1.1) is said to be *oscillatory at* $+\infty$ if it has infinitely many generalized zeros in $\mathbb{N}$, i.e., if for infinitely many positive integer M an interval $(M, M+1]$ contains a generalized zero of x.

  The equation (1.1) is said to be *oscillatory at* $+\infty$ if every solution of (1.1) is oscillatory.

- A solution x of (1.1) is said to be *nonoscillatory at* $+\infty$ if there exists $M_0 \in \mathbb{N}$ such that for every integer $M > M_0$ an interval $(M, M+1]$ contains no generalized zero of x.

  The equation (1.1) is said to be *nonoscillatory at* $+\infty$ if every solution of (1.1) is nonoscillatory.

- In a similar way, the oscillation/nonoscillation at $-\infty$ is defined. By the oscillation of (1.1) in $\mathbb{Z}$ we understand oscillation of (1.1) at $+\infty$ and at $-\infty$.

Lemma 1. *(See [10, Lemma 2].) If (1.1) is nonoscillatory at* $+\infty$, *there exists a nontrivial solution x of (1.1) called recessive at $+\infty$ with the property*

$$\lim_{k \to +\infty} \frac{x_k}{\tilde{x}_k} = 0$$

for any other linearly independent solution $\tilde{x}$ of (1.1). In a similar way we define recessive solution at $-\infty$.

Definition 4. The nonoscillatory equation (1.1) is said to be *of finite type* m, $m \in \mathbb{N}$, if this equation possesses solutions with m generalized zeros in $\mathbb{Z}$ but not with $m+1$ generalized zeros.

Let (1.1) be of finite type m. If its recessive solutions at $-\infty$ and at $+\infty$ are linearly dependent and have $m-1$ generalized zeros in $\mathbb{Z}$, it is said to be *of special kind (m-special).*

Let (1.1) be of finite type m. If its recessive solutions at $-\infty$ and at $+\infty$ are linearly independent and possess $m-1$ generalized zeros on $\mathbb{Z}$, it is said to be *of general kind (m-general).*

Notation 1. In the following text, the interval $[M, N]$ will actually represent the discrete set $[M, N] \cap \mathbb{Z}$, $M, N \in \mathbb{Z}$, $N > M$.

Definition 5. Equation (1.1) is said to be *conjugate in an interval* $[M, N]$ if there exists a solution x of (1.1) which has at least two generalized zeros in $(M-1, N+1]$, i.e., if there exists a solution x and two intervals $(l-1, l]$, $(m, m+1]$, where $l \geq M$, $m \leq N$, such that

$$x_{l-1} \neq 0, \quad r_{l-1}x_{l-1}x_l \leq 0 \qquad \text{and} \qquad x_m \neq 0, \quad r_m x_m x_{m+1} \leq 0.$$

Equation (1.1) is said to be *disconjugate in* $[M, N]$ in the opposite case, i.e., if any solution of this equation has, at most, one generalized zero in $(M-1, N+1]$.

The following theorem is a discrete version of its continuous counterpart (see e.g. [27]). It provides two powerful tools for the investigation of oscillatory properties of (1.1), namely the Riccati technique and the variational principle. This theorem relates *discrete Riccati equation*

$$\Delta w_k + q_k + \frac{w_k^2}{r_k + w_k} = 0, \qquad k \in [M, N+1] \tag{2.9}$$

to the positivity of a *discrete quadratic functional*

$$\mathcal{F}_d(x; M, N) := \sum_{k=M}^{N} [r_k(\Delta x_k)^2 - q_k x_{k+1}^2] \tag{2.10}$$

in the class of nontrivial sequences $x = (x_k)_{k=M}^{N+1}$ satisfying $x_M = 0 = x_{N+1}$ and the *disconjugacy* of equation (1.1).

Using the definition of a generalized zero we can formulate the so-called Roundabout theorem, see e.g. [4, Theorem 1.41] or [13, Theorem 3.1]. This is the main statement of the oscillation theory of (1.1) and relates (1.1), (2.9) and (2.10).

Theorem A. (Roundabout Theorem) *The following statements are equivalent:*

(a) *Equation (1.1) is disconjugate on* $[M, N]$.

(b) *There exists a solution of (1.1) having no generalized zero in* $[M, N+1]$.

(c) *There exists a solution* w *of (2.9) which is defined for every* $k \in [M, N+1]$ *and satisfies* $r_k + w_k > 0$ *for* $k \in [M, N]$.

(d) *The discrete quadratic functional* $\mathcal{F}_d(x)$ *given by* (2.10) *is positive for every nontrivial* x *satisfying* $x_M = 0 = x_{N+1}$.

As a consequence of the Roundabout theorem we easily get Sturm comparsion theorem and Sturm separation theorem, see e.g. [4, Theorem 5.30].

Theorem B. (Sturm Separation Theorem) *Let* x *be a solution of* (1.1) *which contains consecutive generalized zeros in* $(M, M+1]$ *and* $(N, N+1]$*, where* $M, N \in \mathbb{Z}$, $N > M$*. Then every solution of* (1.1) *contains exactly one generalized zero in* $(M, N+1]$.

Remark 2. In a view of Theorem B all solutions of (1.1) have the same oscillatory character, that is, they all have either a finite or an infinite number of generalized zeros on $\mathbb{N}$. It means that Definition 3 can be specified in the following way: the equation (1.1) is said to be *oscillatory (of infinite type)* at $+\infty$ if there exists an oscillatory solution at $+\infty$ and it is said to be *nonoscillatory (of finite type)* if there exists a nonoscillatory solution of (1.1).

Remark 3. Note that equation (1.1) is nonoscillatory at $+\infty$ if and only if there exists an integer M such that (1.1) is disconjugate in $[M, +\infty)$.

Let us illustrate some of the previous concepts by the following examples:

Example 2.

(i) The equation

$$\Delta^2 x_k + x_{k+1} = 0$$

has two linearly independent solutions

$$x_k^{[1]} = \cos \frac{\pi}{3} k \qquad \text{and} \qquad x_k^{[2]} = \sin \frac{\pi}{3} k.$$

Note that the infinitely many intervals $(1+3m, 2+3m]$, $m \in \mathbb{Z}$, *contain a generalized zero of* $x^{[1]}$ *and each interval* $(-1+3m, 3m]$, $m \in \mathbb{Z}$, *contains a generalized zero of* $x^{[2]}$. *It implies that the equation is oscillatory and in addition, it is conjugate in* $[M, M+2]$ *for every integer* M.

(ii) The equation

$$\Delta^2 x_k = 0$$

has two linearly independent solutions

$$x_k^{[1]} = 1 \qquad \text{and} \qquad x_k^{[2]} = k.$$

The solution $x_k^{[1]}$ *is recessive at* $+\infty$ *and at* $-\infty$ *and has no generalized zeros on* $\mathbb{Z}$. *It means that this equation is of type* 1 *of special kind* (1*-special).*

(iii) The equation

$$\Delta^2 x_k - x_{k+1} = 0$$

has two linearly independent solutions

$$x_k^{[1]} = \left(\frac{3+\sqrt{5}}{2}\right)^k \qquad \text{and} \qquad x_k^{[2]} = \left(\frac{3-\sqrt{5}}{2}\right)^k.$$

The solution $x_k^{[1]}$ *is recessive at* $-\infty$ *and* $x_k^{[2]}$ *is recessive at* $+\infty$. *Both solutions have no generalized zero, so the equation is nonoscillatory, especially of type* 1 *of general kind* (1*-general).*

Using Theorem A, one can also obtain various oscillation criteria. One of best-known oscillation criterion proved with the variational method is the discrete analogue of Leighton-Wintner oscillation criterion (see e.g. [19, Chapter XI]).

Theorem C. *(See [13, Theorem 3.2].) Suppose that $r_k > 0$ for large $k \in \mathbb{Z}$ and*

$$\sum^{+\infty} \frac{1}{r_k} = +\infty, \qquad \sum^{+\infty} q_k = +\infty.$$

Then equation (1.1) is oscillatory at $+\infty$.

Now let us turn our attention to the self-reciprocity of equation (1.1). Recall that the equation (1.1) is said to be *self-reciprocal*, if it coincides with its reciprocal equation. The concept of the reciprocity of (1.1) is given in the following lemma.

Lemma 2. *(See [10, §2].) Let $x = (x_k)$ be a solution of (1.1), $r_k \neq 0$ and $q_k \neq 0$ for every $k \in \mathbb{Z}$. Then*

$$y = r\Delta x$$

is a solution of the equation

$$\Delta\left(\frac{1}{q_k}\Delta y_k\right) + \frac{1}{r_{k+1}} y_{k+1} = 0. \tag{2.11}$$

Equation (2.11) *is said to be a reciprocal equation to equation* (1.1)*.*

The form of reciprocal equation (2.11) shows that no difference equation (1.1) is self-reciprocal except the equation $\Delta^2 x_k + x_{k+1} = 0$.

Towards this goal we study equation (1.1) in the form of two first order difference equations called a 2×2 symplectic difference system.

2.3 2×2 symplectic difference system

In this section we recall the concept of 2×2 symplectic difference system. We point out its curicular role in so-called trigonometric transformation which we state in Theorem D.

Definition 6. A 2×2 difference system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \tag{S}$$

$k \in \mathbb{Z}$, where (a_k), (b_k), (c_k) and (d_k) are sequences of real numbers, is said to be *symplectic*, if the matrix $\mathcal{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is symplectic, i.e.,

$$\mathcal{S}^T \mathcal{I} \mathcal{S} = \mathcal{I}, \qquad \mathcal{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.12}$$

where ^{T} stands for the transpose of the matrix indicated. It is not difficult to verify that the 2×2 matrix $\mathcal{S}$ is symplectic if and only if $\det\mathcal{S} = 1$, i.e., $ad - bc = 1$.

Notice that there exists a relation between system (S) and equation (1.1). Equation (1.1) is a special case of symplectic system (S) because it can be viewed as a system of two equations of the first order for $(x,u)=(x,r\Delta x)$

$$\Delta x_k = \frac{1}{r_k}u_k, \qquad \Delta u_k = -q_k x_{k+1}$$

and, in turn, as a symplectic system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{r_k} \\ -q_k & 1-\frac{q_k}{r_k} \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}. \tag{2.13}$$

Definition 7.

- A pair of solutions $z^{[1]} = \binom{x_k}{u_k}^{[1]}$, $z^{[2]} = \binom{x_k}{u_k}^{[2]}$ of (S) with the Casoratian ω, where

  $$\omega \equiv x_k^{[1]}u_k^{[2]} - x_k^{[2]}u_k^{[1]},$$

  is said to be a *basis* of system (S) and we denote it $(z^{[1]}, z^{[2]})$. If $\omega \equiv 1$, it is said to be a *normalized basis.*

- The *reciprocal* system to (S) is the symplectic system with the matrix $\bar{\mathcal{S}} = \mathcal{I}^{-1}\mathcal{S}\mathcal{I}$, i.e., the system

  $$\begin{pmatrix} \bar{x}_{k+1} \\ \bar{u}_{k+1} \end{pmatrix} = \begin{pmatrix} d_k & -c_k \\ -b_k & a_k \end{pmatrix} \begin{pmatrix} \bar{x}_k \\ \bar{u}_k \end{pmatrix}. \tag{S^r}$$

- System (S) is said to be *self-reciprocal* if it coincides with its reciprocal system. Any system of the form

  $$\begin{pmatrix} s_{k+1} \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} p_k & q_k \\ -q_k & p_k \end{pmatrix} \begin{pmatrix} s_k \\ c_k \end{pmatrix}, \tag{S_T}$$

  where $p_k^2+q_k^2=1$ is called a *trigonometric system.* It is easy to see that any symplectic self-reciprocal system (S) takes the form (S_T) and vice versa.

Remark 4. If $\binom{x_k}{u_k}$ is a solution of (S), then $\binom{u_k}{-x_k}$ is a solution of its reciprocal system (S^r).

Let us formulate a definition of a generalized zero of a solution of (S). This term makes possible to describe the oscillation of (S).

Definition 8. An interval $(M, M+1]$, $M \in \mathbb{Z}$, is said to contain a *generalized zero* of a solution $\binom{x}{u}$ of system (S) if

$$x_M \neq 0 \qquad \text{and} \qquad b_M x_M x_{M+1} \leq 0.$$

Oscillation, nonoscillation, classification of nonoscillatory system (S), conjugacy, disconjugacy and recessive solution of (S) we establish in the same way as for (1.1) in Definition 3.

As we claimed above, there is no self-reciprocal equation (1.1) except $\Delta^2 x_k + x_{k+1} = 0$. Thus we study self-reciprocal symplectic systems. The next theorem shows that every symplectic system (S) can be transformed by a transformation preserving oscillatory behavior into a trigonometric system (S_T) which is self-reciprocal.

Theorem D. *(See [6, Theorem 3.1].) Let* $\binom{x_k}{u_k}^{[1]}$, $\binom{x_k}{u_k}^{[2]}$, $k \in \mathbb{Z}$, *be solutions of (S) which form a normalized basis. Let*

$$\mathcal{R}_k = \begin{pmatrix} h_k & 0 \\ g_k & \frac{1}{h_k} \end{pmatrix}, \tag{2.14}$$

where

$$h_k^2 = (x_k^{[1]})^2 + (x_k^{[2]})^2, \qquad g_k = \frac{x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]}}{h_k}. \tag{2.15}$$

Then the trigonometric transformation

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \mathcal{R}_k \begin{pmatrix} s_k \\ c_k \end{pmatrix} \tag{2.16}$$

transforms (S) into the trigonometric system

$$\begin{pmatrix} s_{k+1} \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} p_k & q_k \\ -q_k & p_k \end{pmatrix} \begin{pmatrix} s_k \\ c_k \end{pmatrix}, \tag{S_T}$$

where $p_k^2 + q_k^2 = 1$ *and*

$$p_k = \frac{1}{h_{k+1}} (a_k h_k + b_k g_k), \qquad q_k = \frac{1}{h_k h_{k+1}} b_k \tag{2.17}$$

without changing the oscillatory behavior.

The sequence h satisfying (2.15) *can be chosen in such a way that* $q_k \geq 0$ *and if in addition* $b_k \neq 0$, *then it can be chosen in such a way that* $q_k > 0$ *for* $k \in \mathbb{Z}$.

Theorem D has been formulated by O. Došlý and M. Bohner in [6] for $2n \times 2n$ symplectic systems. The fact that we consider a scalar 2×2 symplectic systems enables to prove Theorem D by a different, simpler way than in [6]. We give a sketch of the proof; the details can be found in the proof of Theorem 1 in [15].

Proof of Theorem D. First we show that the transformation

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \begin{pmatrix} h_k & 0 \\ g_k & \frac{1}{h_k} \end{pmatrix} \begin{pmatrix} y_k \\ z_k \end{pmatrix} \tag{2.18}$$

transforms system (S) into the system

$$\begin{pmatrix} y_{k+1} \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{a}_k & \tilde{b}_k \\ \tilde{c}_k & \tilde{d}_k \end{pmatrix} \begin{pmatrix} y_k \\ z_k \end{pmatrix} \tag{2.19}$$

with the sequences $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ given by the formulas

$$
\begin{aligned}
\tilde{a}_k &= \frac{a_k h_k + b_k g_k}{h_{k+1}}, \\
\tilde{b}_k &= \frac{b_k}{h_k h_{k+1}}, \\
\tilde{c}_k &= -g_{k+1}(a_k h_k + b_k g_k) + h_{k+1}(c_k h_k + d_k g_k), \\
\tilde{d}_k &= \frac{-g_{k+1} b_k + h_{k+1} d_k}{h_k}.
\end{aligned}
$$

Further, we show that system (2.19) is symplectic, i.e., we prove by means of (2.15) and (S) by a direct computation the identities

$$\tilde{a} = \tilde{d} \qquad \text{and} \qquad \tilde{b} = -\tilde{c}.$$

Finally, since $h_k \neq 0$, the transformation (2.18) preserves generalized zeros of (S) and (2.19).

Lemma 3. *(See [10, Lemma 1, Lemma 6].) Let (S_T) be a trigonometric system associated to (S). Then there exists a sequence $\varphi = (\varphi_k)$, $k \in \mathbb{Z}$, such that $\varphi_k \in [0, 2\pi)$ for every k and satisfies*

$$\cos\varphi_k = p_k, \qquad \sin\varphi_k = q_k. \tag{2.20}$$

Further,

$$\begin{pmatrix} s_k \\ c_k \end{pmatrix}^{[1]} = \begin{pmatrix} \cos\left(\sum^{k-1} \varphi_j\right) \\ -\sin\left(\sum^{k-1} \varphi_j\right) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} s_k \\ c_k \end{pmatrix}^{[2]} = \begin{pmatrix} \sin\left(\sum^{k-1} \varphi_j\right) \\ \cos\left(\sum^{k-1} \varphi_j\right) \end{pmatrix} \tag{2.21}$$

form the normalized basis of (S_T).

Chapter 3

Phases of Symplectic Difference Systems and Their Properties

The aim of this chapter is to introduce a phase theory for a symplectic system (S). We start this chapter with a short summary of the establishing of phases for the second order linear differential equations by O. Borůvka in [7]. Then we define the first and second phases for system (S) and we study the relations between them. Some direct consequences of the phase theory to oscillatory properties of system (S) are pointed out as well.

3.1 Phases of Sturm-Liouville differential equations

Consider Sturm-Liouville differential equation (2.1)

$$(r(t)x')' + q(t)x = 0, \qquad r(t) > 0,$$

where $t \in I = (a, b)$, $-\infty \leq a < b \leq \infty$.

The first phase of (2.1) is defined in the following way, see [7, §5.3]: By *first phase of the basis* (x_1, x_2) of the equation (2.1) with the Wronskian $w \equiv r(x_1 x_2' - x_1' x_2)$ we mean any function $\alpha \in C(I)$ which satisfies in this interval the relation

$$\tan \alpha(t) = \frac{x_1(t)}{x_2(t)} \tag{3.1}$$

except at the zeros of x_2. From the fact

$$\left(\frac{x_1}{x_2}\right)' = -\frac{w}{x_2^2}$$

it follows that $\alpha(t)$ of the basis (x_1, x_2) is increasing/decreasing accordingly as w is negative/positive.

By *first phase of the equation* (2.1), we mean the first phase of an arbitrary basis of the equation (2.1).

By *second phase of equation* (2.1) we understand the first phase of its reciprocal equation (2.2) and we denote it by β. For a self-reciprocal equation the first and second phases coincide.

Using the fact that if x is a solution of (2.1), then $y = rx'$ is a solution of (2.2), we have for the second phase β of (2.1)

$$\tan\beta(t) = \frac{rx_1'(t)}{rx_2'(t)} = \frac{x_1'(t)}{x_2'(t)} \tag{3.2}$$

for $t \in I$ except at the zeros of x_2'.

3.2 First phase of symplectic difference system

In this section we present a discrete counterpart of the first phase of (2.1). We define the first phase of system (S) and its connection to trigonometric transformation of (S) given in Theorem D. These results already were studied in [10].

Definition 9. Let $z^{[1]} = \left(\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right)^{[1]}$, $z^{[2]} = \left(\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right)^{[2]}$ form a normalized basis of system (S). By the *first phase of the basis* $(z^{[1]}, z^{[2]})$ we understand any sequence $\psi = (\psi_k)$, $k \in \mathbb{Z}$, such that

$$\triangle\psi_k \in [0, \pi)$$

and

$$\psi_k = \begin{cases} \arctan \frac{x_k^{[2]}}{x_k^{[1]}} & \text{if } x_k^{[1]} \neq 0, \\ \text{odd multiple of } \frac{\pi}{2} & \text{if } x_k^{[1]} = 0. \end{cases} \tag{3.3}$$

The *first phase of system* (S) is any first phase of a normalized basis of this system.

Remark 5. Obviously, if ψ is the first phase of the basis $(z^{[1]}, z^{[2]})$ then $\psi + k\pi$, $k \in \mathbb{Z}$, is the first phase of this basis as well. Conversely, if $\psi^{[1]}$, $\psi^{[2]}$ are two first phases of the basis $(z^{[1]}, z^{[2]})$ then $\psi^{[1]} - \psi^{[2]} \equiv 0 \pmod{\pi}$. Note that in Definition 9 a first phase of (S) is introduced as a nondecreasing sequence. In addition, contary to papers [9, 10, 28] it turned out more convenient to define a first phase of (S) just by the relation (3.3).

Notation 2. In the paper [10] a *phase* is understood as a first phase. Here we will also use a phase if it is evident that we mean the first phase.

Notation 3. By the functions arctan and arccot we understand multivalued functions. The function Arctan will denote a particular branch of the function arctan with the image $(-\pi/2, \pi/2)$ and by the function Arccot will represent the particular branch of arccot with the image $(0, \pi)$, respectively.

In the following theorem we give a relation for a difference of a first phase ψ of system (S). The proof is based on the trigonometric transformation of (S) explained in Theorem D and in Lemma 3.

Theorem E. *(See [10, Theorem 1].) Let* $z^{[1]} = \binom{x_k}{u_k}^{[1]}$ *and* $z^{[2]} = \binom{x_k}{u_k}^{[2]}$ *form the normalized basis of (S), let* ψ *be the first phase of this basis and* (S_T) *trigonometric system associated to (S) with* p, q *satisfying (2.17) and* $q_k \geq 0$, $k \in \mathbb{Z}$. *Then*

$$\sin \Delta\psi_k = q_k, \qquad \cos \Delta\psi_k = p_k,$$

that is,

$$\Delta\psi_k = \begin{cases} \mathit{Arccot}\,\frac{p_k}{q_k} & \text{if } q_k \neq 0, \\ 0 & \text{if } q_k = 0 \end{cases} \tag{3.4}$$

for $k \in \mathbb{Z}$.

Lemma 4. *Let a sequence* (φ_k) *be defined by* (2.20) *and* (ψ_k) *be the first phase of (S). Then*

$$\varphi_k \equiv \Delta\psi_k \pmod{\pi}, \qquad k \in \mathbb{Z}, \tag{3.5}$$

and φ_k *can be chosen from the interval* $[0, \pi)$.

Proof. Let (p_k), (q_k) be sequences given by (2.17), let (φ_k) be a sequence from Lemma 3 and let (ψ_k) be the first phase of system (S) given in Definition 9, $k \in \mathbb{Z}$. According to Lemma 3

$$\frac{p_k}{q_k} = \frac{\cos \varphi_k}{\sin \varphi_k}$$

and by Theorem E

$$\frac{p_k}{q_k} = \frac{\cos \Delta\psi_k}{\sin \Delta\psi_k}.$$

Hence

$$\cot \varphi_k = \cot \Delta\psi_k,$$

which implies (3.5), i.e, $\varphi_k = \Delta\psi_k + m\pi$, where m is an arbitrary integer. According to Definition 9, $\Delta\psi_k \in [0, \pi)$ and taking $m = 0$ we have $\varphi_k \in [0, \pi)$. □

Lemma 5. *Let* $(x_k^{[1]})$ *and* $(x_k^{[2]})$ *be the first components of a normalized basis of system (S), let* (ψ_k) *be the first phase of this basis of (S) and let* (h_k) *be a sequence given by* (2.15). *Then*

$$x_k^{[1]} = h_k \cos \psi_k \qquad \text{and} \qquad x_k^{[2]} = h_k \sin \psi_k. \tag{3.6}$$

Proof. According to Lemma 3 and Lemma 4 $x_k^{[1]}$ and $x_k^{[2]}$ takes the form

$$x_k^{[1]} = h_k \cos\left(\sum^{k-1} \varphi_j\right) \qquad \text{and} \qquad x_k^{[2]} = h_k \sin\left(\sum^{k-1} \varphi_j\right)$$

and so in view of (3.5) we get (3.6). □

Remark 6. Geometrical meaning of a first phase of (1.1) with $r_k > 0$ is similar to a continuous case for (2.1). Let $(x^{[1]}, x^{[2]})$ be a normalized basis of (1.1). An interval $(M, M+1]$ contains no generalized zeros of the solution $x^{[1]}$ if and only if

$$(2n-1)\frac{\pi}{2} < \psi_k < (2n+1)\frac{\pi}{2}, \quad M \le k \le M+1,$$

and it contains no generalized zeros of the solution $x^{[2]}$ if and only if

$$n\pi < \psi_k < (n+1)\pi, \quad M \le k \le M+1,$$

for some $n \in \mathbb{Z}$. In other words, if $\psi_M < (2n+1)\frac{\pi}{2}$ and $\psi_{M+1} \ge (2n+1)\frac{\pi}{2}$, then $x^{[1]}$ has a generalized zero in $(M, M+1]$ and if $\psi_M < n\pi$ and $\psi_{M+1} \ge n\pi$, then $x^{[2]}$ has a generalized zero in $(M, M+1]$. From this reason the first phase is sometimes called the *zero-counting sequence.*

3.3 Second phase of symplectic difference system

In this section we establish the second phase of system (S). We also study its properties and its connection to the first phase of system (S).

Definition 10. By the *second phase* of system (S) we understand the first phase of the reciprocal system (S^r) to (S). We denote it by ϱ.

Proposition 1. *Let $z^{[1]} = \binom{x_k}{u_k}^{[1]}$, $z^{[2]} = \binom{x_k}{u_k}^{[2]}$ form a normalized basis of system (S) and $\varrho = (\varrho_k)$ be the second phase of (S). Then*

$$\Delta\varrho_k \in [0, \pi)$$

and

$$\varrho_k = \begin{cases} \arctan \frac{u_k^{[2]}}{u_k^{[1]}} & \text{if } u_k^{[1]} \ne 0, \\ \text{odd multiple of } \frac{\pi}{2} & \text{if } u_k^{[1]} = 0. \end{cases} \tag{3.7}$$

Proof. According to Remark 4 sequences $\binom{u_k}{-x_k}^{[1]}$ and $\binom{u_k}{-x_k}^{[2]}$ form a normalized basis of (S^r). Thus, by Definition 9 the first phase ϱ of system (S^r), i.e., the second phase of (S), satisfies $\Delta\varrho_k \in [0, \pi)$ and (3.7). □

Proposition 2. *Let $z^{[1]} = \binom{x_k}{u_k}^{[1]}$, $z^{[2]} = \binom{x_k}{u_k}^{[2]}$ form a normalized basis of (S) and let*

$$\bar{h}_k^2 = \big(u_k^{[1]}\big)^2 + \big(u_k^{[2]}\big)^2, \quad \bar{g}_k = -\frac{x_k^{[1]}u_k^{[1]} + x_k^{[2]}u_k^{[2]}}{\bar{h}_k}. \tag{3.8}$$

Let

$$\bar{\mathcal{R}}_k = \begin{pmatrix} \bar{h}_k & 0 \\ \bar{g}_k & \frac{1}{\bar{h}_k} \end{pmatrix}. \tag{3.9}$$

Then the transformation

$$\begin{pmatrix} \bar{x}_k \\ \bar{u}_k \end{pmatrix} = \bar{\mathcal{R}}_k \begin{pmatrix} \bar{s}_k \\ \bar{c}_k \end{pmatrix} \tag{3.10}$$

transforms system (S^r) into the trigonometric reciprocal system

$$\begin{pmatrix} \bar{s}_{k+1} \\ \bar{c}_{k+1} \end{pmatrix} = \begin{pmatrix} \bar{p}_k & \bar{q}_k \\ -\bar{q}_k & \bar{p}_k \end{pmatrix} \begin{pmatrix} \bar{s}_k \\ \bar{c}_k \end{pmatrix}, \tag{S_T^r}$$

where $\bar{p}_k^2 + \bar{q}_k^2 = 1$ *and*

$$\bar{p}_k = \frac{1}{\bar{h}_{k+1}}(d_k \bar{h}_k - c_k \bar{g}_k), \quad \bar{q}_k = -\frac{1}{\bar{h}_k \bar{h}_{k+1}} c_k, \tag{3.11}$$

without changing the oscillatory behavior. The sequence $(\bar{h}_k)$ *satisfying (3.8) can be chosen in such a way that* $\bar{q}_k \geq 0$ *and if in addition* $c_k \neq 0$, *then it can be chosen in such a way that* $\bar{q}_k > 0$ *for* $k \in \mathbb{Z}$.

Further, let ϱ *be the second phase of the basis* $(z^{[1]}, z^{[2]})$ *of (S). Then*

$$\Delta \varrho_k = \begin{cases} \mathit{Arccot}\, \frac{\bar{p}_k}{\bar{q}_k} & \text{if } \bar{q}_k \neq 0, \\ 0 & \text{if } \bar{q}_k = 0 \end{cases} \tag{3.12}$$

for $k \in \mathbb{Z}$.

Proof. Let $(z^{[1]}, z^{[2]})$ be a normalized basis of (S). Then by Remark 4 $\bar{z}^{[1]} = \left(\begin{smallmatrix} u_k \\ -x_k \end{smallmatrix}\right)^{[1]}$, $\bar{z}^{[2]} = \left(\begin{smallmatrix} u_k \\ -x_k \end{smallmatrix}\right)^{[2]}$ form a normalized basis of (S^r). Further, in view of Theorem D, (3.10) transforms the reciprocal system (S^r) into the trigonometric reciprocal system (S_T^r) without changing the oscillatory behavior. By Proposition 1 and Theorem E we give (3.12). ☐

We conclude the above-explained relations between systems (S), (S^r), (S_T) and (S_T^r) with a graphic schema.

Diagram 1.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(\mathrm{S}) \arrow[r, "{\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \mathcal{I} \begin{pmatrix} \bar{x}_k \\ \bar{u}_k \end{pmatrix}}", "\textit{Definition 7}"'] \arrow[d, swap, "{\begin{array}{c} \begin{pmatrix} x_k \\ u_k \end{pmatrix} = \mathcal{R} \begin{pmatrix} s_k \\ c_k \end{pmatrix} \\ \textit{Theorem D} \end{array}}"] & (\mathrm{S}^r) \arrow[d, "{\begin{array}{c} \begin{pmatrix} \bar{x}_k \\ \bar{u}_k \end{pmatrix} = \bar{\mathcal{R}} \begin{pmatrix} \bar{s}_k \\ \bar{c}_k \end{pmatrix} \\ \textit{Proposition 2} \end{array}}"] \\
(\mathrm{S}_T) & (\mathrm{S}_T^r) \\[-2em]
\textit{1-st phase} & \textit{2-nd phase}
\end{tikzcd}$$

The matrices $\mathcal{I}$, $\mathcal{R}$ and $\bar{\mathcal{R}}$ are given by the relations (2.12), (2.14) and (3.9).

In the next theorem we study the relation between first and second phases of system (S), which is a discrete analogy of Borůvka's result (see [7, p. 43]).

Theorem 1. *Let $b_k > 0$, $c_k < 0$, $\psi = (\psi_k)$ be the first phase of system (S) and $\varrho = (\varrho_k)$ the second phase of system (S). Then there exists $n \in \mathbb{N} \cup \{0\}$ with the property*

$$n\pi < |\, \varrho_k - \psi_k \,| < (n+1)\pi, \quad k \in \mathbb{Z}. \tag{3.13}$$

Proof. Let $z^{[1]} = \left(\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right)^{[1]}$ and $z^{[2]} = \left(\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right)^{[2]}$ be two solutions of (S) which form the normalized basis of system (S). By Lemma 5

$$x_k^{[1]} = h_k \cos \psi_k, \quad x_k^{[2]} = h_k \sin \psi_k$$

where $h_k^2 = \left(x_k^{[1]}\right)^2 + \left(x_k^{[2]}\right)^2$, $k \in \mathbb{Z}$. Since by Remark 4 $\left(\begin{smallmatrix} u_k \\ -x_k \end{smallmatrix}\right)^{[1]}$ and $\left(\begin{smallmatrix} u_k \\ -x_k \end{smallmatrix}\right)^{[2]}$ form the normalized basis of system (S^r) with the first phase ϱ, i.e., the second phase of (S), then

$$u_k^{[1]} = \bar{h}_k \cos \varrho_k, \quad u_k^{[2]} = \bar{h}_k \sin \varrho_k$$

where $\bar{h}_k^2 = \left(u_k^{[1]}\right)^2 + \left(u_k^{[2]}\right)^2$, $k \in \mathbb{Z}$. By a direct computation we have

$$\omega = h_k \cos \psi_k \bar{h}_k \sin \varrho_k - h_k \sin \psi_k \bar{h}_k \cos \varrho_k =$$

$$= h_k \bar{h}_k (\cos \psi_k \sin \varrho_k - \sin \psi_k \cos \varrho_k) = h_k \bar{h}_k \sin(\varrho_k - \psi_k).$$

Using the fact that $z^{[1]}$ and $z^{[2]}$ form a normalized basis of (S), i.e., $\omega(z^{[1]}, z^{[2]}) \equiv 1$, we obtain for $k \in \mathbb{Z}$

$$\sin(\varrho_k - \psi_k) \begin{cases} > 0 & \text{if} \quad h_k \bar{h}_k > 0, \\ < 0 & \text{if} \quad h_k \bar{h}_k < 0, \end{cases}$$

i.e., there exists $m \in \mathbb{Z}$ such that

$$2m\pi < \ \varrho_k - \psi_k \ < (2m+1)\pi.$$

This means that there exists $n \in \mathbb{N} \cup \{0\}$ with the property (3.13). $\square$

3.4 Oscillation and nonoscillation in terms of phases

In this section we state the relationship between phases and oscillatory properties of (S).

Theorem 2. *The following statements are equivalent:*

(a) *(S) is oscillatory in $\mathbb{Z}$.*

(b) *(S^r) is oscillatory in $\mathbb{Z}$.*

(c) *(S_T) is oscillatory in $\mathbb{Z}$.*

(d) *(S_T^r) is oscillatory in $\mathbb{Z}$.*

(e) *The first phase ψ of (S) satisfies* $\lim_{k\to+\infty} \psi_k = +\infty$ *and* $\lim_{k\to-\infty} \psi_k = -\infty$.

(f) *The second phase ϱ of (S) satisfies* $\lim_{k\to+\infty} \varrho_k = +\infty$ *and* $\lim_{k\to-\infty} \varrho_k = -\infty$.

(g) *There exist* $k_0 \in \mathbb{N}$ *and* $-l_0 \in \mathbb{N}$ *such that*

$$\sum_{k \geq k_0,\, q_k > 0}^{\infty} \operatorname{Arccot} \frac{p_k}{q_k} = +\infty, \qquad \sum_{-\infty}^{k \leq l_0,\, q_k > 0} \operatorname{Arccot} \frac{p_k}{q_k} = -\infty,$$

where p, q *are given by (2.17).*

Proof. The oscillation in $\mathbb{Z}$ is defined as oscillation at $+\infty$ and $-\infty$, see Definition 3. We prove this statement for the oscillation at $+\infty$, by the similar argument the oscillation at $-\infty$ follows.

"(a) $\Leftrightarrow$ (e) $\Leftrightarrow$ (g)": See [10, Theorem 3].

"(a) $\Leftrightarrow$ (c)": It follows from Theorem D.

"(b) $\Leftrightarrow$ (d)": It follows from Proposition 2.

"(e) $\Rightarrow$ (f)": Suppose that $\varrho_k > \psi_k$, $k \in \mathbb{Z}$. (In the opposite case the proof is analogous.) Let $\lim_{k \to +\infty} \psi_k = +\infty$. By Theorem 1 we have $n\pi + \psi_k < \varrho_k$ for $n \in \mathbb{N} \cup \{0\}$ and by our assumption $\lim_{k \to +\infty}(n\pi + \psi_k) = +\infty$. This means that $\lim_{k \to +\infty} \varrho_k = +\infty$.

"(f) $\Rightarrow$ (e)": Suppose that $\varrho_k > \psi_k$, $k \in \mathbb{Z}$. (In the opposite case the proof is analogous.) Let $\lim_{k \to +\infty} \varrho_k = +\infty$. By Theorem 1 we have $\varrho_k - (n+1)\pi < \psi_k$ for $n \in \mathbb{N}$ and taking into account our assumption $\lim_{k \to +\infty}(\varrho_k - (n+1)\pi) = +\infty$. From here it follows that $\lim_{k \to +\infty} \psi_k = +\infty$.

"(f) $\Leftrightarrow$ (b)": It follows from [10, Theorem 3] and from the fact that ϱ is the first phase of (S^r).

□

Corollary 1. *The following statements are equivalent:*

(a) *System (S) is of finite type.*

(b) *The first phase* ψ *of system (S) is bounded on* $\mathbb{Z}$.

(c) *The second phase* ϱ *of (S) is bounded on* $\mathbb{Z}$.

Proof. It directly follows from Theorem 2, part (a), (e) and (f). □

The following theorem is given in [10, Theorem 4] and describes nonoscillatory properties of symplectic system (S).

Theorem F. *The following statements are equivalent:*

(a) *Symplectic system (S) is of type* m *and of special kind on* $\mathbb{Z}$.

(b) *Any trigonometric system* (S_T) *associated to system (S) is of type* m *and of special kind on* $\mathbb{Z}$.

(c) *Recessive solutions of (S) at* $\pm\infty$ *are linearly dependent and possess* $m-1$ *generalized zeros.*

(d) *Any first phase ψ of system (S) satisfies*

$$\sum_{k=-\infty}^{+\infty} \Delta\psi_k = m\pi.$$

(e) *Any trigonometric system (S_T) associated to system (S) satisfies*

$$\sum_{k=-\infty,\ q_k>0}^{+\infty} \mathit{Arccot}\, \frac{p_k}{q_k} = m\pi.$$

We now illustrate the above-mentioned results by a few examples.

Example 3. *Consider the recurrence relation*

$$x_{k+2} + x_k = 0$$

which can be written as the Sturm-Liouville equation

$$\Delta^2 x_k + 2x_{k+1} = 0,$$

or as the 2×2 *symplectic system*

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \qquad k \in \mathbb{N}. \tag{3.14}$$

This system has two solutions $\left(\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right)^{[1]}$ *and* $\left(\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right)^{[2]}$, *where*

$$\begin{aligned} x_k^{[1]} &= \cos k\frac{\pi}{2} = (0, -1, 0, 1, \dots), \\ u_k^{[1]} &= r_k \Delta x_k^{[1]} = -\cos k\frac{\pi}{2} - \sin k\frac{\pi}{2} = (-1, 1, 1, -1, \dots), \\ x_k^{[2]} &= \sin k\frac{\pi}{2} = (1, 0, -1, 0, \dots), \\ u_k^{[2]} &= r_k \Delta x_k^{[2]} = \cos k\frac{\pi}{2} - \sin k\frac{\pi}{2} = (-1, -1, 1, 1, \dots), \end{aligned}$$

which form a normalized basis.

Using Definition 9 and Proposition 1 the first phase ψ and the second phase ϱ satisfy

$$\psi_k = k\frac{\pi}{2}, \qquad \varrho_k = \frac{\pi}{4} + k\frac{\pi}{2}, \qquad k \in \mathbb{N},$$

respectively. Obviously,

$$\lim_{k\to+\infty} \psi_k = +\infty \qquad \text{and} \qquad \lim_{k\to+\infty} \varrho_k = +\infty,$$

i.e., according to Theorem 2 system (3.14) *is oscillatory at* $+\infty$. *Further,*

$$|\varrho_k - \psi_k| = \frac{\pi}{4}$$

for all $k \in \mathbb{N}$. *This fact illustrates the validity of Theorem 1.*

Example 4. *The recurrence relation*

$$x_{k+2} + 2x_{k+1} + x_k = 0$$

can be viewed as the Sturm-Liouville equation

$$\Delta^2 x_k + 4x_{k+1} = 0$$

or as the symplectic system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \qquad k \in \mathbb{Z}. \tag{3.15}$$

A normalized basis of this system form two solutions $\left(\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right)^{[1]}$ *and* $\left(\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right)^{[2]}$, *where*

$$\begin{array}{rclcrcl} x_k^{[1]} & = & k(-1)^k, & \qquad & x_k^{[2]} & = & (-1)^k, \\ u_k^{[1]} & = & (2k+1)(-1)^{k+1}, & \qquad & u_k^{[2]} & = & 2(-1)^{k+1}. \end{array}$$

The first phase ψ *satisfies*

$$\psi_k = \begin{cases} \arctan \frac{1}{k} & \textit{if} \quad k \in \mathbb{Z} - \{0\}, \\ \frac{\pi}{2} & \textit{if} \quad k = 0 \end{cases} \qquad \textit{and} \qquad \Delta\psi_k \in [0, \pi),$$

i.e.,

$$\psi_k = \begin{cases} \mathit{Arctan}\ \frac{1}{k} + k\pi & \textit{if} \quad k \in \mathbb{Z}_+, \\ \frac{\pi}{2} & \textit{if} \quad k = 0, \\ \mathit{Arctan}\ \frac{1}{k} + (k+1)\pi & \textit{if} \quad k \in \mathbb{Z}_-. \end{cases}$$

The second phase ϱ *satisfies*

$$\varrho_k = \arctan \frac{2}{2k+1} \qquad \textit{with} \qquad \Delta\varrho_k \in [0, \pi),$$

that is,

$$\varrho_k = \begin{cases} \mathit{Arctan}\ \frac{2}{2k+1} + k\pi & \textit{if} \quad k \in \mathbb{Z}_+ \cup \{0\}, \\ \mathit{Arctan}\ \frac{2}{2k+1} + (k+1)\pi & \textit{if} \quad k \in \mathbb{Z}_-. \end{cases}$$

Obviously,

$$\lim_{k\to+\infty} \psi_k = +\infty, \quad \lim_{k\to-\infty} \psi_k = -\infty \qquad \textit{and} \qquad \lim_{k\to+\infty} \varrho_k = +\infty, \quad \lim_{k\to-\infty} \varrho_k = -\infty$$

i.e., according to Theorem 2, system (3.15) *is oscillatory in* $\mathbb{Z}$. *Further,*

$$\varrho_k - \psi_k = \begin{cases} \mathit{Arctan}\ \frac{2}{2k+1} - \mathit{Arctan}\ \frac{1}{k} & \textit{if} \quad k \in \mathbb{Z} - \{0\}, \\ \mathit{Arctan}\ 2 - \frac{\pi}{2} & \textit{if} \quad k = 0. \end{cases}$$

Hence $0 < |\varrho_k - \psi_k| < \pi$, *which illustrates the validity of Theorem 1.*

Example 5. *The recurrence relation*

$$x_{k+2} - 2x_{k+1} + x_k = 0,$$

i.e., the Sturm-Liouville equation

$$\Delta^2 x_k = 0$$

can be viewed as the symplectic system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \qquad k \in \mathbb{Z}. \tag{3.16}$$

This system has two solutions $\left(\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right)^{[1]}$ *and* $\left(\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right)^{[2]}$ *with*

$$\begin{array}{rclcrcl} x_k^{[1]} & = & 1, & \qquad & x_k^{[2]} & = & k, \\ u_k^{[1]} & = & 0, & \qquad & u_k^{[2]} & = & 1, \end{array}$$

which form a normalized basis. The first and second phases

$$\psi_k = \arctan k \qquad \textit{and} \qquad \varrho_k = \frac{\pi}{2}$$

satisfy $0 < |\varrho_k - \psi_k| < \pi$ *and in addition, they are bounded on* $\mathbb{Z}$. *This fact agrees with Corollary 1; i.e., system* (3.16) *is of finite type on* $\mathbb{Z}$.

Open problems.

- ◇ In a continuous case there exists another relation between the first phase α and second phase β of equation (2.1) with $r(t) \equiv 1$ on I. In [7, p.45] the following equivalence is described: Two sufficiently smooth functions α and β defined on I represent a first and second phase of equation (2.1) if and only if

$$\beta = \alpha + \text{Arccot}\, \frac{1}{2} \left(\frac{1}{\alpha} \right)'.$$

  It remains an open problem if there exists a discrete analogue of this theorem for symplectic system (S) or at least for equation (1.1) for such $k \in \mathbb{Z}$ that satisfy $r_k \equiv 1$.

- ◇ Other ways of the investigation could consist in finding applications of a definition and properties of the second phase of system (S). Further, one could investigate *hyperbolic phases* of system (S) as a discrete analogue of hyperbolic phases of equation (2.1) studied in [22, 23]. By means of hyperbolic phases of (S), asymptotic properties of (S) could be described.

Chapter 4

Phases of Sturm-Liouville Difference Equations with Nonzero Casoratian

In this chapter we extend a definition of a first phase of (1.1). Using this, we study some properties and applications of a first phase, especially a construction of second order linear difference equations.

4.1 Extension of phases for difference equations with nonzero Casoratian

First, we establish a first phase of any basis with an arbitrary nonzero Casoratian of equation (1.1). In the next lemma we continue by a trigonometric transformation of (1.1) for an arbitrary basis of (1.1). Finally, by means of that lemma we express the difference of the first phase of an arbitrary basis of (1.1).

Definition 11. By the *first phase of the basis* $(x^{[1]}, x^{[2]})$ of (1.1) with Casoratian ω we mean any sequence $\psi = (\psi_k)$, $k \in \mathbb{Z}$, such that

$$\psi_k = \begin{cases} \arctan \frac{x_k^{[2]}}{x_k^{[1]}} & \text{if } x_k^{[1]} \neq 0, \\ \text{odd multiple of } \frac{\pi}{2} & \text{if } x_k^{[1]} = 0, \end{cases}$$

where

$$\Delta\psi_k \in \begin{cases} [0, \pi) & \text{if } \omega > 0, \\ (-\pi, 0] & \text{if } \omega < 0. \end{cases}$$

Lemma 6. *Let* $(x^{[1]}, x^{[2]})$ *be the basis of* (1.1) *with the Casoratian* ω *and let*

$$h_k^2 = (x_k^{[1]})^2 + (x_k^{[2]})^2, \quad \text{and} \quad g_k = \frac{r_k(x_k^{[1]}\Delta x_k^{[1]} + x_k^{[2]}\Delta x_k^{[2]})}{h_k}. \tag{4.1}$$

Then there exists a solution $\binom{s_k}{c_k}$ of the system (S_T) with p, q satisfying $p_k^2 + q_k^2 = 1$ such that

$$\begin{pmatrix} x_k^{[1]} \\ r_k \Delta x_k^{[1]} \end{pmatrix} = \begin{pmatrix} h_k & 0 \\ g_k & \frac{\omega}{h_k} \end{pmatrix} \begin{pmatrix} c_k \\ -s_k \end{pmatrix}, \quad \begin{pmatrix} x_k^{[2]} \\ r_k \Delta x_k^{[2]} \end{pmatrix} = \begin{pmatrix} h_k & 0 \\ g_k & \frac{\omega}{h_k} \end{pmatrix} \begin{pmatrix} s_k \\ c_k \end{pmatrix}. \tag{4.2}$$

Moreover, p, q in (S_T) are given by relations

$$p_k = \frac{x_k^{[1]} x_{k+1}^{[1]} + x_k^{[2]} x_{k+1}^{[2]}}{h_k h_{k+1}}, \quad q_k = \frac{\omega}{r_k h_k h_{k+1}}. \tag{4.3}$$

Proof. Consider such sequences $s = (s_k)$ and $c = (c_k)$ that

$$s_k = h_k^{-1} x_k^{[2]}, \quad c_k = h_k^{-1} x_k^{[1]}. \tag{4.4}$$

First we show that $\binom{s}{c}$ is a solution of (S_T). Denote

$$z_k = c_k + i s_k = \frac{x_k^{[1]} + i x_k^{[2]}}{h_k}. \tag{4.5}$$

Then

$$z_{k+1} - z_k = \frac{x_{k+1}^{[1]} + i x_{k+1}^{[2]}}{h_{k+1}} - \frac{x_k^{[1]} + i x_k^{[2]}}{h_k} =$$

$$\frac{x_k^{[1]} + i x_k^{[2]}}{h_k} \left[\frac{(x_{k+1}^{[1]} + i x_{k+1}^{[2]}) h_k}{(x_k^{[1]} + i x_k^{[2]}) h_{k+1}} - 1 \right] = z_k \left[\frac{(x_{k+1}^{[1]} + i x_{k+1}^{[2]}) h_k (x_k^{[1]} - i x_k^{[2]})}{h_k^2 h_{k+1}} - 1 \right].$$

Hence

$$z_{k+1} = \frac{x_k^{[1]} x_{k+1}^{[1]} + x_k^{[2]} x_{k+1}^{[2]} + i \frac{\omega}{r_k}}{h_k h_{k+1}} z_k.$$

Put

$$p_k = \Re \left(\frac{z_{k+1}}{z_k} \right) \quad \text{and} \quad q_k = \Im \left(\frac{z_{k+1}}{z_k} \right). \tag{4.6}$$

Then $z_{k+1} = (p_k + i q_k) z_k$ and using (4.5) we have

$$c_{k+1} + i s_{k+1} = (p_k + i q_k)(c_k + i s_k).$$

From here we get that $\binom{s}{c}$ is the solution of (S_T) and (4.3) holds. Since $s_k^2 + c_k^2 = 1$ for $k \in \mathbb{Z}$, we have from (S_T) that $p_k^2 + q_k^2 = 1$.

Using (S_T), (4.6) and (4.4) we have

$$\begin{aligned} r_k \Delta x_k^{[1]} &= r_k (x_{k+1}^{[1]} - x_k^{[1]}) = r_k (h_{k+1} c_{k+1} - h_k c_k) = \\ &= r_k h_{k+1} (-q_k s_k + p_k c_k) - r_k h_k c_k = \\ &= r_k h_{k+1} \Big(-\frac{\omega}{r_k h_k h_{k+1}} s_k + \frac{x_k^{[1]} x_{k+1}^{[1]} + x_k^{[2]} x_{k+1}^{[2]}}{h_k h_{k+1}} c_k \Big) - r_k h_k c_k = \\ &= -\frac{\omega}{h_k} s_k + \frac{r_k}{h_k} c_k (x_k^{[1]} x_{k+1}^{[1]} + x_k^{[2]} x_{k+1}^{[2]} - h_k^2) = -\frac{\omega}{h_k} s_k + g_k c_k. \end{aligned}$$

Similarly,

$$\begin{aligned} r_k \Delta x_k^{[2]} &= r_k(x_{k+1}^{[2]} - x_k^{[2]}) = r_k(h_{k+1}s_{k+1} - h_k s_k) = \\ &= r_k h_{k+1}(p_k s_k + q_k c_k) - r_k h_k s_k = \\ &= r_k h_{k+1}\Big(\frac{x_k^{[1]}x_{k+1}^{[1]} + x_k^{[2]}x_{k+1}^{[2]}}{h_k h_{k+1}} s_k + \frac{\omega}{r_k h_k h_{k+1}} c_k\Big) - r_k h_k s_k = \\ &= \frac{r_k}{h_k} s_k(x_k^{[1]}x_{k+1}^{[1]} + x_k^{[2]}x_{k+1}^{[2]} - h_k^2) + \frac{\omega}{h_k} c_k = g_k s_k + \frac{\omega}{h_k} c_k \end{aligned}$$

and (4.2) holds. □

Remark 7. Lemma 6 extends similar results given for a normalized basis of (S) formulated in Theorem D. Note that we prove this lemma by a completely different way than that given in [6] or [10].

Theorem 3. *Let* $(x^{[1]}, x^{[2]})$ *be the basis of* (1.1) *with the Casoratian* ω *and let* $\psi = (\psi_k)$ *be the first phase of this basis. Then for* $k \in \mathbb{Z}$

$$\Delta\psi_k = \begin{cases} \operatorname{Arccot} \frac{r_k(x_k^{[1]}x_{k+1}^{[1]} + x_k^{[2]}x_{k+1}^{[2]})}{\omega} & \text{if } \omega > 0, \\ \operatorname{Arccot} \frac{r_k(x_k^{[1]}x_{k+1}^{[1]} + x_k^{[2]}x_{k+1}^{[2]})}{\omega} - \pi & \text{if } \omega < 0. \end{cases}$$

Proof. Let $(x^{[1]}, x^{[2]})$ be the basis of (1.1) with the Casoratian ω and ψ be the phase of $(x^{[1]}, x^{[2]})$. By Lemma 6 there exists a solution $\binom{s}{c}$ of trigonometric system (S_T) associated to (1.1) such that (4.2) and (4.3) hold. By Lemma 3 and periodicity of the functions cos and sin there exists $\varphi_k \in [-\pi, \pi)$, $k \in \mathbb{Z}$, such that the sequences p, q in (S_T) take the form

$$\cos\varphi_k = p_k, \qquad \sin\varphi_k = q_k. \tag{4.7}$$

By Lemma 4 $\varphi_k \equiv \Delta\psi_k \pmod{\pi}$. Using this fact, (4.7) gives

$$\cos\Delta\psi_k = p_k, \qquad \sin\Delta\psi_k = q_k,$$

i.e.,

$$\cot\Delta\psi_k = \frac{p_k}{q_k}.$$

If $\omega > 0$ or $\omega < 0$, i.e. $\Delta\psi \in [0, \pi)$ or $\Delta\psi \in [-\pi, 0)$, then

$$\Delta\psi_k = \operatorname{Arccot}\frac{p_k}{q_k} \quad \text{or} \quad \Delta\psi_k = \operatorname{Arccot}\frac{p_k}{q_k} - \pi, \tag{4.8}$$

respectively. Substituting (4.3) into (4.8), the conclusion follows. □

Remark 8. Theorem 3 is a discrete version of a known formula for a phase α of differential equation (2.1), namely

$$\alpha' = \frac{-w}{x_1^2(t) + x_2^2(t)},$$

where w is a Wronskian of solutions x_1 and x_2, see [7, §5.5].

4.2 Construction of Sturm-Liouville difference equations with prescribed properties

In this section we construct a Sturm-Liouville difference equation of which the solutions have a prescribed number of generalized zeros.

Theorem 4. *Let $x^{[1]} = (x_k^{[1]})$, $x^{[2]} = (x_k^{[2]})$ be sequences with the Casoratian $\omega \neq 0$ such that $x^{[1]}$ has $m-1$ generalized zeros on $\mathbb{Z}$ and*

$$\lim_{k\to+\infty} \frac{x_k^{[1]}}{x_k^{[2]}} = 0, \quad \lim_{k\to-\infty} \frac{x_k^{[1]}}{x_k^{[2]}} = 0.$$

Then $x^{[1]}$ and $x^{[2]}$ are linearly independent solutions of equation (1.1) with

$$r_k = \frac{\omega}{x_k^{[1]} x_{k+1}^{[2]} - x_k^{[2]} x_{k+1}^{[1]}}\,, \tag{4.9}$$

$$q_k = \frac{1}{\omega}[-\Delta(r_k \Delta x_k^{[1]}) r_{k+1} \Delta x_{k+1}^{[2]} + \Delta(r_k \Delta x_k^{[2]}) r_{k+1} \Delta x_{k+1}^{[1]}]. \tag{4.10}$$

Such an equation is of type m and of special kind on $\mathbb{Z}$. Moreover, if $\omega = 1$, sequences $x^{[1]}$, $x^{[2]}$ and r satisfy

$$\sum_{k=-\infty}^{+\infty} \operatorname{Arccot}[r_k(x_k^{[1]} x_{k+1}^{[1]} + x_k^{[2]} x_{k+1}^{[2]})] = m\pi. \tag{4.11}$$

Proof. If $x^{[1]}$ and $x^{[2]}$ are solutions of (1.1), then

$$\Delta(r_k \Delta x_k^{[1]}) + q_k x_{k+1}^{[1]} = 0 \tag{4.12}$$

$$\Delta(r_k \Delta x_k^{[2]}) + q_k x_{k+1}^{[2]} = 0. \tag{4.13}$$

According to the Wronskian identity for (1.1)

$$\omega = r_k(x_k^{[1]} \Delta x_k^{[2]} - \Delta x_k^{[1]} x_k^{[2]}) \tag{4.14}$$

it holds (4.9). Let us add $(r_{k+1} \Delta x_{k+1}^{[2]})$-multiple of equation (4.12) to $(-r_{k+1} \Delta x_{k+1}^{[1]})$-multiple of equation (4.13). Using (4.14) we obtain (4.10).

According to Lemma 1, $x^{[1]}$ is a recessive solution at $\pm\infty$. By Theorem F equation (1.1) is of special kind. Since $x^{[1]}$ posseses $m-1$ generalized zeros, (1.1) is of type m on $\mathbb{Z}$. Finally, (6.11) follows from part (d) of Theorem F and from Theorem 3. □

Example 6.

(i) Let $k \in \mathbb{Z}$. Consider the sequences

$$x_k^{[1]} = 1 \qquad \textit{and} \qquad x_k^{[2]} = k + \frac{1}{2}$$

with the Casoratian $\omega = 1$. Obviously, $x^{[1]}$ has no generalized zeros in $\mathbb{Z}$ and $\lim_{k\to\pm\infty} \frac{x_k^{[1]}}{x_k^{[2]}} = \lim_{k\to\pm\infty} \frac{1}{k+\frac{1}{2}} = 0$. By Theorem 4 sequences $x^{[1]}$ and $x^{[2]}$ are linearly independent solutions of equation

$$\Delta^2 x_k^{[1]} = 0,$$

which is of type 1 and of the special kind on $\mathbb{Z}$. Moreover, since $\omega = 1$

$$\sum_{k=-\infty}^{+\infty} \operatorname{Arccotg} \left(k^2 + 2k + \frac{7}{4}\right) = \pi.$$

(ii) Now choose

$$x_k^{[1]} = k + \frac{1}{2}, \qquad x_k^{[2]} = \left(k + \frac{1}{2}\right)\left(k - \frac{1}{2}\right) \qquad \text{and} \qquad \omega = 1.$$

Notice that $x^{[1]}$ has a zero point in $\mathbb{Z}$ and $\lim_{k\to\pm\infty} \frac{x_k^{[1]}}{x_k^{[2]}} = \lim_{k\to\pm\infty} \frac{1}{k-\frac{1}{2}} = 0$. By Theorem 4, $x^{[1]}$ and $x^{[2]}$ form the basis of the equation

$$\Delta\left(\frac{1}{(k+\frac{3}{2})(k+\frac{1}{2})}\Delta x_k^{[1]}\right) + \frac{2}{(k+\frac{5}{2})(k+\frac{3}{2})^2(k+\frac{1}{2})} x_{k+1}^{[1]} = 0,$$

which is of type 2 and of the special kind in $\mathbb{Z}$. Moreover, we have

$$\sum_{k=-\infty}^{+\infty} \operatorname{Arccotg} \left(k^2 + \frac{3}{4}\right) = 2\pi.$$

(iii) More generally, put

$$x_k^{[1]} = \left(k + \frac{1}{2}\right)^{(m-1)}, \qquad x_k^{[2]} = \left(k + \frac{1}{2}\right)^{(m)} \qquad \text{and} \qquad \omega = 1,$$

where $m \geq 3$. Then $x^{[1]}$ has $m-1$ zero points and is the recessive solution at $\pm\infty$. By Theorem 4, $x^{[1]}$ and $x^{[2]}$ form the basis of equation

$$\Delta\left(\frac{1}{(k+\frac{3}{2})(k+\frac{1}{2})^2 \dots (k-m+\frac{7}{2})^2(k-m+\frac{5}{2})}\Delta x_k^{[1]}\right) + \\ + \frac{m!}{(k+\frac{5}{2})(k+\frac{3}{2})^2 \dots (k-m+\frac{7}{2})^2(k-m+\frac{5}{2})} x_{k+1}^{[1]} = 0,$$

which is of type m and of the special kind in $\mathbb{Z}$. In addition, we have

$$\sum_{k=-\infty}^{+\infty} \operatorname{Arccotg} \left[k^2 + (4-2m)k + m^2 - 4m + \frac{19}{4}\right] = m\pi.$$

Open problems.

- ◇ Theorem 4 can be generalized for a construction of a symplectic system (S) which is disconjugate or m-special on $\mathbb{Z}$.
- ◇ It may be interesting to study a connection between any disconjugate symplectic system and so-called *continuous fractions* (for details about continuous fractions see [4, Chapter 2]).

Chapter 5

Conjugacy of Sturm-Liouville Difference Equations

In this chapter we show an application of the phase theory of (1.1) established in Chapter 4. Namely, we investigate conjugacy properties of Sturm-Liouville difference equations (1.1) by means of a first phase of any basis of (1.1). Throughout the whole chapter we suppose

$$r_k > 0$$

for every considered integer k.

5.1 Conjugacy on a finite interval

Let us establish a conjugacy criterion for a finite interval of equation (1.1) using the first phase ψ given by Definition 11 and the Riccati difference equation (2.9). The conjugacy of the equation (1.1) on an interval $[M, N]$ is given by Definition 5.

Theorem 5. *Suppose that there exist real numbers $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\lambda \geq 0$ such that*

$$\sum_{k=0}^{N} \operatorname{Arccot} \frac{2\alpha_k}{\varepsilon_1} \geq \frac{\pi}{4} \tag{5.1}$$

and

$$\sum_{k=M}^{0} \operatorname{Arccot} \frac{2\beta_k}{\varepsilon_2} \geq \frac{\pi}{4}, \tag{5.2}$$

where $N > 1$ and $M < -1$ are arbitrary fixed integers,

$$\alpha_0 = \varepsilon_1 + \lambda + r_0,$$

$$\alpha_k = \left(\lambda + \varepsilon_1 - \sum_{i=0}^{k-1} q_i + r_k\right) \prod_{j=0}^{k-1} \frac{1}{r_j^2} \left(\lambda + \varepsilon_1 - \sum_{i=0}^{j-1} q_i + r_j\right)^2, \quad 1 \leq k \leq N \tag{5.3}$$

and

$$\beta_0 = \varepsilon_2 - \lambda - q_{-1} + r_{-1},$$

$$\beta_k = \left(\varepsilon_2 - \lambda - \sum_{i=k-1}^{-1} q_i + r_{k-1}\right) \prod_{j=k}^{-1} \frac{1}{r_j^2} \left(\varepsilon_2 - \lambda - \sum_{i=j}^{-1} q_i + r_j\right)^2, \qquad M \le k \le -1. \tag{5.4}$$

Then equation (1.1) *is conjugate in* $[M, N]$.

Proof. Let x be the solution of (1.1) given by the initial conditions

$$x_0 = 1, \qquad x_1 = 1 + \frac{\lambda}{r_0}, \quad \lambda \ge 0. \tag{5.5}$$

We will show that x has a generalized zero in the intervals $[2, N]$ and $[M, -2]$.

First, consider the interval $[2, N]$. We will show that there exists $m \in [2, N]$ such that $x_m \neq 0$ and $x_m x_{m+1} \le 0$. Suppose, by contradiction, that x has no generalized zero in $[2, N]$, i.e., $x_k > 0$ for $k \in [2, N+1]$. Let y be another independent solution given by the initial conditions

$$y_0 = 1, \qquad y_1 = 1 + \frac{\lambda + \varepsilon_1}{r_0}.$$

Then the Casoratian of solutions x and y is

$$\omega(x, y) = r_0(x_0 y_1 - y_0 x_1) = \varepsilon_1 > 0.$$

By the Sturm separation theorem we have

$$y_k > x_k > 0 \quad \text{for } 2 \le k \le N+1. \tag{5.6}$$

Let ψ be the first phase of the solutions x and y, i.e.,

$$\psi_k = \arctan \frac{y_k}{x_k} \qquad \text{and} \qquad \Delta\psi_k \in [0, \pi).$$

By Theorem 3 we obtain

$$\Delta\psi_k = \operatorname{Arccot} \frac{r_k(x_k x_{k+1} + y_k y_{k+1})}{\varepsilon_1}.$$

Taking into account $\psi_0 = \pi/4$ and using (5.6) we get

$$\psi_n = \sum_{k=0}^{n-1} \Delta\psi_k + \psi_0 = \sum_{k=0}^{n-1} \operatorname{Arccot} \frac{r_k(x_k x_{k+1} + y_k y_{k+1})}{\varepsilon_1} + \frac{\pi}{4} >$$
$$> \sum_{k=0}^{n-1} \operatorname{Arccot} \frac{2 r_k y_k y_{k+1}}{\varepsilon_1} + \frac{\pi}{4}. \tag{5.7}$$

Next, by means of the Riccati equation, we will estimate the term $2r_k y_k y_{k+1}$ for $k \geq 0$. Denote

$$w_k = \frac{r_k \Delta y_k}{y_k} \tag{5.8}$$

a solution of Riccati equation (2.9). Then

$$w_0 = \lambda + \varepsilon_1, \quad y_1 = \frac{y_0}{r_0}(\lambda + \varepsilon_1 + r_0),$$

and

$$y_k = y_0 \prod_{j=0}^{k-1} \frac{1}{r_j}(w_j + r_j), \quad 2 \leq k \leq N. \tag{5.9}$$

Since $r_k + w_k > 0$, we have from (2.9) that $\Delta w_k + q_k \leq 0$. Thus

$$w_k \leq w_0 - \sum_{j=0}^{k-1} q_j = \lambda + \varepsilon_1 - \sum_{j=0}^{k-1} q_j, \quad 1 \leq k \leq N.$$

From here and (5.9) we get

$$y_k \leq y_0 \prod_{j=0}^{k-1} \frac{1}{r_j}(\lambda + \varepsilon_1 - \sum_{i=0}^{j-1} q_i + r_j), \quad 2 \leq k \leq N.$$

Consequently,

$$2r_0 y_0 y_1 = 2(\lambda + \varepsilon_1 + r_0) = 2\alpha_0$$

and

$$2r_k y_k y_{k+1} \leq 2r_k \prod_{j=0}^{k-1} \frac{1}{r_j}\left(\lambda + \varepsilon_1 - \sum_{i=0}^{j-1} q_i + r_j\right) \prod_{j=0}^{k} \frac{1}{r_j}\left(\lambda + \varepsilon_1 - \sum_{i=0}^{j-1} q_i + r_j\right) =$$
$$= 2\alpha_k, \quad 1 \leq k \leq N.$$

Substituting this estimation to (5.7) we obtain

$$\psi_{n+1} > \sum_{k=0}^{n} \operatorname{Arccot} \frac{2\alpha_k}{\varepsilon_1} + \frac{\pi}{4} \qquad (0 \leq n \leq N).$$

From here and (5.1) we get $\psi_{N+1} > \pi/2$. On the other hand, since y has no generalized zero in $[0, N]$, we have $\psi_{N+1} < \frac{\pi}{2}$, a contradiction. We have proved that x has a generalized zero in $[2, N]$.

Now we will show that the solution x satisfying (5.5) has a generalized zero in $[M, -2]$, i.e. there exists l, $M \leq l \leq 0$, such that $x_{l-1} \neq 0$ and $x_{l-1}x_l \leq 0$. Since x is the solution of (1.1), that is, of the equation

$$r_{k+1}(x_{k+2} - x_{k+1}) - r_k(x_{k+1} - x_k) + q_k x_{k+1} = 0,$$

we have for $k = -1$

$$x_{-1} = 1 - \frac{\lambda + q_{-1}}{r_{-1}}.$$

Let $\bar{y}$ be another independent solution of (1.1) given by the initial conditions

$$\bar{y}_0 = 1, \qquad \bar{y}_{-1} = 1 - \frac{\lambda + q_{-1} - \varepsilon_2}{r_{-1}}.$$

The Casoratian of solutions x and $\bar{y}$ is

$$\bar{\omega}(x, \bar{y}) = r_{-1}(x_{-1}y_0 - y_{-1}x_0) = -\varepsilon_2 < 0.$$

Suppose, by contradiction, that x has no generalized zero in $[M, -2]$, i.e., $x_k > 0$ for $k \in [M-1, -2]$. Then by the Sturm separation theorem $\bar{y}_k > x_k$ for $M - 1 \le k \le -2$, again.

Let $\bar{\psi}$ be the first phase of solutions x and $\bar{y}$, i.e.,

$$\bar{\psi}_k = \arctan \frac{\bar{y}_k}{x_k} \qquad \textit{and} \qquad \Delta\bar{\psi}_k \in (-\pi, 0].$$

By Theorem 3 it holds

$$\Delta\bar{\psi}_k = \operatorname{Arccot} \frac{r_k(x_k x_{k+1} + \bar{y}_k \bar{y}_{k+1})}{-\varepsilon_2} - \pi.$$

Define the backward difference operator $\bar{\Delta}$ by $\bar{\Delta} x_k = x_{k-1} - x_k$. Then

$$\bar{\Delta} x_k = -\Delta x_{k-1}$$

and equation (1.1) takes the form $\bar{\Delta}(r_{k-1}\bar{\Delta}x_k) + q_{k-2}x_{k-1} = 0$.

Taking into account $\bar{\psi}_0 = \pi/4$ we get

$$\begin{aligned}
\bar{\psi}_n = \sum_{k=n+1}^{0} \bar{\Delta}\bar{\psi}_k + \bar{\psi}_0 = -\sum_{k=n+1}^{0} \Delta\bar{\psi}_{k-1} + \bar{\psi}_0 = \\
= -\sum_{k=n+1}^{0} \left[\operatorname{Arccot} \frac{r_{k-1}(x_{k-1}x_k + \bar{y}_{k-1}\bar{y}_k)}{-\varepsilon_2} - \pi\right] + \frac{\pi}{4} = \\
= \sum_{k=n+1}^{0} \operatorname{Arccot} \frac{r_{k-1}(x_{k-1}x_k + \bar{y}_{k-1}\bar{y}_k)}{\varepsilon_2} + \frac{\pi}{4} > \\
\sum_{k=n+1}^{0} \operatorname{Arccot} \frac{2r_{k-1}\bar{y}_{k-1}\bar{y}_k}{\varepsilon_2} + \frac{\pi}{4}.
\end{aligned}$$

Similarly as in the previous part of the proof, we estimate the term $2r_{k-1}\bar{y}_{k-1}\bar{y}_k$ by means of the Riccati equation for $k \le 0$. The sequence

$$\bar{w}_k = \frac{r_{k-1}\bar{\Delta}\bar{y}_k}{\bar{y}_k} \tag{5.10}$$

satisfies the Riccati equation

$$\bar{\Delta}\bar{w}_k + q_{k-2} + \frac{\bar{w}_k^2}{r_{k-1} + \bar{w}_k} = 0.$$

Since $\bar{\Delta}\bar{w}_k \leq -q_{k-2}$ and

$$\bar{w}_0 = \varepsilon_2 - \lambda - q_{-1},$$

we have

$$\begin{aligned} \bar{w}_k &\leq \bar{w}_0 - \sum_{j=k+1}^{0} q_{j-2} = (\varepsilon_2 - \lambda - q_{-1}) - \sum_{j=k+1}^{0} q_{j-2} = \\ &= \varepsilon_2 - \lambda - \sum_{j=k-1}^{-1} q_j, \quad M \leq k \leq -1. \end{aligned}$$

Hence, from here and (5.10) we get

$$\bar{y}_{-1} = \frac{\bar{y}_0}{r_{-1}}(\bar{w}_0 + r_{-1}) = \frac{\bar{y}_0}{r_{-1}}(\varepsilon_2 - \lambda - q_{-1} + r_{-1}),$$

and

$$\bar{y}_k = \bar{y}_0 \prod_{j=k}^{-1} \frac{1}{r_j}(w_{j+1} + r_j) \leq \bar{y}_0 \prod_{j=k}^{-1} \frac{1}{r_j}\left(\varepsilon_2 - \lambda - \sum_{i=j}^{-1} q_i + r_j\right), \quad M \leq k \leq -2.$$

Consequently,

$$2r_{-1}\bar{y}_{-1}\bar{y}_0 = 2(\varepsilon_2 - \lambda - q_{-1} + r_{-1}) = 2\beta_0$$

and

$$\begin{aligned} &2r_{k-1}\bar{y}_{k-1}\bar{y}_k \leq \\ &\leq 2r_{k-1} \prod_{j=k-1}^{-1} \frac{1}{r_j}\left(\varepsilon_2 - \lambda - \sum_{i=j}^{-1} q_i + r_j\right) \prod_{j=k}^{-1} \frac{1}{r_j}\left(\varepsilon_2 - \lambda - \sum_{i=j}^{-1} q_i + r_j\right) = \\ &= 2\beta_k, \quad M \leq k \leq -1. \end{aligned}$$

Thus

$$\bar{\psi}_{n-1} > \sum_{k=n}^{0} \operatorname{Arccot} \frac{2\beta_k}{\varepsilon_2} + \frac{\pi}{4} \qquad (M \leq n \leq 0).$$

From here and (5.2) we get $\bar{\psi}_{M-1} > \pi/2$. Since $\bar{y}$ has no generalized zero on $[M, 0]$, we have $\bar{\psi}_{M-1} < \frac{\pi}{2}$, a contradiction. This completes the proof. □

Remark 9. A closer examination of the proof of Theorem 5 reveals that Theorem 5 can be modified for an arbitrary interval $[A, B]$, where $A, B \in \mathbb{Z}$, $A < B$.

5.2 Conjugacy on $\mathbb{Z}$

Now we state some conjugacy criteria which follow from Theorem 5. They describe the conjugacy of (1.1) on $\mathbb{Z}$. First, putting $N \to +\infty$ and $M \to -\infty$ in Theorem 5, we directly obtain a conjugacy criterion of equation (1.1) in $\mathbb{Z}$.

Corollary 2. *Suppose that there exist real numbers $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\lambda \geq 0$ such that*

$$\sum_{k=0}^{\infty} \operatorname{Arccot} \frac{2\alpha_k}{\varepsilon_1} \geq \frac{\pi}{4}, \quad k \geq 0 \tag{5.11}$$

and

$$\sum_{k=-\infty}^{0} \operatorname{Arccot} \frac{2\beta_k}{\varepsilon_2} \geq \frac{\pi}{4}, \quad k \leq 0, \tag{5.12}$$

where sequences (α_k) or (β_k) are given by (5.3) or (5.4), respectively. Then (1.1) is conjugate in $\mathbb{Z}$.

From Corollary 2 the following conjugacy criterion follows:

Corollary 3. *Let*

$$\liminf_{n\to+\infty} \frac{\sum_{j=0}^{n}\left(\frac{1}{r_j}\sum_{i=0}^{j-1} q_i\right)}{\sum_{j=0}^{n}\frac{1}{r_j}} =: \quad c_1 > 0, \tag{5.13}$$

$$\liminf_{n\to-\infty} \frac{\sum_{j=n}^{-1}\left(\frac{1}{r_j}\sum_{i=j}^{-1} q_i\right)}{\sum_{j=n}^{-1}\frac{1}{r_j}} =: \quad c_2 > 0 \tag{5.14}$$

and

$$\sum_{k=1}^{\infty} \operatorname{Arccot} \frac{r_k}{\exp\left(c_1\sum_{j=0}^{k-1}\frac{1}{r_j}\right)} = \infty, \qquad \sum_{k=-\infty}^{0} \operatorname{Arccot} \frac{r_{k-1}}{\exp\left(c_2\sum_{j=k}^{-1}\frac{1}{r_j}\right)} = \infty. \tag{5.15}$$

Then equation (1.1) is conjugate in $\mathbb{Z}$.

Proof. First, we show that conditions (5.13) and (5.15) imply the validity of (5.11). According to (5.13) there exists $m \in \mathbb{Z}$ such that

$$\frac{\sum\limits_{j=0}^{n}\left(\frac{1}{r_j}\sum\limits_{i=0}^{j-1} q_i\right)}{\sum\limits_{j=0}^{n}\frac{1}{r_j}} > \frac{3}{4}c_1, \quad \text{whenever} \quad n \geq m.$$

Hence

$$\sum_{j=0}^{n}\left[\frac{1}{r_j}\left(\sum_{i=0}^{j-1} q_i - \frac{3}{4}c_1\right)\right] > 0,$$

which yields

$$0 < \exp \sum_{j=0}^{n} \left[\frac{1}{r_j} \left(\frac{3}{4} c_1 - \sum_{i=0}^{j-1} q_i \right) \right] < 1.$$

Consequently,

$$0 < \prod_{j=0}^{n} \exp \left[\frac{1}{r_j} \left(\frac{3}{4} c_1 - \sum_{i=0}^{j-1} q_i \right) \right] < 1. \tag{5.16}$$

Let $d > m+1$. Obviously,

$$\sum_{k=0}^{d} \operatorname{Arccot} \frac{2\alpha_k}{\varepsilon_1} \geq \sum_{k=m+1}^{d} \operatorname{Arccot} \frac{2\alpha_k}{\varepsilon_1}.$$

By (5.3) we have

$$\begin{aligned}
\alpha_k &= (\lambda + \varepsilon_1 + \sum_{i=0}^{k-1} q_i + r_k) \prod_{j=0}^{k-1} \frac{1}{r_j^2} (\lambda + \varepsilon_1 - \sum_{i=0}^{j-1} q_i + r_j)^2 = \\
&= r_k \prod_{j=0}^{k} \frac{1}{r_j} (\lambda + \varepsilon_1 - \sum_{i=0}^{j-1} q_i + r_j) \prod_{j=0}^{k-1} \frac{1}{r_j} (\lambda + \varepsilon_1 - \sum_{i=0}^{j-1} q_i + r_j) \leq \\
&\leq r_k \prod_{j=0}^{k} \exp \left[\frac{1}{r_j} (\lambda + \varepsilon_1 - \sum_{i=0}^{j-1} q_i) \right] \prod_{j=0}^{k-1} \exp \left[\frac{1}{r_j} (\lambda + \varepsilon_1 - \sum_{i=0}^{j-1} q_i) \right].
\end{aligned}$$

Putting $\lambda + \varepsilon_1 = \frac{1}{4} c_1$, we get

$$\alpha_k \leq r_k \prod_{j=0}^{k} \exp \left[\frac{1}{r_j} (\frac{3}{4} c_1 - \sum_{i=0}^{j-1} q_i - \frac{1}{2} c_1) \right] \prod_{j=0}^{k-1} \exp \left[\frac{1}{r_j} (\frac{3}{4} c_1 - \sum_{i=0}^{j-1} q_i - \frac{1}{2} c_1) \right]$$

and using (5.16)

$$\alpha_k < r_k \prod_{j=0}^{k} \exp \frac{-c_1}{2r_j} \prod_{j=0}^{k-1} \exp \frac{-c_1}{2r_j} < r_k \exp \left(-c_1 \sum_{j=0}^{k-1} \frac{1}{r_j} \right).$$

Letting $d \to \infty$ and taking into account (5.15), (5.11) is satisfied.

By a similar way one can check that (5.14) and (5.15) implies the validity of (5.12). The details are omitted. Now, the conclusion follows from Corollary 2. □

Remark 10. If $r_k \equiv 1$ and $\lambda = 0$, then Corollary 2 reduces to [16, Theorem 1]. In addition, (5.15) is satisfied and Corollary 3 reduces to [16, Corollary 1].

Corollary 4. *Suppose, that a sequence* (r_j) *is bounded in* $\mathbb{Z}$ *and*

$$\liminf_{n_1\to-\infty,\ n_2\to+\infty} \sum_{j=n_1}^{n_2} q_j > 0. \tag{5.17}$$

Then (1.1) *is conjugate in* $\mathbb{Z}$.

Proof. Assume (5.17). Then there exists a real-valued sequence (Q_j) such that $Q_j \le q_j$ for every $j \in \mathbb{Z}$ and

$$\sum_{j=-\infty}^{+\infty} Q_j = \liminf_{n_1\to-\infty,\ n_2\to+\infty} \sum_{j=n_1}^{n_2} q_j.$$

Hence there exists $m \in \mathbb{N}$ such that

$$\sum_{j=m}^{+\infty} Q_j > 0 \quad \text{and} \quad \sum_{j=-\infty}^{m-1} Q_j > 0.$$

From here and applying Stolze theorem (see e.g. [1]) we have

$$\liminf_{n\to+\infty} \frac{\sum_{j=m}^{n}\left(\frac{1}{r_j}\sum_{i=0}^{j-1} q_i\right)}{\sum_{j=m}^{n}\frac{1}{r_j}} \ge \liminf_{n\to+\infty} \frac{\sum_{j=m}^{n}\left(\frac{1}{r_j}\sum_{i=0}^{j-1} Q_i\right)}{\sum_{j=m}^{n}\frac{1}{r_j}} \ge \lim_{n\to\infty}\sum_{j=m}^{n} Q_j > 0,$$

i.e., (5.13) holds.

Similarly, one can verify that (5.14) is satisfied. Since (r_j) is bounded, (5.15) holds as well. By Corollary 3 equation (1.1) is conjugate in $\mathbb{Z}$. $\square$

Remark 11. The previous criterion follows from Corollary 3 and is a discrete analogue of the Müller-Pfeiffer criterion for differential equation (2.1) with $r(t) > 0$, see e.g. [1, Remark 4.1.2]:

If

$$\int^{\infty} \frac{1}{r(t)}\, dt = \infty = \int_{-\infty} \frac{1}{r(t)}\, dt$$

and

$$\liminf_{t_1\to-\infty,\ t_2\to+\infty} \int_{t_1}^{t_2} q(t)\, dt \ge 0$$

with $q(t) \not\equiv 0$ on $\mathbb{R}$, then (2.1) is conjugate on $\mathbb{R}$.

Example 7. *Consider the equation*

$$\Delta\left(\frac{1}{(k-\frac{1}{2})(k+\frac{1}{2})}\Delta x_k\right) + \frac{2}{(k-\frac{1}{2})(k+\frac{1}{2})^2(k+\frac{3}{2})} x_{k+1} = 0 \tag{5.18}$$

with $k \in \mathbb{Z}$. *By Corollary 4 this equation is conjugate in* $\mathbb{Z}$. *In addition, one can check that*

$$x_k = k - \frac{1}{2}, \quad y_k = \left(k-\frac{1}{2}\right)\left(k+\frac{1}{2}\right)$$

form the basis of (5.18)*. Since the intervals* $(-1,0]$ *and* $(0,1]$ *contain generalized zeros of* y*, by Definition 5 equation* (5.18) *is conjugate in* $[-1,1]$*.*

Remark 12. The conjugacy criterion stated in Theorem 5 is based on the existence of *focal points* in $[M,N]$. Thus using phases we get *focality criterion* for (1.1): If (5.1) holds with $\lambda = 0$, then (1.1) has a solution x such that $x_0 = x_1 = 1$ having a generalized zero in $[0,N]$.

Open problem. In Corollary 4, can the boundedness of $r = (r_k)$ be replaced by weaker assumptions $\sum_{j=0}^{\infty} \frac{1}{r_j} = \infty$ and $\sum_{j=-\infty}^{-1} \frac{1}{r_j} = \infty$?

Chapter 6

Algebraic Aspects of Symplectic Difference Systems

The aim of the chapter is to show the relation between the symplectic system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \tag{S}$$

the recurrence relation

$$r_{k+1}x_{k+2} + \beta_k x_{k+1} + r_k x_k = 0, \qquad r_k \neq 0, \tag{R}$$

and the tridiagonal symmetric matrix

$$\mathcal{L} = -\begin{pmatrix} \beta_0 & r_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ r_1 & \beta_1 & r_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & r_2 & \beta_2 & r_3 & & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ & & & & & & & \\ 0 & \dots & & 0 & r_{N-3} & \beta_{N-3} & r_{N-2} & 0 \\ 0 & \dots & & & 0 & r_{N-2} & \beta_{N-2} & r_{N-1} \\ 0 & \dots & & & & 0 & r_{N-1} & \beta_{N-1} \end{pmatrix}, \tag{$\mathcal{L}$}$$

where (a_k), (b_k), (c_k), (d_k), (r_k) and (β_k) are suitable real-valued sequences.

6.1 Trigonometric recurrence relations and matrices

In this section we introduce a concept of *trigonometric recurrence relation* and *trigonometric matrix* and we study transformations among (S), (R) and ($\mathcal{L}$) on a finite interval $[0, N]$.

First we formulate a statement which relates (S), (R) and ($\mathcal{L}$).

Proposition 3. *The following statements are equivalent:*

(a) *A sequence $x = (x_k)_{k=0}^{N+1}$ with boundary conditions $x_0 = 0 = x_{N+1}$ is a solution of recurrence relation (R).*

(b) *There exists a sequence $u = (u_k)_{k=0}^{N+1}$ such that $\binom{x_k}{u_k}$ solves (S) with*

$$a_k = 1, \quad b_k = \frac{1}{r_k}, \quad c_k = -r_{k+1} - r_k - \beta_k \quad \text{and} \quad d_k = 1 + \frac{c_k}{r_k}. \tag{6.1}$$

(c) *A sequence $x = (x_k)_{k=1}^{N}$ solves the linear system*

$$\mathcal{L}x = 0.$$

Proof.
(a) $\Longrightarrow$ (b): Suppose that a sequence $x = (x_k)_{k=0}^{N+1}$, $x_0 = 0 = x_{N+1}$ is a solution of recurrence relation (R) with $r_k \neq 0$. Then

$$\begin{aligned} r_{k+1}x_{k+2} &= -\beta_k x_{k+1} - r_k x_k \\ &= (r_{k+1} + r_k - r_{k+1} - r_k - \beta_k)x_{k+1} \\ &\quad -(r_{k+1} + r_k + \beta_k + r_k - r_{k+1} - r_k - \beta_k)x_k. \end{aligned}$$

Put $c_k := -r_{k+1} - r_k - \beta_k$. Then we get

$$\begin{aligned} x_{k+2} &= \left(1 + \frac{r_k}{r_{k+1}} + \frac{c_k}{r_{k+1}}\right)x_{k+1} - \left(1 + \frac{r_k}{r_{k+1}} + \frac{\beta_k}{r_{k+1}} + \frac{r_k}{r_{k+1}} + \frac{c_k}{r_{k+1}}\right)x_k \\ &= x_{k+1} + \frac{1}{r_{k+1}}(-r_{k+1} - r_k - \beta_k)x_k + \frac{1}{r_{k+1}}\left(1 + \frac{c_k}{r_k}\right)r_k(x_{k+1} - x_k). \end{aligned}$$

Hence, putting

$$u_k := r_k(x_{k+1} - x_k) \tag{6.2}$$

$$x_{k+2} = x_{k+1} + \frac{1}{r_{k+1}}\left[(-r_{k+1} - r_k - \beta_k)x_k + \left(1 + \frac{c_k}{r_k}\right)u_k\right]. \tag{6.3}$$

Since (6.2) holds for every integer k, then $x_{k+2} = x_{k+1} + \frac{1}{r_{k+1}}u_{k+1}$. Comparing the last relation with (6.3) we get

$$u_{k+1} = (-r_{k+1} - r_k - \beta_k)x_k + \left(1 + \frac{c_k}{r_k}\right)u_k. \tag{6.4}$$

From (6.2) and (6.4) it follows that $\binom{x_k}{u_k}$ is a solution of the system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{r_k} \\ c_k & 1 + \frac{c_k}{r_k} \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}$$

which is symplectic. Putting $a_k := 1$, $b_k := \frac{1}{r_k}$ and $d_k := 1 + \frac{c_k}{r_k}$ we get the conclusion.
(b) $\Longrightarrow$ (c): If $\binom{x_k}{u_k}$ is a solution of (S) with the sequences given by (6.1), then by a direct computation we prove that the sequence $(x_k)_{k=0}^{N+1}$ is a solution of (R). We get the system of N equations

$$\begin{aligned} r_1x_2 + \beta_0 x_1 &= 0 \\ r_2x_3 + \beta_1 x_2 + r_1 x_1 &= 0 \\ &\vdots \\ r_{N-1}x_N + \beta_{N-2}x_{N-1} + r_{N-2}x_{N-2} &= 0 \\ \beta_{N-1}x_N + r_{N-1}x_{N-1} &= 0, \end{aligned}$$

which means that $(x_k)_{k=1}^N$ solves the system $\mathcal{L}x = 0$, where $\mathcal{L}$ is given by ($\mathcal{L}$).
(c) $\Longrightarrow$ (a): This implication follows immediately by a direct computation. □

Let us define trigonometric concepts which are important to describe transformations of (R) and ($\mathcal{L}$). Trigonometric transformation of system (S) has already been described in Theorem D.

Definition 12. A three-term recurrence relation (R) is said to be *trigonometric*, if

$$|r_k| \geq 1$$

for $k = 0, \ldots, N-1$, and there exists a sequence $e = \{e_k\}_{k=0}^N$, $e_k \in \{-1, 1\}$, such that

$$\beta_k = -e_{k+1}\sqrt{r_{k+1}^2 - 1}\,\operatorname{sgn} r_{k+1} - e_k\sqrt{r_k^2 - 1}\,\operatorname{sgn} r_k. \tag{6.5}$$

A tridiagonal matrix $\mathcal{L}$ of the form (1) is said to be *trigonometric*, if there exist $\varphi_0, \ldots, \varphi_N \in (0, 2\pi)$, $\varphi_j \neq \pi$, $j = 0, \ldots, N$, such that

$$\beta_k = -\operatorname{cotg}\varphi_{k+1} - \operatorname{cotg}\varphi_k \quad \text{and} \quad r_k = \frac{1}{\sin\varphi_k}. \tag{6.6}$$

Now we relate the concepts involved in the previous definition.

Theorem 6. *The following statements are equivalent:*

(a) *The recurrence relation (R) is trigonometric.*

(b) *The symplectic difference system (S_T) is trigonometric.*

(c) *The tridiagonal matrix ($\mathcal{L}$) is trigonometric.*

Proof.
(a) $\Longrightarrow$ (b): Suppose that (R) is trigonometric and put

$$q_k = \frac{1}{r_k}, \quad p_k = \frac{e_k\sqrt{r_k^2 - 1}}{|r_k|}. \tag{6.7}$$

Then obviously $p_k^2 + q_k^2 = 1$ and

$$\frac{p_{k+1}}{q_{k+1}} + \frac{p_k}{q_k} = \frac{e_{k+1} r_{k+1} \sqrt{r_{k+1}^2 - 1}}{|r_{k+1}|} + \frac{e_k r_k \sqrt{r_k^2 - 1}}{|r_k|} = -\beta_k,$$

i.e., (R) can be written in the form

$$\frac{1}{q_{k+1}} x_{k+2} - \left(\frac{p_{k+1}}{q_{k+1}} + \frac{p_k}{q_k} \right) x_{k+1} + \frac{1}{q_k} x_k = 0. \tag{6.8}$$

Put $u_k = \frac{1}{q_k}(x_{k+1} - p_k x_k)$, i.e., $x_{k+1} = p_k x_k + q_k u_k$, and using (6.8)

$$\begin{aligned} u_{k+1} &= \frac{1}{q_{k+1}} (x_{k+2} - p_{k+1} x_{k+1}) \\ &= \frac{p_k}{q_k} x_{k+1} - \frac{1}{q_k} x_k \\ &= \frac{p_k}{q_k} (p_k x_k + q_k u_k) - \frac{1}{q_k} x_k \\ &= \frac{p_k^2 - 1}{q_k} x_k + p_k u_k \\ &= -q_k x_k + p_k u_k, \end{aligned}$$

so, $\binom{x}{u}$ is a solution of (S_T) with p, q given by (6.7).

(b) $\Longrightarrow$ (c): System (S_T) with $q_k \neq 0$ is trigonometric if and only if $p_k^2 + q_k^2 = 1$, i.e., there exists $\varphi_k \in (0, 2\pi)$, $\varphi_k \neq \pi$, such that $\sin \varphi_k = q_k$, $\cos \varphi_k = p_k$. In the previous part of the proof we have shown that if $\binom{x}{u}$ is a solution of (S_T) with $q_k \neq 0$ then $x = (x_k)_{k=0}^{N+1}$ solves (6.8) and hence, if $x_0 = 0 = x_{N+1}$, then $x = (x_k)_{k=1}^{N}$ solves the linear system $\mathcal{L}x = 0$, where

$$r_k = \frac{1}{q_k} = \frac{1}{\sin \varphi_k}, \quad \beta_k = -\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = -\operatorname{cotg} \varphi_{k+1} - \operatorname{cotg} \varphi_k,$$

i.e., the matrix ($\mathcal{L}$) is trigonometric.

(c) $\Longrightarrow$ (a): This implication is an immediate consequence of the relationship between Jacobi matrices and three-term symmetric recurrence relations, see e.g. [21] or [32]. □

In the following diagram we summarize the above-studied concepts and their relations to each other. By (R_T) we mean a trigonometric recurrence relation associated to (R) and by ($\mathcal{L}_T$) trigonometric matrix associated to ($\mathcal{L}$).

Diagram 2.

$$
\begin{tikzcd}[row sep=large, column sep=huge]
(\mathrm{R}) \arrow[d, leftrightarrow, "\textit{Proposition 3}"'] & (\mathrm{R}_T) \arrow[d, leftrightarrow, "\textit{Theorem 6}"] \\
(\mathrm{S}) \arrow[r, "h,g", "\textit{Theorem D}"'] \arrow[d, leftrightarrow, "\textit{Proposition 3}"'] & (\mathrm{S}_T) \arrow[d, leftrightarrow, "\textit{Theorem 6}"] \\
(\mathcal{L}) & (\mathcal{L}_T)
\end{tikzcd}
$$

To complete the previous diagram, in the following two theorems we study relations between (R) and (R_T), and between ($\mathcal{L}$) and ($\mathcal{L}_T$). The next theorem shows that there exists a transformation which transforms the recurrence relation (R) to the trigonometric one.

Lemma 7. *If $(x^{[1]}, x^{[2]})$ is a normalized basis of (R), then there exist sequences $u^{[1]}$ and $u^{[2]}$ such that $z^{[1]} = \binom{x}{u}^{[1]}$ and $z^{[2]} = \binom{x}{u}^{[2]}$ form a normalized basis of (S). Vice versa, if $(z^{[1]}, z^{[2]})$ is a normalized basis of (S), then $(x^{[1]}, x^{[2]})$ form a normalized basis of (R).*

Proof.

"$\Rightarrow$": If $(x^{[1]}, x^{[2]})$ is a basis of (R), then by Proposition 3 there exist sequences $u^{[1]}$ and $u^{[2]}$ such that $z^{[1]} = \binom{x}{u}^{[1]}$ and $z^{[2]} = \binom{x}{u}^{[2]}$ are solutions of (S) with coeffiecients satisfying (6.1). In addition, since $(x^{[1]}, x^{[2]})$ is a normalized basis of (R), taking $u^{[i]}$, $i = 1, 2$, by (6.2), we get the conclusion.

"$\Leftarrow$": If $(z^{[1]}, z^{[2]})$ is a normalized basis of (S), then putting $u^{[i]} = r\Delta x^{[i]}$ for $i = 1, 2$ $(x^{[1]}, x^{[2]})$ is a normalized basis of (R).

$\square$

Theorem 7. *Let $(x_k^{[1]})$ and $(x_k^{[2]})$ be solutions of (R) which form a normalized basis. Then there exist $\varphi_0, \dots, \varphi_N \in (0, 2\pi)$, $\varphi_j \neq \pi$ for $j = 0, \dots, N$ such that the transformation*

$$x_k = h_k y_k,$$

where $h_k = (x_k^{[1]})^2 + (x_k^{[2]})^2$ for $k = 0, \dots, N-1$, transforms (R) into the trigonometric recurrence relation (R_T) of the form

$$\frac{y_{k+2}}{\sin\varphi_{k+1}} - (\operatorname{cotg}\varphi_{k+1} + \operatorname{cotg}\varphi_k)\, y_{k+1} + \frac{y_k}{\sin\varphi_k} = 0. \tag{6.9}$$

Proof. Let $(x^{[1]}, x^{[2]})$ form a normalized basis of (R). By Theorem D the transformation (2.16) transforms (S) into the trigonometric system (S_T). Since by Theorem 6 the recurrence relation (R) and tridiagonal matrix $(\mathcal{L})$ is trigonometric, according to Definition 12 sequences r and β in $(\mathcal{L})$ are of the form (6.6). Together with Proposition 3 it gives (6.9). □

In the next theorem we show that any tridiagonal symmetric matrix of the form $(\mathcal{L})$ can be reduced via a diagonal transformation matrix to a trigonometric matrix.

Theorem 8. *Given a tridiagonal symmetric matrix $\mathcal{L}$ of the form $(\mathcal{L})$, there exists a sequence $(h_k)_{k=1}^N$ such that trigonometric matrix $(\mathcal{L}_T)$ can be expressed in the form*

$$\mathcal{L}_T = \operatorname{diag}\{h_1, \ldots, h_N\}\mathcal{L}\operatorname{diag}\{h_1, \ldots, h_N\}.$$

Proof. Consider a symplectic difference system (S) with $b_k \neq 0$. From the first equation of (S), i.e., $u_k = \frac{1}{b_k}(x_{k+1} - a_k x_k)$ and substituting into the second equation we get the recurrence relation

$$\frac{x_{k+2}}{b_{k+1}} - \left(\frac{a_{k+1}}{b_{k+1}} + \frac{d_k}{b_k}\right) x_{k+1} + \frac{x_k}{b_k} = 0,$$

i.e., (R) with

$$r_k = \frac{1}{b_k} \quad \text{and} \quad \beta_k = -\frac{a_{k+1}}{b_{k+1}} - \frac{d_k}{b_k}. \tag{6.10}$$

Now we transform (S) into the system (2.19). Applying (6.10) we give the recurrence relation with

$$\begin{aligned}
\tilde{r}_k &= \frac{1}{\tilde{b}_k} = \frac{h_k h_{k+1}}{b_k} = h_k h_{k+1} r_k,\\
\tilde{\beta}_k &= -\frac{\tilde{a}_{k+1}}{\tilde{b}_{k+1}} - \frac{\tilde{d}_k}{\tilde{b}_k}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\frac{1}{h_{k+2}}(ah+bg)_{k+1}}{\frac{b_{k+1}}{h_{k+1}h_{k+2}}} - \frac{\frac{1}{h_k}(-g_{k+1}b_k + h_{k+1}d_k)}{\frac{b_k}{h_k h_{k+1}}}\\
&= -h_{k+1}^2\frac{a_{k+1}}{b_{k+1}} - h_{k+1}^2\frac{d_k}{b_k}\\
&= h_{k+1}^2\beta_k.
\end{aligned}$$

Now, by Theorem D, there exist sequences h, g such that (2.16) transforms (S) into (S_T) and this fact, coupled with the statement of Theorem 6, gives the required result. □

We conclude this section with a diagram which illustrates all relations explained above:
Diagram 3.

$$
\begin{tikzcd}[column sep=huge, row sep=large]
(\mathrm{R}) \arrow[r, "h", "\textit{Theorem 7}"'] \arrow[d, leftrightarrow, "\textit{Proposition 3}"'] & (\mathrm{R}_T) \arrow[d, leftrightarrow, "\textit{Theorem 6}"] \\
(\mathrm{S}) \arrow[r, "{h, g}", "\textit{Theorem D}"'] \arrow[d, leftrightarrow, "\textit{Proposition 3}"'] & (\mathrm{S}_T) \arrow[d, leftrightarrow, "\textit{Theorem 6}"] \\
(\mathcal{L}) \arrow[r, "h", "\textit{Theorem 8}"'] & (\mathcal{L}_T)
\end{tikzcd}
$$

6.2 Positive definiteness of trigonometric matrices

In this section we introduce the statement which is a discrete analogue of a result for equation (2.1): The trigonometric quadratic functional (with a positive function p)

$$\mathcal{F}(x; a, b) = \int_a^b \left[\frac{1}{p(t)} x'^2 - p(t) x^2 \right] dt$$

is positive over the class of nontrivial differentiable functions satisfying $x(a) = 0 = x(b)$ if and only if

$$\int_a^b p(t)\, dt < \pi,$$

see e.g. [7].

Theorem 9. *The trigonometric matrix ($\mathcal{L}$), where (β_k), (r_k) are given by* (6.6) *with $\varphi_k \in (0, \pi)$, is positive definite if and only if*

$$\sum_{k=0}^{N-1} \varphi_k < \pi. \tag{6.11}$$

In particular, the m-th principal minor D_m of $\mathcal{L}$, $m = 1, \dots, N-1$, is given by the formula

$$D_m = \frac{\sin\left(\sum_{j=0}^{m} \varphi_j\right)}{\prod_{i=0}^{m} \sin \varphi_i}. \tag{6.12}$$

Proof. The matrix is positive definite if all its principal minors are positive. First we prove that each principal minor is given by (6.12). We proceed by induction. Denote $q_k = \sin\varphi_k$, $p_k = \cos\varphi_k$. Then we have

$$D_1 = \frac{p_0}{q_0} + \frac{p_1}{q_1} = \operatorname{cotg}\varphi_0 + \operatorname{cotg}\varphi_1 = \frac{\sin(\varphi_0+\varphi_1)}{q_0 q_1}.$$

Suppose that (6.12) holds for any $k \in \mathbb{N}$ and we show that it holds for $k+1$.

Consider the determinant of the $(k+1)$-th order D_{k+1}. Expanding D_{k+1} by the $(k+1)$-th row (using the Laplace rule), we get

$$\begin{aligned} D_{k+1} &= \left(\frac{p_k}{q_k} + \frac{p_{k+1}}{q_{k+1}}\right) D_k - \frac{1}{q_k^2} D_{k-1} \\ &= \frac{\sin(\varphi_k + \varphi_{k+1})}{q_k q_{k+1}} \frac{\sin\left(\sum_{j=0}^{k}\varphi_j\right)}{q_0\cdots q_k} - \frac{1}{q_k^2}\frac{\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)}{q_0\cdots q_{k-1}}. \end{aligned}$$

Denote $G_{k+1} = (q_0\cdots q_{k-1} q_k^2 q_{k+1}) D_{k+1}$. Then we have

$$\begin{aligned} G_{k+1} &= \sin(\varphi_k+\varphi_{k+1}) \sin\left(\sum_{j=0}^{k-1}\varphi_j\right) - \sin\varphi_{k+1}\sin\left(\sum_{j=0}^{k-1}\varphi_j\right) \\ &= (\sin\varphi_k\cos\varphi_{k+1} + \cos\varphi_k\sin\varphi_{k+1})\left[\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\cos\varphi_k + \cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_k\right] \\ &\quad - \sin\varphi_{k+1}\sin\left(\sum_{j=0}^{k-1}\varphi_j\right) \\ &= \sin\varphi_k\cos\varphi_{k+1}\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\cos\varphi_k \\ &\quad + \cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_k\sin\varphi_k\cos\varphi_{k+1} + \cos\varphi_k\sin\varphi_{k+1}\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\cos\varphi_k \\ &\quad + \cos\varphi_k\sin\varphi_{k+1}\cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_k - \sin\varphi_{k+1}\sin\left(\sum_{j=0}^{k-1}\varphi_j\right) \\ &= \sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_{k+1}(\cos^2\varphi_k - 1) + \cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin^2\varphi_k\cos\varphi_{k+1} \\ &\quad + \sin\varphi_k\cos\varphi_{k+1}\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\cos\varphi_k + \cos\varphi_k\sin\varphi_{k+1}\cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_k \end{aligned}$$

$$
\begin{aligned}
&= \sin^2\varphi_k \left[-\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_{k+1} + \cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\cos\varphi_{k+1}\right] \\
&\quad + \sin\varphi_k\cos\varphi_k\left[\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\cos\varphi_{k+1} + \cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_{k+1}\right] \\
&= \sin^2\varphi_k\cos\left(\sum_{j=0}^{k-1}\varphi_j + \varphi_{k+1}\right) + \sin\varphi_k\cos\varphi_k\sin\left(\sum_{j=0}^{k-1}\varphi_j + \varphi_{k+1}\right) \\
&= \sin\varphi_k\sin\left(\sum_{j=0}^{k+1}\varphi_j\right).
\end{aligned}
$$

It means that

$$D_{k+1} = \frac{\sin\left(\sum_{j=0}^{k+1}\varphi_j\right)}{q_0\cdots q_{k+1}}$$

which proves (6.12).

"$\Rightarrow$": Let matrix $(\mathcal{L})$ be positive definite, i.e., for every $m = 1, \ldots, N-1$

$$D_m > 0. \tag{6.13}$$

If for every $k = 0, \ldots, N-1$ $\varphi_k \in (0, \pi)$, then from (6.13) follows (6.11).

"$\Leftarrow$": Obviously, (6.11) implies (6.13) with $\varphi_k \in (0, \pi)$.

$\square$

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