MASARYK UNIVERSITY

Faculty of Science

Department of Mathematics and Statistics

Jan Pavlík

ON CATEGORIES OF ALGEBRAS

Ph.D. Thesis

Supervisor: prof. RNDr. Jiří Rosický, DrSc. **BRNO, 2010**

Bibliographical identification

Name and surname of the author: Jan Pavlík

Title of dissertation: On Categories of Algebras

Program: Algebra, Number Theory and Mathematical Logic

Supervisor: prof. RNDr. Jiří Rosický, DrSc.

Year of defense: 2010

Keywords: Free algebra, Coalgebra, Variety, Adjunction, Monad, Codensity Monad

2000 Mathematics Subject Classification: 08C05, 18A40, 18C05, 18C20.

Abstrakt

Práce se zabývá teorií kategorií. Studujeme v ní konkrétní kategorie, zejména kategorie algeber a koalgeber. Tyto objekty jsou zobecněním pojmu algebra v klasickém smyslu univerzální algebry na množinách. Uvedeme několik druhů algeber a vztahy mezi jejich kategoriemi. Zavedeme pojem l-algebraických kategorií, který zahrnuje mnoho přirozených případů.

Připomeneme definici variety od J. Adámka a H. E. Porsta na kokompletní kategorii založenou na řetězcové konstrukci volné algebry. Jedná se o kategoriální protějšek rovnicového zadání variety v klasické univerzální algebře. Ukážeme jiný přístup k těmto třídám včetně důkazu ekvivalence s výše uvedeným. Uvedeme také důležité vztahy těchto tříd k jiným třídám algeber. Zejména dokážeme, že každá varieta je algebraická, což spojuje dva různé kategoriální přístupy k varietám.

Dále se zaměříme na otázku volných algeber. Protože řetězcová konstrukce nám pro vhodné funktory dává volné algebry, ukážeme, jak z ní pomocí výsledků G. M. Kellyho z r. 1980 můžeme odvodit existenci volné algebry ve varietě.

Poslední kapitola se věnuje vztahům mezi existencí volné algebry, Alg-univerzálních funktorů a kohustotních monád příslušného zapomínajícího funktoru. Výsledkem je další charakterizační věta pro monadické kategorie.

Abstract

The thesis is on category theory. Concrete categories are investigated, namely the categories of algebras and coalgebras. These objects are category-theoretical generalizations of algebras in the sense of the classical universal algebra on sets. Numerous kinds of algebras are recalled showing the relations between their categories. A concept of l-algebraic categories is introduced including many natural examples.

For the cocomplete base category, the definition of variety by J. Adámek and H. E. Porst is recalled. It is based on the concept of free-algebra chain construction which enables a categorical counterpart to the equational presentation of varieties in the classical universal algebra. A different approach to these classes is presented proving its equivalence with the original one. Moreover, their relationships to other kinds of algebras are shown. Namely, each variety is proved to be algebraic, which connects two different categorical approaches to varieties.

Next focus is on the concept of freeness. Since the free-algebra chain construction yields the free algebra for suitable functors, it is shown how to use this fact together with the results of G. M. Kelly from 1980 to obtain the corresponding property for the free algebras in a variety.

The final chapter is devoted to the problem of the relationship between the existences of free objects, Alg-universal functors and codensity monads for a given concrete category. This results in another characterization of monadic categories.

Contents

0.1 Introduction

This thesis deals with category theory - an abstract mathematical theory developed for carrying the results from one mathematical field to another. Such fields are described by categories consisting of objects and morphisms. The reader is expected to be familiar with the foundations of this theory. Our interest is in the *concrete categories* which are equivalent to categories whose objects can be thought of as "structured objects" of some *base category*.

An important family of concrete categories is that of *algebras* and *coalgebras*. The well-known notion of universal algebra is generalized in an arbitrary base category by replacing the underlying set by an object and the operations by certain morphisms. There are many ways to implement such a generalization. Some of them may even generalize of a *variety* of algebras, i.e., classes of algebras satisfying certain additional properties. One of the aims of this thesis is to provide an overview of the generalizations of a universal algebra and of their mutual relations. The results on coalgebras may be then obtained by duality on the base category.

As we know from the classical universal algebra, there is an important concept of *free algebras*. Such algebras can be used to simulate all the computations in algebras. On the other hand, the coalgebras are intimately connected to the theory of automata with cofree coalgebras playing the role of universal automata, which enable the simulation of all automata of a given kind.

Both freeness and cofreeness are easy to carry over to a general base category since it can be described using the concept of *adjunction*. Then one may ask about the existence of the free algebras in general. Since its general existence is not very likely to happen for any kind of algebras, we may be satisfied with some sufficient conditions. These are usually conditions determining the some of categories and functors. In some cases, they enforce the preservation of some colimits by some functors, in other cases, they can be expressed in terms of the existence of *codensity monads*, which is generally a weaker tool than the adjunction.

Thesis Structure

In the following paragraphs, the expressions written in boldface refer to the features that are considered to be new.

The thesis is divided into two parts and two appendices. In the first part we are concerned with categories of algebras. We recall the definitions of algebras for a signature, functor and monad and f-algebraic and monadic categories. Then we introduce the concept of **l-algebraic categories** which includes, in addition to the ones mentioned above, the categories of algebras for a diagram of functors or monads and newly introduced **polymeric categories**. Further we recall algebraic categories of Linton and Rosický consisting of algebras for a type. From their concept of equational theory, we derive a useful concept of **functorial theory** and we show their **equivalence**. We present a full version of Reiterman theorem converting f-algebraic categories into algebraic ones. Then we **extend** it **for** all **polymeric categories**.

The second chapter of the first part deals with categories of algebras over a cocomplete category. We recall Adámek's free algebra chain construction and his (and Porst's) concept of equational classes. We show another way of defining of these generalizations of classical varieties - by means of **natural identities** - and we show the **equivalence** of these concepts; these classes are further called varieties. This new concept enables easy conversion of a **variety** to l-categorical presentation **as** a certain **limit of f-algebraic categories**. However, we prove that every variety is an intersection of an ordinal chain of polymeric categories. This makes it possible to prove that **every variety is algebraic**, which connects the approach of Adámek - Porst with that of Linton - Rosický.

The second part deals with the problem of free algebras. After outlining the standard notions and results including theorems over adjunction, Eilenberg-Moore categories, Beck's theorem and a very important Kelly's theorem, we recall the construction of a free algebra in an f-algebraic category via the colimit of chains. This procedure is then **extended to varieties**. Moreover, we use polymeric varieties for an **alternative presentation** of Eilenberg-Moore categories for a free monad.

The final chapter aims to describe the existence of free algebras in Beck categories in terms of **Alg -universality** and codensity monads. We prove that all **Beck categories with pointwise codensity monads** and **l-categories with codensity monads** are **monadic** and that, generally, this result **cannot be strengthened**. Finally we show an overview of the relationships obtained and a few remaining **open questions**.

The appendices recall basic notions and facts of concrete categories, universality, adjunctions, Kan extensions and codensity monads.

Preliminaries

We use the standard language of category theory used namely in books [24], [7] and lecture notes [26]. Since authors use different notions, in cases where it may be confusing we refer to the Appendix or to the bibliographic reference. The reader is expected to be familiar with the basic notions and the principles of category theory and with discrete mathematics generally.

We work with collections of four size levels: sets, classes, metaclasses and hyperclasses. Sets and classes lie in the model of Gödel-Bernays set theory GBC, while metaclasses are collections of classes. Anything "bigger" is called hyperclass here, but we use this level rarely.

To explain the basic categorical notions, the categories are collections of objects and mappings between the objects. According to the collection size level we have the following:

- *small category* - objects form a set and morphisms form a class
- *locally small category* - morphisms between each two objects form a set
- *category* - both objects and morphisms form a class
- *metacategory* - both objects and morphisms form a metaclass
- *hypercategory* - both objects and morphisms form a hyperclass.

Due to these definitions, we have also several levels of functors and natural transformations, but we do not distinguish these in terminology. However, in some cases we emphasize the size of entities we work with, by stating a "Metacategorical remark". Entities which do not fit within GBC are called *illegitimate*.

Throughout this thesis we also use an object-free approach to categories. From this point of view, each category is a collection of morphisms with the unit element and a partial binary operation defined whenever the obvious condition of domain-codomain matching is satisfied. We point out whenever we deal with this presentation of a category.

Notation

The empty set and an empty mapping with the codomain A are denoted by $\emptyset$, $\varnothing_A$, respectively. The identity morphism on an object A and the identity functor on a category $\mathcal{C}$ are written as id_A, $\mathrm{Id}_{\mathcal{C}}$, respectively. Moreover we set $\varnothing = \varnothing_\emptyset = \mathrm{id}_\emptyset$. By $\mathbf{1}$ we denote the category with only one element 0 and identity while category $\mathbf{2}$ contains two objects 0 and 1 and morphisms $\iota : 0 \to 1$, id_0, and id_1.

The categories are usually denoted by capital calligraphic letters such as $\mathcal{A}, \mathcal{B}, \mathcal{C}$, objects are usually typed in capital roman italic from the begining of the alphabet A, B, C, while the functors have various notation, but usually also capital roman italic starting from $E, F, G \ldots$. The exceptions are in the case of diagrams - these functors are often labeled by D with the objects in the domain category denoted by small italic letters c, d as well as elements of almost any other set. From most of the expression we usually drop the brackets and composition symbol $\circ$.

The class of all objects and morphisms of a category $\mathcal{C}$ (in short: the $\mathcal{C}$-objects, the $\mathcal{C}$-morphisms) will be denoted by $\mathrm{Ob}(\mathcal{C})$, $\mathrm{Mor}(\mathcal{C})$, respectively. Each morphism $f : A \to B$ has its *domain* $\mathrm{dom}(f) = A$ and *codomain* $\mathrm{cod}(f) = B$. The set of all morphisms between two $\mathcal{C}$-objects A, B will be denoted by $\hom_{\mathcal{C}}(A, B)$ or just $\hom(A, B)$. For an object A, we denote its covariant, contravariant hom-functor by $\hom(A, -)$, $\hom(-, A)$, respectively. Given functors $F, G : \mathcal{A} \to \mathcal{B}$, the class of natural transformations between F and G is denoted by $\mathbf{Nat}(F, G)$.

The constant functor $\mathcal{C} \to \mathcal{A}$ mapping all morphisms onto an id_A for an $\mathcal{A}$-object A will be denoted by C_A. Given a morphism $f : A \to B$, the corresponding *constant transformation* is denoted by $C_f : C_A \to C_B$ - it clearly satisfies $C_{f,M} = f$ for every object M. The domain of the constant functor is usually considered to be $\mathbf{1}$, but it may occasionally be redefined.

Let $\{A_i | i \in I\}$ be a set of objects in some category $\mathcal{C}$. Their product is usually denoted by $\prod_{i\in I} A_i$ with projections $p_i : \prod_{i\in I} A_i \to A_i$ for $i \in I$. Dually, the coproduct is denoted by $\coprod_{i\in I} A_i$ and the canonical injections are usually written as $u_i : A_i \to \coprod_{i\in I} A_i$. If we have only two objects A, B, we denote their product and coproduct by $A \times B$ and $A + B$, respectively. Given morphisms $f : A \to C$ and $g : B \to D$, by $[f, g] : A + B \to C + D$, we denote a "coproduct of morphisms" $f : A \to C$ and $g : B \to D$, i.e., a unique morphism such that $[f, g] \circ u_A = u_C \circ f$ and $[f, g] \circ u_B = u_D \circ g$.

The category of sets and mappings will be denoted by **Set**. By P we denote the powerset functor $\mathbf{Set} \to \mathbf{Set}$ assigning to each set the set of its subsets. Class of ordinal numbers is denoted by Ord. It is often considered also as a category with the usual order-induced structure.

Note 1 *Given a category $\mathcal{D}$, by $\mathcal{D}^*$ we denote the category made from $\mathcal{D}$ by adding a new terminal object 1 and terminal morphisms $t_d : d \to 1$ for each $d \in \mathrm{Ob}\mathcal{D}$. Dually, we construct a category $\mathcal{D}_*$ by adding, to the category $\mathcal{D}$, a new initial object 0 together with morphisms $\iota_q : 0 \to q$ for each $q \in \mathrm{Ob}\mathcal{D}$.*

Acknowledgements

I would like to thank my supervisor Jiří Rosický for guiding and directing my research, for numerous discussions, and especially for his patience. I appreciate also the discussions with Jiří Adámek, Libor Barto, David Kruml, Michal Kunc, Marc Olschok, Jan Paseka, Ladislav Skula, Sergejs Solovjovs, Josef Šlapal, Věra Trnková and I am grateful for the support from many others.

Part I

Categories of Algebras

Chapter 1

Algebras over a General Category

This chapter deals with both algebras and coalgebras - the essential entities for this thesis. The first section provides an overview of the meanings of the word "algebra", namely of those which are used later on in this thesis. Further, we recall the well-known free-algebra chain construction introduced by Jiří Adámek in [5]. It is a chain of functors, here called "term-functors", which may be seen as approximations of the free algebra - this topic is discussed in Part II. In this chapter, we focus on another application of this chain, which makes it possible to define the equational class of algebras on any cocomplete category as shown in [9]. Such a class is an analogue of the variety in its classical meaning in the universal algebra on $\mathcal{S}et$.

There is a disadvantage to the free-algebra construction - it uses several kinds of colimits, hence it tends to restrict the research on cocomplete categories. To avoid this, we introduce a more general concept of l-algebraic categories which, as we will see in Part II., enables the general treatment concerning free objects even without the need of colimits.

Moreover, we recall an approach to variety-like classes introduced by Linton in [25] and Rosický in [33]. Finally, we present examples of algebras and coalgebras and of some varieties including the polymeric ones.

1.1 Algebras and Coalgebras

Algebra is the fundamental mathematical notion. The word comes from the ancient arabic word "al jabr" which originally meant the modification of sides of an equation in order to simplify it. During centuries, it was transformed to "algebra" and its meaning shifted to today's "use of operations on set" and a derived meaning of this word also denotes a "set with operations". These are *algebras for a signature* on category of sets, from our point of view. In this thesis however, we do not have to restrict ourselves to this category and a use more general definition of *algebra for a functor*. Then we show several more sophisticated definitions which tend to reflect the notions of *variety of algebras*. However, that will be the topic of the next section.

In this chapter we deal with algebras and, in each case *the coalgebras*, can be defined in a dual way. In some cases, if useful, we write its explicit definition.

1.1.1 Algebras for a Signature

First we recall one way of generalizing the classical concept of universal algebra - *many-sorted algebras*. A finitary S*-sorted signature* Σ is a set of *operation symbols* together with an *arity function* $\mathrm{ar} : \Sigma \to S^*$ with range in the set of finite sequences over set S. For $\sigma \in \Sigma$, the assignment $\sigma \mapsto s_1 s_2 \dots s_k s$ is usually written in the form $\sigma : s_1 s_2 \dots s_k \to s$ and interpreted as a k-ary operation with i-th variable of sort s_i and with the sort of value s.

Definition 1.1.1 (Finitary many-sorted algebra for a signature) *Let S be a set of sorts and Σ be S-sorted signature. Let $\mathcal{C}$ be a category with finite products.*

*An S-*sorted algebra *$\underline{A}$ of a signature Σ (a Σ-*algebra *in shortly) is an S-sorted $\mathcal{C}$-object $\{A_s, s \in S\}$ such that, for each operation symbol $\sigma : s_1 s_2 \dots s_k \to s$, there is a morphism $\sigma_{\underline{A}} : A_{s_1} \times A_{s_2} \times \cdots \times A_{s_k} \to A_s$.*

A homomorphism of S-sorted algebras *$\phi : \underline{A} \to \underline{B}$ is an S-sorted $\mathcal{C}$-morphism satisfying, for each operation symbol σ of arity $s_1 s_2 \dots s_k \to s$, the equality $\sigma_{\underline{B}} \circ (\phi_{s_1}, \phi_{s_2}, \dots, \phi_{s_k}) = \phi_s \circ \sigma_{\underline{A}}$. The category of algebras for S-sorted signature Σ will be denoted by* $\mathbf{Alg}\,\Sigma$.

Remark 1.1.1 *By a slight modification of the definition, the algebras can be defined also for infinitary signatures.*

1.1.2 Algebras for a Functor

The fundamental notion for this thesis will be algebra for a functor. Let $F : \mathcal{C} \to \mathcal{C}$ be a functor.

Definition 1.1.2 (Algebra for a functor) *An F-*algebra *is a pair (A, α) consisting of an object A and of a morphism $\alpha : FA \to A$ in $\mathcal{C}$. Given two F-algebras (A, α) and (B, β) and a morphism $f : A \to B$, then we say that ϕ is F-*algebra morphism *if $\beta \circ Ff = f \circ \alpha$.*

The category of F-algebras and F-algebras morphisms will be denoted by $\mathbf{Alg}_{\mathcal{C}}F$ *or just by* $\mathbf{Alg}\,F$ *if we do not need to emphasize the base category. A category isomorphic to* $\mathbf{Alg}\,F$*, for some functor* $F : \mathcal{C} \to \mathcal{C}$*, will be called* f-algebraic*.*

Remark 1.1.2

- $\mathbf{Alg}\,F$ *is concrete over* $\mathcal{C}$ *via the forgetful functor* U_F *given by* $(A, \alpha) \mapsto A$*.*
- *Dually, we define* F-coalgebras *and the category which they form is denoted by* $\mathbf{Coalg}_{\mathcal{C}}\,F$*. It can be also described as the dual category of* $\mathbf{Alg}_{\mathcal{C}^{op}}F^{op}$ *where* F^{op} *is the corresponding endofunctor on* $\mathcal{C}^{op}$*.*

Remark 1.1.3 *Single-sorted algebras on the category of sets may be seen as algebras for a functor. Indeed, if* Σ *is a single-sorted signature, i.e. the set of sorts is* $S = 1$*, we can define the polynomial functor*

$$h_\Sigma = \coprod_{\sigma \in \Sigma} \hom(ar(\sigma), -) : \mathcal{C} \to \mathcal{C},$$

Σ*-algebras are in one-to-one natural correspondence with* h_Σ*-algebras, thus*

$$\mathbf{Alg}\,\Sigma \cong_{\mathcal{S}et} \mathbf{Alg}\,h_\Sigma.$$

We recall some well-known facts about categories of algebras (see [9]).

Remark 1.1.4 *Let* $F : \mathcal{C} \to \mathcal{C}$ *be a functor and* $U_F : \mathbf{Alg}\,F \to \mathcal{C}$ *be the forgetful functor. Then the following holds.*

1. U_F *creates limits.*
2. U_F *creates all colimits preserved by* F*.*
3. *If* $\mathcal{C}$ *is complete, then so is* $\mathbf{Alg}_{\mathcal{C}}F$*.*

Metacategorical remark 1.1.5 *Given a category* $\mathcal{C}$*, the metaclass of* $\mathcal{C}$*-concrete categories and* $\mathcal{C}$*-concrete functors will be denoted by* $\mathbf{Con}\,\mathcal{C}$ *and referred to as* metacategory of $\mathcal{C}$-concrete categories*. The operator* $\mathbf{Alg}$ *may be considered contravariant functor* $\mathbf{End}\mathcal{C} \to \mathbf{Con}\,\mathcal{C}$ *defined on the metacategory of endofunctors on* $\mathcal{C}$*. It assigns, to a natural transformation* $\phi : G \to F$*, the concrete functor* $\mathbf{Alg}\,\phi : \mathbf{Alg}\,F \to \mathbf{Alg}\,G$ *given by equality* $(\mathbf{Alg}\,\phi)(A, \alpha) = (A, \alpha \circ \phi_A)$*. This functor translates colimits to limits, i.e., e.g.,* $\mathbf{Alg}\,(F + G) \cong \mathbf{Alg}\,F \times_{\mathcal{C}} \mathbf{Alg}\,G$ *as defined in Remark 1.2.4.*

See the Definition A.1.4 for the explanation of the notion of Beck category. As a direct consequence of Remark 1.1.4 we get another well known fact, which will be used later on.

Proposition 1.1.6 *Every f-algebraic category is Beck.*

1.1.3 Algebras for a Monad

Another important definition of algebra arises from the theory of adjunctions and monads.

Definition 1.1.3 (Algebra for a monad) *Given a monad* $\underline{M} = (M, \eta, \mu)$ *on* $\mathcal{C}$*, we define a* category $\underline{M}-\mathbf{alg}$ of $\underline{M}$-algebras *(also called Eilenberg-Moore category) as a full subcategory of* $\mathbf{Alg}\, M$ *consisting of all* M*-algebras* (A, α) *satisfying the* Eilenberg-Moore identities*:*

$$\alpha \circ \eta_A = \mathrm{id}_A, \tag{1.1}$$

$$\alpha \circ M\alpha = \alpha \circ \mu_A, \tag{1.2}$$

i.e., the following diagrams commute

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\eta_A"] & A \\ MA \arrow[ur, "\alpha"'] & \end{tikzcd} \qquad \begin{tikzcd} M^2A \arrow[r, "\mu_A"] \arrow[d, "M\alpha"] & MA \arrow[d, "\alpha"] \\ MA \arrow[r, "\alpha"] & A. \end{tikzcd}$$

A category concretely isomorphic to $\underline{M}-\mathbf{alg}$ *for some monad* $\underline{M}$ *is called* monadic*.*

Metacategorical remark 1.1.7 *Consider the metacategory* $\mathbf{Monad}\, \mathcal{C}$ *of monads over* $\mathcal{C}$ *with monads as obejcts and monad transformations as the morphisms (see Appendix A). Now the assignment* $\underline{M} \mapsto \underline{M}-\mathbf{alg}$ *can be seen as a contravariant functor* $-\mathbf{alg} : \mathbf{Monad}\, \mathcal{C}^{op} \to \mathbf{Con}\, \mathcal{C}$*. Since* $\mathbf{Monad}\, \mathcal{C}$ *is a concrete metacategory over* $\mathbf{End}\mathcal{C}$ *via some forgetful functor* Z*, the functor* $-\mathbf{alg}$ *is clearly a subfunctor of the composition* $\mathbf{Alg} \circ Z : \mathbf{Monad}\, \mathcal{C}^{op} \to \mathbf{Con}\, \mathcal{C}$*.*

The proof of the following proposition can be found in [17].

Proposition 1.1.8 *The assignment* $-\mathbf{alg} : \underline{M} \mapsto \underline{M}-\mathbf{alg}$ *induces a contravariant equivalence between monads with monad morphisms and monadic categories with concrete functors.*

Metacategorical remark 1.1.9 *Let* $\mathbf{Mon}\, \mathcal{C}$ *be the full submetacategory of* $\mathbf{Con}\mathcal{C}$ *made of monadic categories. Now we can express the above correspondence using the metacategorical language: there is a contravariant equaivalence between the metacategories* $\mathbf{Monad}\, \mathcal{C}$ *and* $\mathbf{Mon}\, \mathcal{C}$*.*

In chapter 4.1 we will recall more results on monads with connection to free algebras.

1.2 Limits of Concrete Categories

In this section we will work with diagrams of concrete categories over a base category $\mathcal{C}$. By a (small) *diagram D of concrete categories* we mean a functor D form some (small) category $\mathcal{D}$ to a metacategory of concrete categories over $\mathcal{C}$ and concrete functors.

Metacategorical remark 1.2.1 *Since there is an obvious "forgetful functor" to the metacategory* $\mathbf{CAT}$ *of categories given by* $(\mathcal{A}, U) \mapsto \mathcal{A}$*, we may consider* $\mathbf{Con}\,\mathcal{C}$ *to be a submetacategory of the "slice-metacategory"* $\mathbf{CAT}/\mathcal{C}$ *containing those pairs* $(\mathcal{A}, U)$ *where* U *is faithful. Since* $\mathbf{CAT}$ *is complete w.r.t. small diagrams, so is* $\mathbf{CAT}/\mathcal{C}$ *(Remark A.2.2). We will show that* $\mathbf{Con}\,\mathcal{C}$ *is complete, too.*

Recall the assignment $\mathcal{D} \mapsto \mathcal{D}^*$ adding a new terminal object to a category - see Note 1.

Lemma 1.2.2 *Let* $\mathcal{D}$ *be a small category and* $D : \mathcal{D}^* \to \mathbf{CAT}$ *a diagram with* $D(1) = \mathcal{C}$ *and let* $D(t_d) = U_d$ *be faithful for every* $d \in \mathrm{Ob}\mathcal{D}$*. Let* $\mathcal{A}$ *and* $\{L_d : \mathcal{A} \to Dd | d \in \mathrm{Ob}\mathcal{D}^*\}$ *be the limit and the limit cone, respectively, of* D*. Then* L_1 *is faithful.*

Proof: Let $\alpha, \beta : X \to Y$ be a pair of $\mathcal{A}$-morphisms with $L_1(\alpha) = L_1(\beta)$. Let d be an object in $\mathcal{D}$. Since $L_1 = U_d \circ L_d$ and U_d is faithful, we have $L_d(\alpha) = L_d(\beta)$. Consider the category $\mathbf{2}$ and define a functor $M_d : \mathbf{2} \to Dd$ such that $M_d(\iota) = L_d(\alpha)$. The collection of all functors M_j for $j \in \mathrm{Ob}\mathcal{D}$ now forms the D-compatible cone. Certainly, both assignments $R_\alpha : \iota \mapsto \alpha$ and $R_\beta : \iota \mapsto \beta$ define the functors $\mathbf{2} \to \mathcal{A}$ satisfying $L_d \circ R_- = M_d$. However, the factorization over the limit cone is unique, hence $\alpha = \beta$. □

Corollary 1.2.3 *The limits in* $\mathbf{Con}\,\mathcal{C}$ *exist.*

Proof: Due to the Metacategorical remark 1.2.1 we have the limits in $\mathbf{CAT}/\mathcal{C}$ in the form as described in Remark A.2.2. Since $\mathbf{Con}\,\mathcal{C}$ contains those pairs $(\mathcal{A}, U) \in \mathrm{Ob}(\mathbf{CAT}/\mathcal{C})$ where the structure arrow is faithful, by Lemma 1.2.2, the terminal-object component of the limit cone is faithful as well, i.e., the limit in $\mathbf{CAT}/\mathcal{C}$ of the diagram in $\mathbf{Con}\,\mathcal{C}$ is in $\mathbf{Con}\,\mathcal{C}$. □

Remark 1.2.4 *In particular, there exist products in* $\mathbf{Con}\mathcal{C}$*. Given two concrete categories* $\mathcal{A}$*,* $\mathcal{B}$*, their concrete product will be denoted by* $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$*, and can be obtained as a fibre-wise product, i.e.,* $\mathbf{Fib}\,(\mathcal{A} \times_{\mathcal{C}} \mathcal{B})(A) = \mathbf{Fib}\,(\mathcal{A})(A) \times \mathbf{Fib}\,(\mathcal{B})(A)$*. The products of an infinite number of concrete categories can be described analogously. The terminal object, i.e., the product over the empty index set, is the base category itself and the terminal morphisms are forgetful functors.*

Metacategorical remark 1.2.5 *It is easy to see, that, moreover, the intersection of a large collection of subcategories exists. Clearly, it contains all objects and morphisms, which occur in all categories of collection.*

Lemma 1.2.6 *A limit of a (possibly large) diagram of Beck categories is Beck.*

Proof: Let $(\mathcal{A}, U)$ be the concrete limit of a diagram $D : \mathcal{D} \to \mathbf{Con}\,\mathcal{C}$ with $Dd = (\mathcal{A}_d, U_d)$ being a Beck category for every object d and let L_d, $d \in \mathcal{D}$ be the limit cone. To show $\mathcal{A}$ is Beck, consider a category $\mathcal{Q}$ and a diagram $G : \mathcal{Q} \to \mathcal{A}$ such that there is a limit $A = \lim UG$ in $\mathcal{C}$ and a limit cone λ_q, $q \in \mathcal{Q}$. Let $\mathcal{G} = \mathrm{Im}G$. Consider the category $\mathcal{G}_*$ by adding obtained from $\mathcal{G}$ by adding a new initial object - see Note 1. We define $Z : \mathcal{G}_* \to \mathcal{C}$ by $Z_{|\mathcal{G}} = U_{|\mathcal{G}}$ and by $Z(\iota_q) = \lambda_q$ which makes $\mathcal{G}_*$ a concrete category. Let $d \in \mathcal{D}$. Since $(\mathcal{A}_d, U_d)$ is Beck and $UG = U_d L_d G$, the forgetful functor U_d creates the limit A_d and the limit cone λ_q^d of the diagram $L_d G$. Then there is a concrete functor $L'_d : \mathcal{G}_* \to (\mathcal{A}_d, U_d)$, such that $L'_{d|\mathcal{G}} = L_{d|\mathcal{G}}$ and $L'_d(\iota_q) = \lambda_q^d$. Now, given another object $d' \in \mathcal{D}$, we get the functor $L'_{d'} : \mathcal{G}_* \to Dd'$ and due to the uniqueness of creation of limits by each U_-, for each $\mathcal{D}$-morphism $f : d \to d'$, the compatibility condition $Df \circ L'_d = L'_{d'}$ is satisfied. Hence, L'_- forms a D-compatible cone, therefore, there is a unique functor $P : \mathcal{G}_* \to \mathcal{A}$ with $L_- \circ P = L'_-$. Thus, $P(0)$ is the limit and $P\iota_q, q \in \mathcal{Q}$ is the limit cone for G.

The proof of creation of absolute coequalizers follows the same line. $\square$

1.2.1 Algebras for a Concrete Diagram

Definition 1.2.1 *A limit, denoted by* $\mathbf{Alg}\,D$*, of a (possibly large) diagram* $D : \mathcal{D} \to \mathbf{Con}\,\mathcal{C}$ *with every object mapped on an f-algebraic category, will be called an* l-algebraic category*. If the category* $\mathcal{D}$ *has a weakly initial object, we say* $\mathbf{Alg}\,D$ *is* homogenous*.*

Remark 1.2.7 *Let* $D : \mathcal{D} \to \mathbf{End}\mathcal{C}$ *be a diagram with* $D(x) = \mathbf{Alg}F_x$*, where* $F_x : \mathcal{C} \to \mathcal{C}$ *is a functor for every object* $x \in \mathcal{D}$*. The objects of category* $\mathbf{Alg}\,D$ *may be seen as* algebras for the diagram D*, i.e., the collections of* $\mathcal{C}$*-morphisms* $\{\alpha_x : F_x A \to A | x \in \mathcal{D}\}$ *satisfying, for each* $f : x \to y$ *in* $\mathcal{D}$*, the* D*-compatibility condition* $D(f)(A, \alpha_x) = (A, \alpha_y)$*. The morphisms in* $\mathbf{Alg}\,D$ *are the morphisms of algebras for each* x*, i.e.,* $\phi : (A, \alpha) \to (B, \beta)$ *is a morphism if* $\phi \circ \alpha_x = \beta_x \circ F_x(\phi)$ *for every* $x \in \mathrm{Ob}\mathcal{D}$*.*

The concrete product $\mathbf{Alg}\,F \times_{\mathcal{C}} \mathbf{Alg}\,G$ for some endofunctors F, G on $\mathcal{C}$ is an example of l-algebraic category. Its objects are $\mathcal{C}$-objects with two structure arrows - for both F and G. However, if $\mathcal{C}$ has coproducts, then the category **Alg** $F \times_{\mathcal{C}}$ **Alg** G is isomorphic to **Alg** $(F+G)$. Generally, if $\mathcal{C}$ is cocomplete and the diagram in the definition maps all morphisms on "f-algebraic functors", i.e., those concrete functors which are **Alg** -images of natural transformations, then, due to Metacategorical remark 1.1.5, the obtained category is an f-algebraic category. Therefore we will focus especially on the diagrams which do not factorize over functor **Alg** .

Here we show the basic property of each l-algebraic category.

Lemma 1.2.8 *Every l-algebraic category is Beck.*

Proof: The statement is a direct consequence of Lemma 1.2.6 and Proposition 1.1.6. □

In Part II, Example 5.2.1 we show a Beck category which is not an l-algebraic category. Hence, the above implication cannot be reversed.

1.2.2 Algebras for a Diagram of Monads

If we substitute the monadicity for f-algebraicity in the concept above, we get the category of algebras for a diagram of monads. In fact, a concrete diagram D of monadic categories may be, due to Lemma 1.1.8, seen as $D = -\mathbf{alg} \circ D'$ for some diagram $D' : \mathcal{D} \to (\mathbf{Monad}\, \mathcal{C})^{op}$. Hence it makes sense to define algebras for a diagram of monads directly by the diagram $\mathcal{D} \to \mathbf{Monad}\, \mathcal{C}$ as follows.

Definition 1.2.2 (Algebra for a diagram of monads) *Let $D : \mathcal{D} \to \mathbf{Monad}\mathcal{C}$ be a diagram and $D(x) = (M_x, \eta^x, \mu^x)$ for every object $x \in \mathcal{D}$. We define a category $D-$**alg** of* algebras for a diagram D of monads *whose objects are collections of $\mathcal{C}$-morphisms $\{\alpha_x : M_x A \to A | x \in \mathcal{D}\}$ where α_x is in $D(x)$–**alg** for every object $x \in \mathcal{D}$ and for each $f : x \to y$ in $\mathcal{D}$ the D-compatibility condition $\alpha_y \circ D(f)_A = \alpha_x$ is satisfied. The morphisms in $D-$**alg** are the morphisms of algebras for each x, i.e., $\phi : (A, \alpha) \to (B, \beta)$ is a morphism if $\phi \circ \alpha_x = \beta_x \circ M_x(\phi)$ for every x.*

To avoid the use of illegitimate notions, one may replace **Monad** $\mathcal{C}$ by its suitable subcategory. The category of algebras for a diagram of monads was introduced by Kelly in [20]. The reason was the investigation of an *algebraic colimit of monad digram D*. This colimit is a monad $\underline{K}$, such that $\underline{K}-\mathbf{alg} \cong_{\mathcal{C}} D-\mathbf{alg}$. Kelly's result from this research will be very important in Part II.

1.2.3 Polymeric Categories

In the text bellow we will introduce an important class of l-algebraic categories. The way they are obtained will be later used analogously to define varieties. Let F be an endofunctor on a category $\mathcal{C}$.

At first, we introduce the notion of *polymer* (see [29]).

Definition 1.2.3 *Let (A, α) be an F-algebra. Given $n \in \omega$, a n-*polymer *of an algebra (A, α) is the morphism $\alpha^{(n)} : F^n(A) \to A$ in $\mathcal{C}$ defined recursively:*

$$\alpha^{(0)} = \mathrm{id}_A, \alpha^{(n+1)} = \alpha \circ F\alpha^{(n)}.$$

Remark 1.2.9 *The assignment $(A, \alpha) \mapsto (A, \alpha^{(n)})$ clearly defines a functor $P_n : \mathbf{Alg}\, F \to \mathbf{Alg}\, F^n$.*

Now we have a sequence of functor functors F^n together with functors between the categories of their algebras. This lead us to the definition:

Definition 1.2.4 *Let n be an ordinal and G be an endofunctor on $\mathcal{C}$. A natural transformation $\phi : G \to F^n$ is called* n-ary polymeric G-term *in category of F-algebras. A pair $(\phi, \psi)_p$ of polymeric G-terms of arities m,n, respectively, is called* polymeric identity *of* arity-pair (m, n) *with* domain *G)(*polymeric G-identity *in short). If $m = n$, we say that $(\phi, \psi)_p$ has an* arity *of n. Moreover, for an F-algebra (A, α), we define*

$$(A, \alpha) \models (\phi, \psi)_p \quad \overset{def}{\Leftrightarrow} \quad \alpha^{(m)} \circ \phi_A = \alpha^{(n)} \circ \psi_A,$$

and we say that the F-algebra (A, α) satisfies *the polymeric G-identity $(\phi, \psi)_p$.*

For a class $\mathcal{I}$ of polymeric identities we define a polymeric variety of F-algebras *as the class of all algebras satisfying all $(\phi, \psi)_p \in \mathcal{I}$. The corresponding full subcategory of $\mathbf{Alg}\, F$ is denoted by $\mathbf{Alg}\,(F, \mathcal{I})$. If $\mathcal{I}$ is a set or singleton, we say that $\mathbf{Alg}\,(F, \mathcal{I})$ is set-induced or single-induced, respectively. A category concretely isomorphic to $\mathbf{Alg}\,(F, \mathcal{I})$ for some F and $\mathcal{I}$ will be called* polymeric*.*

Lemma 1.2.10 *Every polymeric category is l-algebraic and homogenous.*

Proof: It is easy to see that a single-induced polymeric variety $\mathbf{Alg}(F, (\phi, \psi)_p)$, where $(\phi, \psi)_p$ is (m, n)-ary polymeric G-identity, is an equalizer of the pair of concrete functors obtained by the following compositions:

$$\begin{tikzcd}
 & \mathbf{Alg}\, F^m \arrow[dr, "\mathbf{Alg}\, \phi"] & \\
\mathbf{Alg}\, F \arrow[ur, "P_m"] \arrow[dr, "P_n"'] & & \mathbf{Alg}\, G \\
 & \mathbf{Alg}\, F^n. \arrow[ur, "\mathbf{Alg}\, \psi"'] &
\end{tikzcd}$$

A polymeric variety induced by a class of polymeric identities is an intersection of polymeric varieties induced by single polymeric identities, hence an l-algebraic category. □

The following lemma shows the presentation of monadic categories by polymeric identities.

Lemma 1.2.11 *Every monadic category is polymeric.*

Proof: Given a monad $\underline{M} = (M, \eta, \mu)$, the Eilenberg-Moore category $\underline{M}-$**alg** is a polymeric variety of M-algebras induced by polymeric identities $(\eta, \mathrm{id}_{\mathrm{Id}})_p$, $(\mathrm{id}_{\underline{M}^2}, \mu)_p$ of domains Id, $\underline{M}^2$, respectively, and arity-pairs $(1,0)$, $(2,1)$, respectively.

$$\begin{tikzcd} \mathrm{Id} \arrow[r, "\mathrm{id}"] \arrow[dr, "\eta"'] & \mathcal{M}^0 \\ & \mathcal{M}^1, \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}^2 \arrow[r, "\mu"] \arrow[dr, "\mathrm{id}_{\mathcal{M}^2}"'] & \mathcal{M}^1 \\ & \mathcal{M}^2. \end{tikzcd}$$

□

Thus we have a conclusion:

Corollary 1.2.12 *Every monadic category is l-algebraic.*

1.3 Algebraic Categories

1.3.1 Algebras for a Type

Another kind of algebras on a general category was developed by F. E. Linton (see [25]) and J. Rosický in [33]. We start with the definition.

Definition 1.3.1 (Algebra for a type) *Given a category $\mathcal{C}$ and a class Ω of* operation symbols*, a* type *on $\mathcal{C}$ with the domain Ω is a mapping*

$$t : \Omega \to (\mathrm{Ob}\mathcal{C})^2.$$

Given $\sigma \in \Omega$, $t(\sigma) = (t_0(\sigma), t_1(\sigma))$ is called an arity-pair *for ω. If Ω is a set, we say that the type is* bounded.

An algebra for a type *t (a t-algebra in short) is a pair (A, S) made up of a $\mathcal{C}$-object A and a mapping $S : \Omega \to \mathrm{Mor}\mathcal{S}et$ such that $S(\sigma) : \hom(t_0(\sigma), A) \to \hom(t_1(\sigma), A)$ for each $\sigma \in \Omega$. A morphism of t-algebras $f : (A, S) \to (B, T)$ is a morphism $f : A \to B$ such that, for every $\sigma \in \Omega$, the following diagram commutes:*

$$\begin{tikzcd} \hom(t_0(\sigma), A) \arrow[r, "S(\sigma)"] \arrow[d, "\hom(t_0(\sigma), f)"] & \hom(t_1(\sigma), A) \arrow[d, "\hom(t_1(\sigma), f)"] \\ \hom(t_0(\sigma), B) \arrow[r, "T(\sigma)"] & \hom(t_1(\sigma), B), \end{tikzcd}$$

i.e., $f \circ S(\sigma)(m) = T(\sigma)(f \circ m)$ *for every* $m : t_0(\sigma) \to A$*. The metacategory of* t*-algebras and their morphisms will be denoted by* $t-\mathbf{alg}$.

1.3.2 Equational Theories

Type algebras enable a variety-like treatment. Although the varieties will be studied in the following chapter, now we will show how they are defined for type algebras. Let t be a type on a category $\mathcal{C}$ with the domain Ω for the rest of this section.

Definition 1.3.2 *The* terms of type t *(or* t-terms*) and their* arity-pairs *will be defined recursively:*

1. σ *is the term of arity-pair* $t(\sigma)$ *for every* $\sigma \in \Omega$*.*
2. *For every* $\mathcal{C}$*-morphism* f*, there is a term* $\overline{f}$ *of arity-pair* $(\mathrm{cod}(f), \mathrm{dom}(f))$ *- such a term will be called* morphism-constant*.*
3. *Given terms* q*,* p *of arity-pairs* (Z, Y) *and* (Y, X)*, respectively, by its composition* $q \cdot p$ *we denote another term of arity-pair* (Z, X)*.*
4. *Pairs of terms* $\overline{f} \cdot \overline{g}$ *and* $\overline{g \circ f}$ *are considered to be equal for every pair of composable morphisms* g, f*.*
5. *Every term is formed by a finite sequence of the four steps above.*

The class of all terms of type t *will be denoted by* $\mathcal{T}(t)$*. By* (X, Y)-ary t-equation *we mean a pair of* t*-terms of the arity-pair* (X, Y)*.*

Remark 1.3.1 *Given a type* $t : \Omega \to (\mathrm{Ob}\mathcal{C})^2$*,* $\mathcal{T}(t)$ *can be defined as a free partial algebra with binary operation* $\cdot$ *and a class of morphism-constants with generators in the class* Ω *with the operations defined whenever the types of elements are compatible, i.e. composable in the sense of domain-codomain pairs. Hence there is* universal inclusion $h : \Omega \to \mathcal{T}(t)$ *satisfying the following property. For every partial algebra* A *with these operations and with a "type":* $A \to (\mathrm{Ob}\mathcal{C})^2$ *and for every type-preserving mapping* $m : \Omega \to A$*, there exists a unique algebra-homomorphism* $\underline{m} : \mathcal{T}(t) \to A$ *such that*

$$\underline{m} \circ h_t = m.$$

Given an object A *in* $\mathcal{C}$*, we may get such a partial algebra on the class* $\mathrm{Mor}\mathcal{C}$ *with the usual composition and morphism-constants of the form* $\hom_{\mathcal{C}}(f, A)$*. Hence, for each algebra* (A, S)*, the mapping* $S : \Omega \to \mathrm{Mor}\mathcal{C}$ *induces a partial-algebra homomorphism* $\underline{S} : \mathcal{T}(t) \to \mathrm{Mor}\mathcal{C}$ *such that* $\underline{S} \circ h = S$*. This property will be used later on.*

We will define such a mapping more explicitly.

Definition 1.3.3 *Let (A, S) be a t-algebra. An* evaluation of term *on (A, S) is an extension of the mapping S on the class of all t-terms $\underline{S} : \mathcal{T}(t) \to$* Mor$\mathcal{S}et$ *given by the following recursion:*

- $\underline{S}(\sigma) = S(\sigma)$ *for every* $\sigma \in \Omega$.
- *For every $\mathcal{C}$-morphism f,* $\underline{S}(\overline{f}) = \hom(f, A)$.
- *For the composition $q{\cdot}p$ of terms q and p, we define* $\underline{S}(q{\cdot}p) = \underline{S}(q) \circ \underline{S}(p)$.

Remark 1.3.2 *Note that, for each term p of arity-pair (X, Y), the evaluation on an algebra (A, S) defines a mapping* $\underline{S}(p) : \hom(X, A) \to \hom(Y, A)$.

Now we can define a variety-like class of type-algebras ([33]).

Definition 1.3.4 *Let (A, S) be a t-algebra and (p, q) be a t-equation. We say that (A, S)* satisfies *the (p, q) if*

$$\underline{S}(p) = \underline{S}(q).$$

Then we write

$$(A, S) \models (p, q).$$

For a class $\mathcal{E}$ of t-equations, the pair $(t, \mathcal{E})$ is called an equational theory *over $\mathcal{C}$ and we define an* algebraic metaclass of t-algebras *as the class of all algebras satisfying all equations $(p, s) \in \mathcal{E}$. The corresponding full submetacategory of $t-$**alg** will be denoted by $(t, \mathcal{E})-$**alg**. A (meta)category will be called* algebraic, *if it is isomorphic to $(t, \mathcal{E})-$**alg** for some equational theory $(t, \mathcal{E})$. Two theories are said to be* algebraically equivalent *if they induce the same classes of algebras.*

Note 2 *We point out the incompatibility of terminology: the notion of* algebraic category *has a different meaning in [7].*

Recall the usual inference rules for equational logic: reflexivity, symmetry, transitivity and composition. Another inference rule of the composition of morphism-constants is implicitly given by property 4. in the Definition 1.3.2. Theory closed under these rules will be called *closed*. Then, for each equational theory, there exists its *closure* - the smallest of its closed superclasses of equations. Clearly, every equational theory is algebraically equivalent to its closure.

Definition 1.3.5 *Let $(t, \mathcal{E})$ and $(t', \mathcal{E}')$ be closed equational theories with the domains of types Ω, Ω', and universal inclusions h, h',respectively. An* interpretation *of $(t, \mathcal{E}) \to (t', \mathcal{E}')$ is a mapping $i : \Omega \to \Omega'$ such that the induced morphism $\mathcal{T}(i) = \underline{h' \circ i} : \mathcal{T}(t) \to \mathcal{T}(t')$ between the classes of terms yields a mapping $(\mathcal{T}(i), \mathcal{T}(i))_{|\mathcal{E}} : \mathcal{E} \to \mathcal{E}'$.*

Since the composition of interpretations is obviously an interpretation, we get a metacategory $\mathbf{Eq}\,\mathcal{C}$ of closed equational theories and their interpretations. The following relationship is derived from the results in [33].

Lemma 1.3.3 *The assignment $(t, \mathcal{E}) \mapsto (t, \mathcal{E})-\mathbf{alg}$ can be extended to a contravariant functor $-\mathbf{alg}^e : \mathbf{Eq}\,\mathcal{C} \to \mathbf{Con}\,\mathcal{C}$*

Proof: Let $i : (t, \mathcal{E}) \to (t', \mathcal{E}')$ be an interpretation of closed equational theories. Then there is a concrete functor $i-\mathbf{alg} : (t', \mathcal{E}')-\mathbf{alg} \to (t, \mathcal{E})-\mathbf{alg}$ given by the assignment $(A, H) \mapsto (A, H \circ i)$, which preserves the satisfaction of equations. The rest is clear. $\square$

Definition 1.3.6 *A concrete functor between algebraic categories will be called* algebraically concrete *if it is isomorphic to $i-\mathbf{alg}^e$ for some interpretation of equational theories. A diagram $D : \mathcal{D} \to \mathbf{Con}\,\mathcal{C}$ is called algebraically concrete if it factorizes over $-\mathbf{alg}^e$.*

Canonical Theories

Linton in [25] and Rosický in [33] studied algebraic categories of a certain kind which play the role of the universal algebraic categories over a concrete (meta)category. Although they will not be of our interest, to make the overeview complete we state the definitions and main results.

Definition 1.3.7 *Let $(\mathcal{A}, U)$ be a concrete category. By an* algebraic metaclosure *of $(\mathcal{A}, U)$, we mean an algebraic metacategory $(\mathcal{D}, Z) = (t_U, \mathcal{E}_U)-\mathbf{alg}$ together with a concrete functor $T : (\mathcal{A}, U) \to (\mathcal{D}, Z)$, such that for every algebraic metacategory $(t', \mathcal{E}')-\mathbf{alg}$ and every functor $S : (\mathcal{A}, U) \to (t', \mathcal{E}')-\mathbf{alg}$, there is a unique interpretation $e : (t', \mathcal{E}') \to (t, \mathcal{E}_U)$ between the corresponding closed equational theories such that $S = e-\mathbf{alg} \circ \mathcal{T}$.*

If $(\mathcal{D}, Z)$ is a category, we call it an algebraic closure *of $(\mathcal{A}, U)$.*

The following description of algebraic metaclosure is from [33].

Definition 1.3.8 *Let $(\mathcal{A}, U)$ be a concrete category. Let*

$$\Omega_{[X,Y]} = \mathbf{Nat}(\hom(X, -)U, \hom(Y, -)U)$$

be the class of operation symbols of arity-pair (X, Y). Then we have a canonical type t_U *defined on $\Omega = \bigcup_{X,Y \in \mathrm{Ob}\mathcal{C}} \Omega_{[X,Y]}$. Let $\mathcal{E}_0$ consist of equations*

1. $(f, \hom(f,-)U)$ *for every morphism f in $\mathcal{C}$*

2. $(\sigma \cdot \rho, \sigma \circ \rho)$ *for every composable pair of natural transformations $\sigma, \rho \in \Omega$.*

Let $\mathcal{E}_U$ be a closure of $\mathcal{E}_0$. The theory $(t_U, \mathcal{E}_U)$ is called U-canonical *and the resulting category $(t_U, \mathcal{E}_U)-$**alg** *will be denoted by $U-$**alg**.*

The following statement is proved in [33].

Theorem 1.3.4 (Rosický's theorem) *Let $(\mathcal{A}, U)$ be a concrete category. Then $U-$**alg** is the algebraic metaclosure of $(\mathcal{A}, U)$.*

Linton proved in [25] the following property:

Theorem 1.3.5 (Linton's theorem) *Let $(\mathcal{A}, U)$ be a concrete category with a pointwise codensity monad for U. Then the category $U-$**alg** is monadic.*

For an overview of facts over codensity monads, see Appendix B, B.2.2. The concrete categories with codensity monads will be studied in Chapter 5.

1.3.3 Functorial Theories

Object-Identical Functors

Definition 1.3.9 *Let $\mathcal{P}, \mathcal{Q}$ be categories built on the same class of objects. A functor $S : \mathcal{P} \to \mathcal{Q}$ is called* object-identical *if $(\forall X \in \mathrm{Ob}\mathcal{C}) S(X) = X$. Object-identical functors will be denoted by $\mathcal{P} \Rightarrow \mathcal{Q}$.*

Remark 1.3.6 *Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be categories with the same class of objects and let there be functors $S : \mathcal{P} \to \mathcal{Q}$, $T : \mathcal{Q} \to \mathcal{R}$. If any two functors of S, T, TS are object-identical, then so is the third.*

Metacategorical remark 1.3.7 *We will be interested in object-identical functors with the values in a metacategory. If the objects form a class, it just means that we work with a not necessarily locally small category. Given a category $\mathcal{P}$, the metaclass of $\mathcal{P}$-object-identical functors (object-identical functors defined on $\mathcal{P}$) and object-identical functors as morphisms will be denoted by $\mathcal{P} \Downarrow$* **METACAT** *and referred to as* hypercategory of $\mathcal{P}$-object-identical functors. *Since there is an obvious "forgetful functor" to the hypercategory* **METACAT** *of metacategories given by $(\mathcal{Q}, S) \mapsto \mathcal{Q}$, we may consider $\mathcal{P} \Downarrow$* **METACAT** *to be a submetacategory of the "co-slice-metacategory" $\mathcal{P}\backslash$***METACAT** *containing those pairs $(\mathcal{Q}, S)$ where S is object-identical. Since* **METACAT** *is cocomplete w.r.t. small diagrams, so is $\mathcal{P}\backslash$***METACAT** *due to Metacategorical Remark A.2.2 and duality. It is straightforward to show that $\mathcal{P} \Downarrow$* **METACAT** *has colimits, too.*

Metacategorical remark 1.3.8 *Moreover, even some large colimits, such as those of* Ord*-directed chain, exist. Consider an ordinal chain of* $S_{i,j} : (\mathcal{Q}_i, T_i) \Rightarrow (\mathcal{Q}_j, T_j)$ *object-identical functors with each* $S_{i,j}$*,* $i \leq j$*, full. Then there is a category* $\mathcal{Q}$ *with* $\mathrm{Ob}\mathcal{Q} = \mathrm{Ob}\mathcal{P}$ *and, for objects* X, Y*,*

$$\hom_{\mathcal{Q}}(X, Y) = \coprod_{i \in \mathrm{Ord}} \hom_{\mathcal{Q}_i}(X, Y)/\sim$$

where $\sim$ *is the equivalence generated by* $\rho_i \sim \sigma_j \Leftrightarrow (\exists n \in \mathrm{Ord}) S_{i,n}(\rho) = S_{j,n}(\sigma)$*. The composition s given simply by* $[\sigma] \circ [\tau] = [\sigma \circ \tau]$*. Clearly, there are functors* $S'_i : \mathcal{Q}_i \to \mathcal{Q}$ *such that* $S'_j S_{i,j} = S'_i$ *for* $i \leq j$*. Let* $T = S_0 T_0$*. Then* $(\mathcal{Q}, T)$ *is the colimit of the corresponding chain as we show bellow. Let* $U : \mathcal{P} \Rightarrow \mathcal{R}$ *be a functor together with a collection of object-identical functors* $V_i : (\mathcal{Q}_i, T_i) \Rightarrow (\mathcal{R}, U)$ *such that* $V_j S_{i,j} = V_i$ *for* $i \leq j$*. Then there is a unique functor* $Z : \mathcal{Q} \Rightarrow \mathcal{R}$ *assigning, to each* $[\sigma]$*,* $\sigma \in \mathrm{Mor}\mathcal{Q}_i$*, the morphism* $V_i(\sigma)$*. The correctness is easy to prove. Hence* $ZS'_i = V_i$ *for every* i *and* $(\mathcal{Q}, T)$ *is the colimit.*

Functorial Theories

In paper [25], F. E. Linton studied various categorical representations of category $U-\mathbf{alg}$. We will extend his approach to an arbitrary theory. We start with the notion of clone of operations - see [25].

Remark 1.3.9 *Let* $T : \mathcal{P} \to \mathcal{Q}$ *be a functor. All hom-functors will be considered on category* $\mathcal{Q}$*.*

A clone of operations on T *is the category* $\mathbf{Cl}\, T$ *with* $\mathrm{Ob}(\mathbf{Cl}\, T) = \mathrm{Ob}\mathcal{P}$ *with each morphism* $X \to Y$ *in* $\mathbf{Cl}\, T$ *being a natural transformation*

$$\hom(X, -) \circ T \to \hom(Y, -) \circ T.$$

Then there is an obvious functor

$$\exp_T : \mathcal{Q}^{op} \Rightarrow \mathbf{Cl}\, T$$

given by $\exp_T(f) = \hom(f, -)T$*.*

Given an object A *in* $\mathcal{Q}$*, we use the notation* $\mathbf{Cl}\, A = \mathbf{Cl}\, C_A$*,* $\exp_A = \exp_{C_A}$ *(here* C_A *is the constant functor defined on* $\mathcal{P}$*). There is a functor* $\Sigma_A : \mathbf{Cl}\, A \to \mathcal{S}et$ *given by an assignment* $X \mapsto \hom(X, A)$ *on objects and "identity" on morphisms since each morphism in* $\mathbf{Cl}\, A$ *is actually a mapping* $\hom(X, A) \to \hom(Y, A)$*. Then the functor* Σ_A *clearly satisfies*

$$\Sigma_A \circ \exp_A = \hom(-, A). \tag{1.3}$$

Now let $\mathcal{C}$ be a fixed category.

Definition 1.3.10 *Let* $\mathcal{Q}$ *be a metacategory with* $\mathrm{Ob}\mathcal{Q} = \mathrm{Ob}\mathcal{C}$. *A pair* $(\mathcal{Q}, T)$ *with*

$$T : \mathcal{C}^{op} \Rightarrow \mathcal{Q}$$

is called functorial theory *over the* base category $\mathcal{C}$ *with the* type-carrier $\mathcal{Q}$. *The hypercategory*

$$\mathbf{Th}\,\mathcal{C} = \mathcal{C}^{op} \Downarrow \mathbf{METACAT}$$

will be called hypercategory of functorial theories over $\mathcal{C}$.

From now on, $\mathbf{Th}\,\mathcal{C}$ and the codomains of its objects will be referred to as categories if we do not need to emphasize their actual size.

According to the Remark 1.3.9, for each object A in $\mathcal{C}$, there is a functorial theory $(\mathbf{Cl}\,A, \exp_A)$, which we will call *A-constant theory.* Now we can define the notion of algebra.

Definition 1.3.11 (Algebra for a functorial theory)
Let $(\mathcal{Q}, T)$ *be a functorial theory over* $\mathcal{C}$. *A pair* (A, G) *consisting of a* $\mathcal{C}$*-object* A *and a morphism* $G : (\mathcal{Q}, T) \to (\mathbf{Cl}\,A, \exp_A)$ *of functorial theories is called a* $(\mathcal{Q}, T)$-algebra. *Given* $(\mathcal{Q}, T)$*-algebras* $(A, G), (B, H)$, *a morphism* $f : A \to B$ *is called* $(\mathcal{Q}, T)$-algebra morphism *if there is a natural transformation* $f^* : \Sigma_A G \to \Sigma_B H$ *such that*

$$f^* T = \hom_{\mathcal{C}}(-, f).$$

The metacategory of $(\mathcal{Q}, T)$*-algebras and their morphisms will be denoted by* $(\mathcal{Q}, T)-\mathbf{alg}$.

Remark 1.3.10 Coalgebras for functorial theories *are defined as algebras for functorial theories over a dual category. Hence we need* co-theories *of the form* $\mathcal{C} \Rightarrow \mathcal{Q}$ *to define coalgebras.*

The following property is a generalization of Linton's observation.

Proposition 1.3.11 *Given a functorial theory* $T : \mathcal{C}^{op} \Rightarrow \mathcal{Q}$, *then* $(\mathcal{Q}, T)-\mathbf{alg}$ *is a pullback of the diagram*

$$\begin{tikzcd}
(\mathcal{Q}, T)-\mathbf{alg} \arrow[r, dashed, "J"] \arrow[d, dashed, "Z"] & \hom(\mathcal{Q}, \mathcal{S}et) \arrow[d, "\hom(T, \mathcal{S}et)"] \\
\mathcal{C}^{op} \arrow[r, "Y"] & \hom(\mathcal{C}, \mathcal{S}et)
\end{tikzcd}$$

where Y *is the Yoneda embedding.*

Proof: Let Z be the forgetful functor and J be given by the assignment $(A,G) \mapsto \Sigma_A G$ on algebras and by $f \mapsto f^*$ on morphisms. Then we have for every algebra

$$\begin{aligned}\hom(T,\mathcal{S}et) \circ J(A,G) &= \hom(T,\mathcal{S}et)(\Sigma_A G) = \Sigma_A GT \\ &= \Sigma_A \exp_A = \hom(-,A) = Y(A) = YZ(A,G)\end{aligned}$$

and

$$\begin{aligned}\hom(T,\mathcal{S}et) \circ J(f) &= \hom(T,\mathcal{S}et)(f^*) = f^*T \\ &= \hom(-,f) = Y(f) = YZ(f)\end{aligned}$$

on morphisms.

Now, suppose there is a metacategory $\mathcal{D}$ with functors $Z' : \mathcal{D} \to \mathcal{C}$ and $J' : \mathcal{D} \to \hom(\mathcal{Q},\mathcal{S}et)$ with $YZ' = \hom(T,\mathcal{S}et)J'$. For each object $D \in \mathcal{D}$, there is an object $A_D = Z'(D)$ and a functor $H_D = J'(D) : \mathcal{Q} \to \mathcal{S}et$ such that $H_D T = \hom(-,A_D)$, hence for each object $X \in \mathrm{Ob}\mathcal{C}$, we have $H_D T(X) = \hom(X,A_D)$. Therefore, each morphism $\sigma : X \to Y$ in $\mathcal{Q}$ is mapped on $H_D(\sigma) : \hom(X,A_D) \to \hom(Y,A_D)$, thus it is an Σ_{A_D}-image of $\sigma' : X \to Y$ in $\mathbf{Cl}\, A_D$. Therefore we have a functor $H'_D : \mathcal{Q} \to \mathbf{Cl}\, A_D$ such that $\Sigma_{A_D} \circ H'_D = H_D$. By analogy, we get the property for morphisms which implies that the assignment $(q : D \to D') \mapsto Z'q : (A_D, H_D) \to (A'_D, H'_D)$ defines a unique functor $M : \mathcal{D} \to (\mathcal{Q},T){-}\mathbf{alg}$. Clearly, it satisfies $JM = J'$ and $ZM = Z'$. Hence $(\mathcal{Q},T){-}\mathbf{alg}$ is the pullback of $(Y, \hom(T,\mathcal{S}et))$. $\square$

Remark 1.3.12 *There is an obvious contravariant functor* $-\mathbf{alg}^f : \mathbf{Th}\,\mathcal{C} \to \mathbf{Con}\,\mathcal{C}$ *given by*

$$(S : (\mathcal{Q},T) \to (\mathcal{Q}',T')) \mapsto (S{-}\mathbf{alg} : (\mathcal{Q}',T'){-}\mathbf{alg} \to (\mathcal{Q},T){-}\mathbf{alg})$$

with $S{-}\mathbf{alg}(A,G) = (A,GS)$.

Lemma 1.3.13 *The contravariant functor* $-\mathbf{alg}^f$ *turns colimits of theories into concrete limits of categories.*

Proof: Let $\{F_f : (\mathcal{Q}_i,T_i) \Rightarrow (\mathcal{Q}_j,T_j) | f : i \to j \in \mathcal{D}\}$ be a diagram of functorial theories. Let $(\mathcal{Q},T)$ be its colimit with the colimit cocone $S_i : \mathcal{Q}_i \Rightarrow \mathcal{Q}$ and let $Z_i : (\mathcal{Q}_j,T_j){-}\mathbf{alg} \to \mathcal{C}$ be the forgetful functor for each i. According to Proposition 1.3.11, $(\mathcal{Q},T){-}\mathbf{alg}^f$ is a pullback of

$$\mathcal{C} \xrightarrow{Y} \hom(\mathcal{C}^{op},\mathcal{S}et) \xleftarrow{\hom(T,\mathcal{S}et)} \hom(\mathcal{Q},\mathcal{S}et).$$

Since $\hom_{\mathbf{CAT}}(-,\mathcal{S}et) = \hom_{\mathbf{CAT}^{op}}(\mathcal{S}et,-)$, which preserves limits defined in $\mathbf{CAT}^{op}$, the functor $\hom(-,\mathcal{S}et)$ turns colimits into limits. Hence the diagram in question can be mapped contravariantly by $\hom(-,\mathcal{S}et)$ with the limit pair $(\hom(\mathcal{Q},\mathcal{S}et), \hom(T,\mathcal{S}et))$. Since the limits commute with limits, the pair $((\mathcal{Q},T){-}\mathbf{alg}, Z)$ in the pullback considered is the limit of pairs $((\mathcal{Q}_i,T_i){-}\mathbf{alg}, Z_i)$. $\square$

1.3.4 Equational vs. Functorial Theories

We will show that both functorial and equational concepts are equivalent. We still work with the fixed category $\mathcal{C}$. Throughout this section, we identify categories with the classes of their morphisms, i.e. in some cases, we consider their object-free presentation. Hence, e.g., instead of a functor $T : \mathcal{Q} \to \mathcal{S}et$ we may write a morphism-mapping $T : \mathrm{Mor}\mathcal{Q} \to \mathrm{Mor}\mathcal{S}et$ and vice versa.

Lemma 1.3.14 *For every closed equational theory $(t, \mathcal{E})$, there exists a functorial theory $(\mathcal{Q}, T)$ such that $(t, \mathcal{E})-\mathbf{alg} \cong_{\mathcal{C}} (\mathcal{Q}, T)-\mathbf{alg}$.*

Note 3 *As in the Remark 1.3.9, all hom-functors will be considered on the base category $\mathcal{C}$ unless explicitly denoted by a subscript such as $\hom_{\mathcal{Q}}$.*

Proof: Let $(t, \mathcal{E})$ be a closed equational theory with the type-domain Ω. Consider $\mathcal{T}(t)$ as a category with object-identical functor $V : \mathcal{C}^{op} \Rightarrow \mathcal{T}(t)$ embedding the morphism-constants. The required functorial theory will be obtained by prolonging the functor V. We will define a category $\mathcal{Q} = \mathcal{Q}_{(t,\mathcal{E})}$ on the class of objects $\mathrm{Ob}\mathcal{C}$. For a pair of $\mathcal{C}$-objects X, Y, let $\Omega_{[X,Y]}$, $\mathcal{E}_{[X,Y]}$, $\mathcal{T}(t)_{[X,Y]}$ be the classes of operation symbols, t-equations and t-terms of arity-pair (X, Y), respectively. Since $\mathcal{E}$ is clearly a type-preserving congruence on the partial algebra $\mathcal{T}(t)$, we may apply a factorization process obtaining the natural surjection $N : \mathcal{T}(t) \to \mathcal{T}(t)/\mathcal{E}$ and still having a mapping $t' : \mathcal{T}(t)/\mathcal{E} \to (\mathrm{Ob}\mathcal{C})^2$. Let $\mathrm{Mor}\mathcal{Q} = \mathcal{T}(t)/\mathcal{E}$ and $t' = (\mathrm{dom}, \mathrm{cod})$. Then $\hom_{\mathcal{Q}}(X, Y) = \mathcal{T}(t)_{[X,Y]}/\mathcal{E}_{[X,Y]}$. The composition is clearly induced by the factorization and identities are given by the morphism-constants $\overline{\mathrm{id}_X}$. Let $T = N \circ V$, then $T(f) = N(\overline{f})$ for every morphism f in $\mathcal{C}^{op}$.

We will show $(t, \mathcal{E}) - \mathbf{alg} \cong_{\mathcal{C}} (\mathcal{Q}, T) - \mathbf{alg}$. Let (A, G) be a t-algebra satisfying $\mathcal{E}$. Then the term evaluation $\underline{G} : \mathcal{T}(t) \to \mathrm{Mor}\mathcal{S}et$ considered as a functor $\mathcal{T}(t) \Rightarrow \mathcal{S}et$ factorizes over N and assigns, to each term τ of arity-pair (X, Y), a mapping $\underline{G}(\tau) : \hom(X, A) \to \hom(Y, A)$ which can be seen as a morphism $X \to Y$ in $\mathbf{Cl}\, A$. Hence, we have a functor $\underline{\underline{G}} : \mathcal{Q} \Rightarrow \mathbf{Cl}\, A$ such that $\Sigma_A \circ \underline{\underline{G}} \circ N = \underline{G}$. Therefore, $\underline{\underline{G}} : (\mathcal{Q}, T) \to (\mathbf{Cl}\, A, \exp_A)$ is a morphism in $\mathbf{Th}\, \mathcal{C}$, i.e. $K(A, G) = (A, \underline{\underline{G}})$ is a $(\mathcal{Q}, T)$-algebra.

$$
\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, Rightarrow, "V"] \arrow[d, Rightarrow, "T"'] \arrow[dr, Rightarrow, "\exp_A"' near end] & \mathcal{T}(t) \arrow[dl, Rightarrow, "N" near end] \arrow[drr, "\underline{G}"] & & \Omega \arrow[ll, "h"'] \arrow[d, "G"] \\
\mathcal{Q} \arrow[r, Rightarrow, "\underline{\underline{G}}"'] & \mathbf{Cl}\, A \arrow[rr, "\Sigma_A"] & & \mathcal{S}et
\end{tikzcd}
$$

Let $\phi : (A, G) \to (B, H)$ be a morphism in $(t, \mathcal{E})-\mathbf{alg}$. Then $\phi : A \to B$ is a morphism in $\mathcal{C}$ and, for every $\sigma \in \Omega$ of arity-pair (X, Y), we have

$\mathrm{hom}(Y,\phi) \circ G(\sigma) = H(\sigma) \circ \mathrm{hom}(X,\phi)$. Since $\mathcal{Q}$ is generated by the elements of Ω by compositions and morphism-constants, the diagram

$$\begin{tikzcd} \mathrm{hom}(X,A) \arrow[r, "\underline{G}(\tau)"] \arrow[d, "\mathrm{hom}(X,\phi)"] & \mathrm{hom}(Y,A) \arrow[d, "\mathrm{hom}(Y,\phi)"] \\ \mathrm{hom}(X,B) \arrow[r, "\underline{H}(\tau)"] & \mathrm{hom}(Y,B), \end{tikzcd}$$

commutes for every (X,Y)-ary t-term τ. Note that, for $f : X \to Y$ in $\mathcal{C}$, the morphism-constant $\overline{f} = T(f)$ is mapped by $\underline{G}$ onto the natural transformations $\mathrm{hom}(f,A) = \exp_A(f)$ and by $\underline{H}$ analogously.

Let $\phi^*_- = \mathrm{hom}(-,\phi)$. Since $(A,G),(A,H) \models \mathcal{E}$, we have the induced functors $\underline{\underline{G}} : \mathcal{Q} \to \mathbf{Cl}\,A$, $\underline{\underline{H}} : \mathcal{Q} \to \mathbf{Cl}\,B$ satisfying $\mathrm{hom}(W,A) = \Sigma_A\underline{\underline{G}}(W)$ and $\mathrm{hom}(W,B) = \Sigma_B\underline{\underline{H}}(W)$ for every object W in $\mathcal{C}$, and get the commutative diagram

$$\begin{tikzcd} \Sigma_A\underline{\underline{G}}(X) \arrow[r, "\underline{\underline{G}}(\tau)"] \arrow[d, "\phi^*_X"] & \Sigma_A\underline{\underline{G}}(Y) \arrow[d, "\phi^*_Y"] \\ \Sigma_B\underline{\underline{H}}(X) \arrow[r, "\underline{\underline{H}}(\tau)"] & \Sigma_B\underline{\underline{H}}(Y) \end{tikzcd}$$

hence we have a natural transformation $\phi^* : \Sigma_A\underline{\underline{G}} \to \Sigma_B\underline{\underline{H}}$ with $\phi^*T = \mathrm{hom}(-,\phi)$. Therefore, $\phi : (A,\underline{\underline{G}}) \to (B,\underline{\underline{H}})$ is a morphism of $(\mathcal{Q},T)$-algebras. We have defined a concrete functor $K : (t,\mathcal{E})-\mathbf{alg} \to (\mathcal{Q},T)-\mathbf{alg}$ and we will find its inverse.

Let (A,G) be a $(\mathcal{Q},T)$-algebra. Define $G' = \Sigma_A GNh$, the resulting algebra $K'(A,G) = (A,G')$ clearly satisfies $\mathcal{E}$ since $\underline{\underline{G'}} = \Sigma_A GN$ due to the universality of h. If there is a $(\mathcal{Q},T)$-algebra morphism $\phi : (A,G) \to (A,H)$, we have a natural transformation $\phi^* : \Sigma_A G \to \Sigma_B H$ with $\phi^*T = \mathrm{hom}(-,\phi)$. Then, for every $\sigma \in \Omega$, we have

$$\begin{aligned} \mathrm{hom}(Y,\phi) \circ G'(\sigma) &= \mathrm{hom}(Y,\phi) \circ \Sigma_A GNh(\sigma) \\ &= \Sigma_B HNh(\sigma) \circ \mathrm{hom}(X,\phi) = H'(\sigma) \circ \mathrm{hom}(X,\phi) \end{aligned}$$

hence the diagram

$$\begin{tikzcd} \mathrm{hom}(X,A) \arrow[r, "G'(\sigma)"] \arrow[d, "\mathrm{hom}(X,\phi)"] & \mathrm{hom}(Y,A) \arrow[d, "\mathrm{hom}(Y,\phi)"] \\ \mathrm{hom}(X,B) \arrow[r, "H'(\sigma)"] & \mathrm{hom}(Y,B), \end{tikzcd}$$

commutes and ϕ is the morphism of t-algebras $(A,G') \to (B,H')$. Therefore $K' : (\mathcal{Q},T)-\mathbf{alg} \to (t,\mathcal{E})-\mathbf{alg}$

We will show that the assignments K and K' are mutually inverse. Since, for a t-algebra (A, G), we have $(\underline{\underline{G}})' = \Sigma_A \underline{\underline{G}} N h = \underline{G} h = G$, we get $(\underline{\underline{G}})' = G$, thus $K'K = \mathrm{Id}_{(t,\mathcal{E})\text{–}\mathbf{alg}}$. Conversely, for every $(\mathcal{Q}, T)$-algebra (B, H), we have the functor $\underline{\underline{H'}} : \mathcal{Q} \to \mathbf{Cl}\, B$ satisfying $\Sigma_B \underline{\underline{H'}} N h = \underline{\underline{H'}} = \Sigma_B H N h$. Hence, by the universality of h, $\Sigma_B \underline{\underline{H'}} N = \Sigma_B H N$ and since N and Σ_B are clearly epi, mono, respectively, we get $\underline{\underline{H'}} = H$, thus $KK' = \mathrm{Id}_{(\mathcal{Q},T)\text{–}\mathbf{alg}}$. Therefore $(t, \mathcal{E})-\mathbf{alg}$ and $(\mathcal{Q}, T)-\mathbf{alg}$ are concretely isomorphic. $\square$

Remark 1.3.15 *Given a concrete category $(\mathcal{A}, U)$, its canonical theory $(t_U, \mathcal{E}_U)$ is converted into the functorial theory $(\mathbf{Cl}\, U, \exp_U)$.*

Lemma 1.3.16 *There is a functor $L : \mathbf{Eq}\, \mathcal{C} \to \mathbf{Th}\, \mathcal{C}$ such that*

$$\text{–}\mathbf{alg}^f \circ L = \text{–}\mathbf{alg}^e.$$

Proof: All we need is to extend the assignment $(t, \mathcal{E}) \mapsto (\mathcal{Q}_{(t,\mathcal{E})}, T_{(t,\mathcal{E})})$ on morphisms. Let $i : (t, \mathcal{E}) \to (t', \mathcal{E}')$ be an interpretation between two closed equational theories with domains Ω, Ω' and canonical functors h_t, N_E and $h_{t'}, N_{E'}$, respectively. Then the mapping $N_{E'} \circ h_{t'} \circ i$ factorizes over $N_E \circ h_t$ and there exists a unique $L(i) : \mathcal{Q}_{t,E} \to \mathcal{Q}_{t',E'}$ such that

$$L(i) \circ N_E \circ h_t = N_{E'} \circ h_{t'} \circ i.$$

We will show that the assignment L carries $i-\mathbf{alg}^e$ onto $L(i)-\mathbf{alg}^f$. Let $i : (t, \mathcal{E}) \to (t', \mathcal{E}')$ be an interpretation of closed equational theories. Then, given (A, G) in $(t, \mathcal{E}')-\mathbf{alg}$, we have

$$\begin{array}{rcl} \Sigma_A \circ \underline{\underline{G}} \circ L(i) \circ N_E \circ h_t & = & \Sigma_A \circ \underline{\underline{G}} \circ N_{E'} \circ h_{t'} \circ i \\ & = & G \circ i \\ & = & \Sigma_A \circ \underline{\underline{G \circ i}} \circ N_E \circ h_t \end{array}$$

hence by universality of h_t, Σ_A being mono and N_t being epi we get $\underline{\underline{G}} \circ L(i) = \underline{\underline{G \circ i}}$, thus the diagram

$$\begin{tikzcd} (A, G) \arrow[r, mapsto, "i\text{–}\mathbf{alg}"] \arrow[d, "K_{(t,\mathcal{E})}"] & (A, G \circ i) \arrow[d, "K_{(t',\mathcal{E}')}"] \\ (A, \underline{\underline{G}}) \arrow[r, mapsto, "L(i)\text{–}\mathbf{alg}"] & (A, \underline{\underline{G}} \circ L(i)) \end{tikzcd}$$

commutes, and the property is satisfied. $\square$

Remark 1.3.17 *By an* enlargement *of an equational theory $(t, \mathcal{E})$ we mean an interpretation* $\mathrm{id}_{\mathrm{dom}(t)} : (t, \mathcal{E}) \to (t, \mathcal{E}')$ *for some $\mathcal{E}' \supseteq \mathcal{E}$. The enlargement is converted by the functor L to a surjective morphism of functorial theories $L(i) : \mathcal{Q}_{(t,\mathcal{E})} \Rightarrow \mathcal{Q}_{(t,\mathcal{E}')}$ given by factorizing of the classes of morphisms.*

Lemma 1.3.18 *For every functorial theory* $(\mathcal{Q}, T)$*, there exists an equational theory* $(t, \mathcal{E})$ *such that* $(t, \mathcal{E})-\mathbf{alg} \cong_{\mathcal{C}} (\mathcal{Q}, T)-\mathbf{alg}$.

Proof: Let $(\mathcal{Q}, T)$ be a functorial theory over $\mathcal{C}$. Let $\Omega = \mathrm{Mor}\mathcal{Q}$, $t_{\mathcal{Q}} : \Omega \to (\mathrm{Ob}\mathcal{C})^2$ be a mapping given by the assignment $t_{\mathcal{Q}}(\sigma) = (\mathrm{dom}(\sigma), \mathrm{cod}(\sigma))$ and let $\mathcal{E}_0$ consist of the following equations

1. $(Tm, \overline{m})$ for every morphism m in $\mathcal{C}^{op}$ (here $\overline{m}$ is the morphism-constant for $t_{\mathcal{Q}}$)

2. $(\rho \cdot \sigma, \rho \circ \sigma)$ for every pair of composable $\mathcal{Q}$-morphisms σ, ρ

Let $\mathcal{E}_T$ be the closure of $\mathcal{E}_0$, then $(t_{\mathcal{Q}}, \mathcal{E}_T)$ will be the required equational theory. It easy to show that $(\mathcal{Q}, T)-\mathbf{alg} \cong_{\mathcal{C}} (t_{\mathcal{Q}}, \mathcal{E}_T)-\mathbf{alg}$. In fact, $(\mathcal{Q}, T)$-algebras (A, H) are in a natural one-to-one correspondence with $(t_{\mathcal{Q}}, E_T)$-algebras (A, G) by

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, Rightarrow, "V"] \arrow[d, Rightarrow, "T"] \arrow[dr, Rightarrow, "\exp_A"' near end] & \mathcal{T}(t) \arrow[dl, Rightarrow, "N" near end] & & \Omega \arrow[ll, "h"'] \arrow[d, "G"] \\
\mathcal{Q} \arrow[r, Rightarrow, "H"] & \mathbf{Cl}\, A \arrow[rr, "\Sigma_A"] & & \mathcal{S}et
\end{tikzcd}$$

as well as in the Lemma 1.3.14, but here the composition $N \circ h$ is even the isomorphism on object-free categories. The rest is straightforward. $\square$

Lemma 1.3.19 *There is a functor* $P : \mathbf{Th}\, \mathcal{C} \to \mathbf{Eq}\, \mathcal{C}$ *such that*

$$-\mathbf{alg}^e \circ P = -\mathbf{alg}^f.$$

Now we see that the equational and functorial approaches are equivalent. Hence we can state the following.

Corollary 1.3.20 *A concrete category is algebraic iff it is concretely isomorphic to* $(\mathcal{Q}, T)$–$\mathbf{alg}$ *for some functorial theory* $(\mathcal{Q}, T)$*. A concrete functor between algebraic categories is algebraically concrete iff it isomorphic to* $S-\mathbf{alg}$ *for some morphism* S *of functorial theories. A diagram* $D : \mathcal{D} \to \mathbf{Con}\, \mathcal{C}$ *is algebraically concrete iff it factorizes over* $-\mathbf{alg}^f$.

Remark 1.3.21 *One can prove an even stronger relationship. The functors* L*,* P *are mutually adjoint* $L \dashv P$ *so that:*

- $\mathbf{Th}\, \mathcal{C}$ *is a reflective subcategory in* $\mathbf{Eq}\, \mathcal{C}$*,*

- $\mathbf{Th}\, \mathcal{C}$ *is monadic over* $\mathbf{Eq}\, \mathcal{C}$*.*

1.3.5 Algebraic vs. L-algebraic Categories

First we state the relation to Beck categories - see [33].

Proposition 1.3.22 *Every algebraic category is Beck.*

Our aim is now to compare algebraic and l-algebraic categories.

Lemma 1.3.23 *Let $\mathcal{C}$ be a category with copowers. Then every algebraic category over $\mathcal{C}$ is l-algebraic.*

Proof: At first we prove that each metacategory $t-\mathbf{alg}$ is the limit of a (possibly large) limit of f-algebraic categories. Let $t : \Omega \to (\mathrm{Ob}\mathcal{C})^2$ be a type. Given a t-algebra (A, Q), we have, for every $\sigma \in \Omega$, a mapping $Q(\sigma) : \hom(t_0(\sigma), A) \to \hom(t_1(\sigma), A)$. Since $\mathcal{C}$ has copowers, the functor $\hom(t_1(\sigma), -)$ has the left adjoint $- \bullet t_1(\sigma)$ (see Remark B.1.7). Now we use the functors $G_{Y,X}$ defined in Remark B.2.8 for obejcts X, Y in $\mathcal{C}$. Hence the algebra is uniquely determined by the object A and by the collection of morphisms $\widetilde{Q(\sigma)} : G_{t_1(\sigma),t_0(\sigma)}(A) \to A$ in $\mathcal{C}$. We get structure arrows of algebras in $\mathbf{Alg}\, G_{t_1(\sigma),t_0(\sigma)}$. Hence,

$$t-\mathbf{alg} \cong_{\mathcal{C}} \prod_{\sigma\in\Omega} \mathbf{Alg}\, G_\sigma$$

where $G_\sigma = G_{t_1(\sigma),t_0(\sigma)}$ for every $\sigma \in \Omega$. Note that if Ω is a class we get a metacategory. We will show that the t-terms are in the following one-to-one correspondence with some concrete functors defined on $\prod_{\sigma\in\Omega} \mathbf{Alg}\, G_\sigma$. For each t-term of arity-pair (X, Y), the corresponding functor will have the values in $\mathbf{Alg}\, G_{Y,X}$. Let $\mathcal{A} = \prod_{\sigma\in\Omega} \mathbf{Alg}\, G_\sigma$.

For $\sigma \in \Omega$, let $P_\sigma : \mathcal{A} \to \mathbf{Alg}\, G_\sigma$ be the projection functor.

Given a morphism $f : Y \to X$, then $P_{\overline{f}} : \mathcal{A} \to \mathbf{Alg}\, G_{Y,X}$ is defined by $(A, H) \mapsto (A, \widetilde{\hom(f, A)})$ where $\widetilde{\hom(f, A)} : G_{Y,X}(A) = \hom(X, A) \bullet Y \to A$ is derived from $\hom(f, A)$ by the adjunction $- \bullet Y \dashv \hom(Y, -)$.

Now consider a composition $q \cdot p$ of terms q and p of arity-pairs (X, Y), (Y, Z), respectively and suppose both $P_q : \mathcal{A} \to \mathbf{Alg}\, G_{Y,X}$ and $P_p : \mathcal{A} \to \mathbf{Alg}\, G_{Z,Y}$ are defined. Then there is an obvious functor $(P_q, P_p) : \mathcal{A} \to \mathbf{Alg}\, G_{Y,X} \times_{\mathcal{C}} \mathbf{Alg}\, G_{Z,Y}$. The unit $\eta^Y : \mathrm{Id}_{\mathcal{S}et} \to \hom(Y, -) \circ (- \bullet Y)$ of the adjunction $- \bullet Y \dashv \hom(Y, -)$ yields the transformation $h : G_{Z,X} \to G_{Z,Y} G_{Y,X}$, $h = (- \bullet Z)\eta^Y \hom(X, -)$. Then we have a functor $S_{Z,Y,X} : \mathbf{Alg}\, G_{Y,X} \times_{\mathcal{C}} \mathbf{Alg}\, G_{Z,Y} \to \mathbf{Alg}\, G_{Z,X}$ defined on an object (A, α, β) by a $G_{Z,X}$-algebra $(A, \beta \circ G_{Z,Y}\alpha \circ h_A)$. We define $P_{q\cdot p} = S_{Z,Y,X} \circ (P_q, P_p)$.

The assignment $s \mapsto P_s$ together with algebra reformulation $(A, H) \rightsquigarrow (A, \widetilde{H})$ yields, for each term s in $\mathcal{T}(t)$, the structure arrow of $P_s(A, \widetilde{H})$ being

$$\alpha_{(H,s)} = \widetilde{\underline{H}(s)}.$$

The proof leads through an induction. If s is in Ω or is a morphism-constant, then it holds by definition of $\widetilde{H}$ and P_s. If $s = q \cdot p$ and q are p terms from the previous paragraph satisfying this property, then

$$\begin{aligned}
\alpha_{(H,q\cdot p)} &= \alpha_{(H,q)} \circ G_{Z,Y}\alpha_{(H,p)} \circ h_A = \widetilde{\underline{H}(q)} \circ G_{Z,Y}\widetilde{\underline{H}(p)} \circ h_A \\
&= \widetilde{\underline{H}(q)} \circ (- \bullet Z)(\hom(Y, \widetilde{\underline{H}(p)}) \circ \eta^Y \hom(X, A)) \\
&= \widetilde{\underline{H}(q)} \circ (- \bullet Z)(\underline{H}(p)) = \widetilde{\underline{H}(q) \circ \underline{H}(p)} = \widetilde{\underline{H}(q \cdot p)}
\end{aligned}$$

where the last but one equality follows from adjunction as in B.1.4. Hence the property holds for $q \cdot p$, too. Then the t-algebra (A, H) satisfies an identity (p, q) iff its counterpart $(A, \widetilde{H})$ in $\mathcal{A}$ satisfies $P_p(A, \widetilde{H}) = P_q(A, \widetilde{H})$. Hence $(A, \widetilde{H})$ lies in the equalizer of P_p and P_q. Therefore, each category $(t, \mathcal{I})-\mathbf{alg}$ is the intersection of some collection of equalizers on the category $\prod_{\sigma\in\Omega} \mathbf{Alg}\, G_\sigma$, i.e., $(t, \mathcal{I})-\mathbf{alg}$ is an l-algebraic category. $\square$

We may state also a converse relation. In fact, J. Reiterman proved, but did not publish, the following result:

Theorem 1.3.24 (Reiterman's theorem)
Every f-algebraic category is algebraic.

Proof: The paper of Kurz and Rosický [22] presents the dual version of this theorem with the base category $\mathcal{S}et$. By modification of its proof we get the following:
Given a functor $F : \mathcal{C} \to \mathcal{C}$, let Ω contain symbols σ_X of arity-pair (X, FX) for every object X in $\mathcal{C}$ and let $\mathcal{I}$ be the closure of a class of equations

$$(\overline{Ff} \cdot \sigma_X, \sigma_Y \cdot \overline{f})$$

labeled by all morphisms $f : Y \to X$ in $\mathcal{C}$. Now we have an assignment $R : \mathbf{Alg}\, F \to (t, \mathcal{I})-\mathbf{alg}$ given by $(A, \alpha) \mapsto (A, R_\alpha)$ for an F-algebra (A, α). Here R_α denotes the following assignment: given an object X and morphism $h : X \to A$ we set

$$R_\alpha(\sigma_X)(h) = \alpha \circ Fh.$$

If $g : (A, \alpha) \to (B, \beta)$ is a morphism of algebras, then, also, g is a morphism $(A, R_\alpha) \to (B, R_\beta)$. In fact, $\beta \circ Fg = g \circ \alpha$ and, for every object X in $\mathcal{C}$, the diagram

$$\begin{tikzcd}
\hom(X,A) \arrow[r, "R_\alpha(\sigma_X)"] \arrow[d, "\hom(X,g)"] & \hom(FX,A) \arrow[d, "\hom(FX,g)"] \\
\hom(X,B) \arrow[r, "R_\beta(\sigma_X)"] & \hom(FX,B),
\end{tikzcd}$$

commutes since, for every $h : X \to A$, we have

$$\begin{aligned}
(R_\beta(\sigma_X) \circ \hom(X,g))(h) &= R_\beta(\sigma_X)(g \circ h) = \beta \circ F(g \circ h) \\
&= \beta \circ Fg \circ Fh = g \circ \alpha \circ Fh \\
&= g \circ R_\alpha(\sigma_X)(h) = (\hom(FX,g) \circ R_\alpha(\sigma_X))(h).
\end{aligned}$$

We will prove that every (A, R_α) satisfies all equations in $\mathcal{I}$. Let $f : Y \to X$ be a $\mathcal{C}$-morphism, then, for every $h : X \to A$, we get

$$\begin{aligned}
\underline{R_\alpha}(\overline{Ff} \cdot \sigma_X)(h) &= (\underline{R_\alpha}(\overline{Ff}) \circ R_\alpha(\sigma_X))(h) = \underline{R_\alpha}(\overline{Ff})(\alpha \circ Fh) \\
&= \alpha \circ Fh \circ Ff = \alpha \circ F(h \circ f) \\
&= R_\alpha(\sigma_Y)(h \circ f) = R_\alpha(\sigma_Y)(\hom(f,A)(h)) \\
&= \underline{R_\alpha}(\sigma_Y \cdot \overline{f})(h),
\end{aligned}$$

hence $(A, R_\alpha) \models (\overline{Ff} \cdot \sigma_X, \sigma_Y \cdot \overline{f})$, therefore $(A, R_\alpha) \models \mathcal{I}$ since $\mathcal{I}$ is the closure of the class of all such equations.

Now, we need to define a functor $S : (t, \mathcal{I})\mathbf{-alg} \to \mathbf{Alg}\, F$ which will play the role of R^{-1}. Let

$$S(A,H) = (A, H(\sigma_A)(\mathrm{id}_A)).$$

Let $g : (A,H) \to (B,G)$ be a morphism of t-algebras. Then g is a morphism of F-algebras since $(B,G) \models (\overline{Fg} \cdot \sigma_B, \sigma_A \cdot \overline{g})$ and

$$\begin{aligned}
g \circ H(\sigma_A)(\mathrm{id}_A) &= (\hom(FA,g) \circ H(\sigma_A))(\mathrm{id}_A) = (G(\sigma_A) \circ \hom(A,g))(\mathrm{id}_A) \\
&= G(\sigma_A)(\hom(A,g)(\mathrm{id}_A)) = G(\sigma_A)(g) \\
&= G(\sigma_A)(\hom(g,A)(\mathrm{id}_B)) = \underline{G}(\sigma_A \cdot \overline{g})(\mathrm{id}_B) \\
&= \underline{G}(\overline{Fg} \cdot \sigma_B)(\mathrm{id}_B) = (\underline{G}(\overline{Fg}) \circ G(\sigma_B))(\mathrm{id}_B) \\
&= (\hom(Fg,B) \circ G(\sigma_B))(\mathrm{id}_B) = G(\sigma_B)(\mathrm{id}_B) \circ Fg.
\end{aligned}$$

Hence S is a functor. To conclude the proof, we need to show that it is inverse to R. Let (A, α) be an F-algebra, then

$$\begin{aligned}
SR(A,\alpha) &= S(A, R_\alpha) = (A, R_\alpha(\sigma_A)(\mathrm{id}_A)) \\
&= (A, \alpha \circ F\mathrm{id}_A) = (A, \alpha).
\end{aligned}$$

For an algebra (A, H) in $(t, \mathcal{I})-\mathbf{alg}$ let $\alpha = H(\sigma_A)(\mathrm{id}_A)$. Then $RS(A, H) = (A, \alpha) = (A, R_\alpha)$ and $R_\alpha(\sigma_X) = H(\sigma_X)$ for every $X \in \mathrm{Ob}\mathcal{C}$, since $(A, H) \models (\overline{Fh} \cdot \sigma_A, \sigma_X \cdot \overline{h})$ and, for each morphism $h : X \to A$, we have

$$\begin{aligned} R_\alpha(\sigma_X)(h) &= \alpha \circ Fh = H(\sigma_A)(\mathrm{id}_A) \circ Fh \\ &= \mathrm{hom}(Fh, A)(H(\sigma_A)(\mathrm{id}_A)) = (\underline{H}(\overline{Fh}) \circ H(\sigma_A))(\mathrm{id}_A)) \\ &= \underline{H}(\overline{Fh} \cdot \sigma_A)(\mathrm{id}_A) = \underline{H}(\sigma_X \cdot \overline{h})(\mathrm{id}_A) \\ &= H(\sigma_X)(\mathrm{hom}(h, A)(\mathrm{id}_A)) = H(\sigma_X)(h). \end{aligned}$$

Therefore, $RS = \mathrm{Id}_{(t,\mathcal{I})-\mathbf{alg}}$ and $SR = \mathrm{Id}_{\mathbf{Alg}\,F}$ and R is an isomorphism. $\square$

Definition 1.3.12 *Given a functor $F : \mathcal{C} \to \mathcal{C}$, then the equational theory $(t_F, \mathcal{I}_F) = (t, \mathcal{I})$ and the functor $R_F = R : \mathbf{Alg}\,F \to (t, \mathcal{I})-\mathbf{alg}$ given by $(A, \alpha) \mapsto (A, R_\alpha)$ and used in the proof above will be called* Reiterman theory *and* Reiterman isomorphism*, respectively.*

This isomorphism enables even more. We may use it to classify some l-algebraic categories. Namely, the polymeric categories will be proved to be algebraic by the enlargement of the corresponding Reiterman theory. Moreover, if we modify the Reiterman theory for a *set of $\mathcal{C}$-endofunctors*, we may redefine the Reiterman isomorphism even for some non-homogenous l-algebraic categories such as products of f-algebraic categories. Generally, this concept will be called *Reiterman conversion*.

We will show the proof for polymeric categories explicitly. Let F be an endofunctor on $\mathcal{C}$ for the rest of this section. We will need the following notion:

Definition 1.3.13 *Given an object X in $\mathcal{C}$ and $k \in \mathbb{N}_0$, then we define* k-polymeric t_F-term $\tau_X^{(k)}$ *by recursion: $\tau_X^{(0)} = \overline{\mathrm{id}_X}$ and $\tau_X^{(n+1)} = \sigma_{F^n X} \cdot \tau_X^{(n)}$ for $n \in \mathbb{N}_0$.*

Remark 1.3.25 *Observe that the terms $\tau_X^{(k)}$ have arity pairs of $(X, F^k X)$.*

Lemma 1.3.26 *Let X be a $\mathcal{C}$-object and $k \in \mathbb{N}$, then*

$$\underline{R_\alpha}(\tau_X^{(k)})(h) = \alpha^{(k)} \circ F^k h$$

for every F-algebra (A, α) and morphism $h : X \to A$. Here R_α is the structure arrow of $R_F(A, \alpha)$.

Proof: Let (A, α) be an F-algebra and $h : X \to A$ be a morphism. By induction we have:

Initial step: $\underline{R_\alpha}(\tau_X^{(0)})(h) = R_\alpha(\overline{\mathrm{id}_X})(h) = h \circ \mathrm{id}_X = \mathrm{id}_A \circ h = \alpha^{(0)} \circ F^0 h$.
Inductive step: Let $n \in \mathbb{N}_0$ and $\underline{R_\alpha}(\tau_X^{(n)})(h) = \alpha^{(n)} \circ F^n h$ be satisfied. Then

$$\begin{aligned}
\underline{R_\alpha}(\tau_X^{(n+1)})(h) &= \underline{R_\alpha}(\sigma_{F^n X} \cdot \tau_X^{(n)})(h) \\
&= (R_\alpha(\sigma_{F^n X}) \cdot \underline{R_\alpha}(\tau_X^{(n)}))(h) \\
&= R_\alpha(\sigma_{F^n X})(\underline{R_\alpha}(\tau_X^{(n)})(h)) \\
&= R_\alpha(\sigma_{F^n X})(\alpha^{(n)} \circ F^n h) \\
&= \alpha \circ F(\alpha^{(n)} \circ F^n h) \\
&= \alpha \circ F\alpha^{(n)} \circ F^{n+1} h \\
&= \alpha^{(n+1)} \circ F^{n+1} h.
\end{aligned}$$

□

Lemma 1.3.27 *Given a functor $G : \mathcal{C} \to \mathcal{C}$, $k \in \mathbb{N}_0$ and $\phi : G \to F^k$ be a natural transformation. Then*

$$\underline{R_\alpha}(\overline{\phi_X} \cdot \tau_X^{(k)})(h) \quad = \quad \alpha^{(k)} \circ \phi_A \circ Gh \tag{1.4}$$

for every F-algebra (A, α) and every morphism $h : X \to A$.

Proof: Let (A, α) be an F-algebra and $h : X \to A$ be a morphism. If $k = 0$, then $\phi : G \to \mathrm{Id}$ and we have

$$\underline{R_\alpha}(\overline{\phi_X} \cdot \tau_X^{(k)})(h) = \underline{R_\alpha}(\overline{\phi_X})(h) = h \circ \phi_X = \phi_A \circ Gh = \alpha^{(0)} \circ \phi_A \circ Gh.$$

Now suppose $k \geq 0$. Then we have

$$\begin{aligned}
\underline{R_\alpha}(\overline{\phi_X} \cdot \tau_X^{(k)})(h) &= \hom(\phi_X, A)(\underline{R_\alpha}(\tau_X^{(k)})(h)) = \underline{R_\alpha}(\tau_X^{(k)})(h)) \circ \phi_X \\
&= \alpha^{(k)} \circ F^k h \circ \phi_X = \alpha^{(k)} \circ \phi_A \circ Gh.
\end{aligned}$$

□

Lemma 1.3.28 *Given a functor $G : \mathcal{C} \to \mathcal{C}$ and an (m, n)-ary polymeric G-identity $(\phi, \psi)_p$ in the category of F-algebras, then, for every F-algebra (A, α),*

$$(A, \alpha) \models (\phi, \psi)_p \Leftrightarrow (\forall X \in \mathrm{Ob}\mathcal{C})\ (A, R_\alpha) \models (\overline{\phi_X} \cdot \tau_X^{(m)}, \overline{\psi_X} \cdot \tau_X^{(n)}).$$

Proof: Let (A, α) be an F-algebra.

$\Rightarrow$ Let $(A, \alpha) \models (\phi, \psi)_p$, then $\alpha^{(m)} \circ \phi_A = \alpha^{(n)} \circ \psi_A$ and we have

$$\underline{R_\alpha}(\overline{\phi_X} \cdot \tau_X^{(m)})(h) \overset{(1.4)}{=} \alpha^{(m)} \circ \phi_A \circ Gh = \alpha^{(n)} \circ \psi_A \circ Gh$$

for every $h : X \to A$. Hence due to symmetry of calculation we get

$$\underline{R_\alpha}(\overline{\phi_X} \cdot \tau_X^{(m)})(h) = \underline{R_\alpha}(\overline{\psi_X} \cdot \tau_X^{(n)})(h).$$

Hence $(A, R_\alpha) \models (\phi_X \cdot \tau_X^{(m)}, \psi_X \cdot \tau_X^{(n)})$ and the right-hand side holds.

$\Leftarrow$ Now suppose $(A, R_\alpha) \models (\overline{\phi_X} \cdot \tau_X^{(m)}, \overline{\psi_X} \cdot \tau_X^{(n)})$ for every object X in $\mathcal{C}$. Then the satisfaction of $(\overline{\phi_A} \cdot \tau_A^{(m)}, \overline{\psi_A} \cdot \tau_A^{(n)})$ yields $\underline{R_\alpha}(\overline{\phi_A} \cdot \tau_A^{(m)})(\mathrm{id}_A) = \underline{R_\alpha}(\overline{\psi_A} \cdot \tau_A^{(n)})(\mathrm{id}_A)$. And since $\underline{R_\alpha}(\overline{\phi_A} \cdot \tau_A^{(m)})(\mathrm{id}_A) = \alpha^{(m)} \circ \phi_A \circ G\mathrm{id}_A = \alpha^{(m)} \circ \phi_A$, we get

$$\alpha^{(m)} \circ \phi_A = \alpha^{(n)} \circ \psi_A,$$

hence $(A, \alpha) \models (\phi, \psi)_p$.

$\square$

Corollary 1.3.29 *Every polymeric category is algebraic.*

Proof: Consider a class $\mathcal{P} = \{p_i | i \in I\}$ of polymeric identities in the category of F-algebras. Due to Reiterman's theorem and Lemma 1.3.28 we have, for each $p_i \in \mathcal{P}$, a class $\widetilde{P_i}$ of t_F-identities $\widetilde{p_{i,X}}$ such that a subcategory $\mathcal{A}$ of $\mathbf{Alg}F$ satisfies p iff the R_F-image of the category $\mathcal{A}$ satisfies every $\widetilde{p}_X$. Hence a class $\mathcal{P}$ is, in this sense, algebraically equivalent to the class $\widetilde{\mathcal{P}} = \bigcup_{i \in I} \widetilde{P_i}$ of t_F-identities. This implies that the Reiterman isomorphism can be restricted as

$$\mathbf{Alg}\,(F, \mathcal{P}) \cong_{\mathcal{C}} (t_F, \mathcal{J})-\mathbf{alg},$$

where $\mathcal{J}$ is the closure of $\mathcal{I}_F \cup \widetilde{\mathcal{P}}$. $\square$

By *functorial Reiterman conversion* we mean an application of the Reiterman isomorphism composed with the conversion into the functorially induced algebraic categories, i.e., the assignment $F \mapsto L(t_F, \mathcal{I}_F)$.

The functorial Reiterman conversion has the following couniversal property. Let $F^{op} : \mathcal{C}^{op} \to \mathcal{C}^{op}$ be the dual functor for F. Consider a discrete category $\mathcal{D}$ on $\mathrm{Ob}\mathcal{C}$ with identities being the only morphisms. Then there is an embedding $U : \mathcal{D} \to \mathcal{C}^{op}$ and we still have the morphism-constant embedding $V_F : \mathcal{C}^{op} \to \mathcal{T}(t_F)$. Now the collection $\{\sigma_X | X \in \mathrm{Ob}\mathcal{C}\}$ defines a natural transformation $\sigma : VU \to VF^{op}U$.

Lemma 1.3.30 *Under the above assumptions, the factorization of $\mathcal{T}(t_F)$ over the Reiterman theory $\mathcal{I}_F$ yields a category $\mathcal{Q}_F$ and a functor N_F such that*

1. *$N_F\sigma : N_FV \to N_FVF^{op}$ is a natural transformation*

2. *For every $N' : \mathcal{T}(t_F) \Rightarrow \mathcal{Q}'$ with $N'\sigma$ being a natural transformation $N'V \to N'VF^{op}$ there exists a unique $R : \mathcal{Q} \Rightarrow \mathcal{Q}'$ such that $N' = RN_F$.*

Proof: In fact, the collection of equations $(\overline{Ff} \cdot \sigma_X, \sigma_Y \cdot \overline{f})$, for $\mathcal{C}$-morphisms $f : Y \to X$, yields exactly that σ is natural transformation in the corresponding factor-category. The factorization property is obvious. $\square$

We will show that it can be extended on natural transformations.

Lemma 1.3.31 *Let $\phi : F \to G$ be a natural transformation between $\mathcal{C}$-endofunctors. Then $\mathbf{Alg}\phi : \mathbf{Alg}F \to \mathbf{Alg}G$ is algebraically concrete functor.*

Proof: Consider closed Reiterman equational theories $(t_F, \mathcal{I}_F)$, $(t_G, \mathcal{I}_G)$ with type domains $\Omega_F = \{\sigma_X^F | X \in \mathrm{Ob}\mathcal{C}\}$, $\Omega_G = \{\sigma_X^G | X \in \mathrm{Ob}\mathcal{C}\}$, Reiterman isomorphisms R_F, R_G, the conversion L form Lemma 1.3.19 and isomorphisms K_F, K_G from Lemma 1.3.14 for F and G, respectively. Then $L(t_F, \mathcal{I}_F) = (\mathcal{Q}_F, T_F)$, $L(t_G, \mathcal{I}_G) = (\mathcal{Q}_G, T_G)$ and we have a diagram

$$\begin{tikzcd} \mathbf{Alg}\, G \arrow[r, "R_G"] \arrow[d, "\mathbf{Alg}\,\phi"] & (t_G, \mathcal{I}_G)-\mathbf{alg} \arrow[r, "K_G"] & (\mathcal{Q}_G, T_G)-\mathbf{alg} \arrow[d, "M"] \\ \mathbf{Alg}\, F \arrow[r, "R_F"] & (t_F, \mathcal{I}_F)-\mathbf{alg} \arrow[r, "K_F"] & (\mathcal{Q}_F, T_F)-\mathbf{alg} \end{tikzcd}$$

with the induced functor $M = K_F R_F \mathbf{Alg}\phi R_G^{-1} K_G^{-1}$. We will find a morphism $\mathcal{Q}_\phi : (\mathcal{Q}_F, T_F) \to (\mathcal{Q}_G, T_G)$ such that $M = \mathcal{Q}_\phi - \mathbf{alg}$. Let V_F, V_G be the morphism-constant embeddings.

Let $\phi^{op} : G^{op} \to F^{op}$ be the natural transformation gained by dualization of ϕ and consider the type-preserving mapping $P_\phi : \Omega_F \to \mathcal{T}(t_G)$ given by $P_\phi(\sigma_X^F) = \overline{\phi_X^{op}} \cdot \sigma_X^G$. Each morphism-constant $\overline{\phi_X^{op}}$ in $\mathcal{T}(t_G)$ can be seen as a component of $V_G \phi^{op}$. The induced functor $\underline{P_\phi} : \mathcal{T}(t_F) \to \mathcal{T}(t_G)$ preserves morphism-constants, hence $\underline{P_\phi} V_F = V_G$ and using the natural transformation $\sigma^F : V_F U \to V_F F^{op} U$ we get a natural transformation

$$\underline{P_\phi}\sigma^F = V_G \phi^{op} \circ \sigma^G : V_G U = \underline{P_\phi} V_F U \to \underline{P_\phi} V_F F^{op} U = V_G F^{op} U.$$

Now we shift it by N_G to $\mathcal{Q}_G$ and get

$$N_G \underline{P_\phi} \sigma^F = N_G V_G \phi^{op} \circ N_G \sigma^G : N_G V_G U \to N_G V_G F^{op} U.$$

But $N_G V_G \phi^{op} \circ N_G \sigma^G$ is a natural transformation $N_G V_G \to N_G V_G F^{op}$ since it is a composition of $N_G \sigma^G : N_G V_G \to N_G V_G G^{op}$ and $N_G V_G \phi^{op} : N_G V_G G^{op} \to N_G V_G F^{op}$. Hence the functor $N' = N_G \underline{P_\phi} : \mathcal{T}(t_F) \Rightarrow \mathcal{Q}_G$ yields a natural transformation $N'\sigma^F : N' V_F \to N' V_F F^{op}$. Thus, by Lemma 1.3.30 there exists a unique $\mathcal{Q}_\phi : \mathcal{Q}_F \Rightarrow \mathcal{Q}_G$ such that $\mathcal{Q}_\phi N_F = N' = N_G \circ \underline{P_\phi}$. Therefore, we have a morphism of theories $\mathcal{Q}_\phi : (\mathcal{Q}_F, T_F) \to (\mathcal{Q}_G, T_G)$.

It remains to show that $\mathcal{Q}_\phi$–$\mathbf{alg} = M$. For every $(\mathcal{Q}_G, T_G)$-algebra (A, H) we have

$$\begin{array}{rcll} M(A,H) & = & K_F R_F \mathbf{Alg}\,\phi R_G^{-1} K_G^{-1}(A,H) & \\ & = & K_F R_F \mathbf{Alg}\,\phi R_G^{-1}(A,H') & H' = \Sigma_A H N_G h_G \\ & = & K_F R_F \mathbf{Alg}\,\phi(A, H'(\sigma_A^G)(\mathrm{id}_A)) & \\ & = & K_F R_F (A, H'(\sigma_A^G)(\mathrm{id}_A) \circ \phi_A) & \\ & = & K_F(A, R_{F,\alpha}) & \alpha = H'(\sigma_A^G)(\mathrm{id}_A) \circ \phi_A \\ & = & (A, \underline{\underline{\underline{R_{F,\alpha}}}}) & \end{array}$$

We will show that $R_{F,\alpha}(\sigma^F) = \underline{H'}(\overline{\phi_X} \cdot \sigma^G_X)$ for every object X in $\mathcal{C}$. Consider a morphism $f : X \to A$ in $\mathcal{C}$. Then

$$\begin{array}{rcl} R_{F,\alpha}(\sigma^F_X)(f) & = & \alpha \circ Ff = H'(\sigma^G_A)(\mathrm{id}_A) \circ \phi_A \circ Ff \\ & = & \underline{H'}(\sigma^G_A)(\mathrm{id}_A) \circ Gf \circ \phi_X \\ & = & \hom(\phi_X, A)(\hom(Gf, A)(\underline{H'}(\sigma^G_A)(\mathrm{id}_A))) \\ & = & \hom(\phi_X, A)(\underline{H'}(\overline{Gf} \cdot \sigma^G_A)(\mathrm{id}_A))) \\ & = & \hom(\phi_X, A)(\underline{H'}(\sigma^G_X \cdot \overline{f})(\mathrm{id}_A))) \\ & = & \hom(\phi_X, A)(\underline{H'}(\sigma^G_X)(\mathrm{id}_A \circ f)) \\ & = & \underline{H'}(\overline{\phi_X} \cdot \sigma^G_X)(f). \end{array}$$

(Note that we do not distinguish the symbols σ^F_X from $h_F(\sigma^F_X)$ here.) Therefore $R_{F,\alpha} = \underline{H'} \circ P_\phi = \Sigma_A H N_G \underline{P_\phi} h_F = \Sigma_A H \mathcal{Q}_\phi N_F h_F$. Hence $\underline{\underline{(R_{F,\alpha})}} = H \circ \mathcal{Q}_\phi$ and we have $(A, \underline{\underline{(R_{F,\alpha})}}) = (A, H \circ \mathcal{Q}_\phi) = \mathcal{Q}_\phi\mathbf{-alg}(A, H)$ $\square$

Corollary 1.3.32 *There is a functor* $\mathrm{Th} : \mathbf{End}\mathcal{C} \to \mathbf{Th}\,\mathcal{C}$ *such that*

$$\mathbf{-alg}^f \circ \mathrm{Th} = \mathbf{Alg}\,.$$

Chapter 2

Algebras over a Cocomplete Category

Let $\mathcal{C}$ be a cocomplete category throughout this chapter and $F : \mathcal{C} \to \mathcal{C}$ be a functor. Our aim is to define a category of certain F-algebras which replaces a variety of algebras in a classical sense.

2.1 Chain Constructions

In order to define a variety, we need to recall the concept of *free-algebra chain construction* (introduced in [5], generalized in [9]). Its connection to free algebras will be studied in the second part. We will show the definition in the functorial form.

Definition 2.1.1 *We will use transfinite induction to define* term functors $F_n : \mathcal{C} \to \mathcal{C}$, *for* $n \in \mathrm{Ord}$ *and natural transformations* $w_{m,n} : F_m \to F_n$, *for* $m \leq n$*:*

Initial step: $F_0 = \mathrm{Id}_{\mathcal{C}}$, $w_{0,0} = \mathrm{id}$

Isolated step: *Let* $F_{n+1} = FF_n + \mathrm{Id}_{\mathcal{C}}$, *the transformations* $w_{0,n+1} = \iota_{n+1}$ *and* $q_n : FF_n \to F_{n+1}$ *are the canonical injections of* $\mathrm{Id}_{\mathcal{C}}$ *and* FF_n, *respectively, into the coproduct and* $w_{m+1,n+1} = [Fw_{m,n}, \mathrm{id}_{\mathrm{Id}_{\mathcal{C}}}]$ *for*

$m \leq n$ is defined by

$$\begin{tikzcd}
FF_m \arrow[r, "Fw_{m,n}"] \arrow[d, "q_m"'] & FF_n \arrow[d, "q_n"] \\
F_{m+1} \arrow[r, "w_{m+1,n+1}"] & FF_n + \mathrm{Id}_{\mathcal{C}} = F_{n+1} \\
\mathrm{Id}_{\mathcal{C}} \arrow[u, "\iota_{m+1}"] \arrow[ur, "\iota_{n+1}"'] &
\end{tikzcd} .$$

If m is a limit ordinal, then we define $w_{m,m+1}$ as the unique factorization of $\{w_{k,m+1}|k < m\}$ over the colimit cocone $\{w_{k,m}|k < m\}$.

Limit step: *$F_n = \operatorname{colim}_{m<n} F_m$ and $w_{m,n}$ is the corresponding component of the colimit cocone.*

The construction gives rise to the transformation $y_n : F \to F_n$ defined by

$$y_n = w_{1,n} \circ q_0$$

for every ordinal $n > 0$.

To distinguish the transformations for different functors we put the name of the functor in the superscript: $w^F_{m,n}$, q^F_n, ι^F_n, y^F_n.

For every $m \leq n$, the construction yields the property:

$$w_{m,n} \circ q_m \quad = \quad q_n \circ Fw_{m,n}. \tag{2.1}$$

Note 4 *Despite the name "n-ary term functor" refers to the arity of a term, its universal-algebraic counterpart for a given type is a "set of terms of* depth *of n".*

Remark 2.1.1 *If we substitute* Id_C *for C_0 in initial step of construction, we get an equivalent concept.*

As a consequence of the definition we get the following properties (see [9]).

Remark 2.1.2 *Given an F-algebra (A, α), for every $n \in \mathrm{Ord}$, there is a morphism (a* term-evaluation *on (A, α))*

$$\epsilon_{n,(A,\alpha)} : F_n A \to A$$

defined recursively by: $\epsilon_{0,(A,\alpha)} = \mathrm{id}_A$*,* $\epsilon_{n+1,(A,\alpha)} = [\alpha \circ F\epsilon_{n,(A,\alpha)}, \mathrm{id}_A]$*,*

$$\begin{tikzcd}
FF_nA \arrow[r, "F\epsilon_{n,(A,\alpha)}"] \arrow[d, "q_{n,A}"'] & FA \arrow[d, "\alpha"] \\
F_{n+1}A \arrow[r, "\epsilon_{n+1,(A,\alpha)}"] & A \\
A \arrow[u, "\iota_{n+1,A}"] \arrow[ur, "\mathrm{id}_A"'] &
\end{tikzcd}$$

and by $\epsilon_{l,(A,\alpha)} = \operatorname*{colim}_{m<l} \epsilon_{m,(A,\alpha)}$ *for a limit ordinal* l*. Then, for every* n*,* $m \leq n$*, we have:*

$$\epsilon_{n+1,(A,\alpha)} \circ q_{n,A} = \alpha \circ F\epsilon_{n,(A,\alpha)} \tag{2.2}$$
$$\epsilon_{n,(A,\alpha)} \circ \iota_{n,A} = \mathrm{id}_A \tag{2.3}$$
$$\epsilon_{m,(A,\alpha)} = \epsilon_{n,(A,\alpha)} \circ w_{m,n,A} \tag{2.4}$$
$$\epsilon_{n,(A,\alpha)} \circ y_{n,A} = \alpha, \tag{2.5}$$

where the last property requires $n > 0$*. We write the name of the functor in the superscript* $\epsilon_{k,(A,\alpha)} = \epsilon^F_{k,(A,\alpha)}$ *if necessary.*

2.2 Varieties

There are several ways how to define a class of algebras called "variety". Its classical meaning is "an equationally presentable class of algebras". An "equation" in universal algebra, also called "identity", is a pair of terms of corresponding language. An algebra satisfies this identity iff these terms have the same evaluation on this algebra for each evaluation of variables. This is how one can define a *variety of algebras for a signature* but we are not concerned with this case and we refer to it just in the Examples section. Instead, we focus on algebras for a functor and describe the varieties for such a case. We show two ways how to define them and prove that these concepts are equivalent.

2.2.1 Equational Classes

We recall here the notion of equational category of F-algebras introduced in [9].

Definition 2.2.1 *Let* X *be an object of* $\mathcal{C}$*,* $n \in \mathrm{Ord}$ *. An* equation arrow *of arity* n *over* X *is defined as a regular epimorphism* $e : F_nX \to E$*. The object* X *is called a* variable-object *of* e*.*

We say that an F-algebra (A, α) satisfies *an equation arrow* $e : F_n X \to E$ *if for every* $f : X \to A$ *there is a morphism* $h : E \to A$ *such that*

$$\epsilon_{n,(A,\alpha)} \circ F_n f = h \circ e.$$

For a class $\mathcal{E}$ *of equation arrows, we define an* equational class of F-algebras induced by $\mathcal{E}$ *as the class of all algebras satisfying all equations* $e \in \mathcal{E}$. *Viewed as a full subcategory of* $\mathbf{Alg}\, F$, *it is called an* equational category *and denoted by* $\mathbf{Alg}\,(F, \mathcal{E})$. *Equational category presentable in this way by a single equation arrow is called* single-based.

Two equation arrows are said to be algebraically equivalent *iff they define the same equational class. By analogy, we define algebraic equivalence for the classes of equation arrows.*

As shown in [9], this approach generalizes the classical universal algebra on sets since every identity uniquely determines the regular epimorphism on the set of all terms which is given by identifying the terms included in the identity.

2.2.2 Naturally Induced Classes

Now we introduce a concept of algebras induced by natural transformations. (see [28]).

Definition 2.2.2 *Let n be an ordinal and G be a $\mathcal{C}$-endofunctor. A natural transformation* $\phi : G \to F_n$ *is called a* natural term, *more precisely an* n-ary G-term. *By* G-identity *we mean a pair of G-terms. Such pairs are called* natural identities.

Let ϕ and ψ be m-ary and n-ary G-terms, respectively. The functor G is called a domain *and* (m, n) *is an* arity-pair *of identity* (ϕ, ψ). *If $m = n$, we say that* (ϕ, ψ) *has an* arity *of n.*

We say that an F-algebra (A, α) satisfies *the identity* (ϕ, ψ) *if*

$$\epsilon_{m,(A,\alpha)} \circ \phi_A = \epsilon_{n,(A,\alpha)} \circ \psi_A.$$

Then we write

$$(A, \alpha) \models (\phi, \psi).$$

For a class $\mathcal{I}$ *of natural identities we define a* naturally induced class of F-algebras *as the class of all algebras satisfying all identities* $(\phi, \psi) \in \mathcal{I}$. *The corresponding full subcategory of* $\mathbf{Alg}\, F$ *is denoted by* $\mathbf{Alg}\,(F, \mathcal{I})$. *If it is expressable with* $\mathcal{I}$ *being a singleton, we say that* $\mathbf{Alg}\,(F, \mathcal{I})$ *is* single-induced.

Two natural identities are said to be algebraically equivalent iff they induce the same classes of F-algebras. Analogously, we define the algebraic equivalence of classes of natural identities. The algebraic equivalence relation will be denoted by $\approx$.

Remark 2.2.1

1. *Arities of components of a natural identity can be arbitrarily raised. Clearly, for an identity (ϕ, ψ) of arity-pair (m_1, m_2) we have $(\phi, \psi) \approx (w_{m_1,n} \circ \phi, w_{m_2,n} \circ \psi)$ for every $n \geq \max\{m_1, m_2\}$. Hence, every natural identity is algebraically equivalent to the identity consisting of natural terms of the same arity .*
2. *Every set $N = \{(\phi_i, \psi_i) | i \in I\}$ of n-ary natural identities is algebraically equivalent to a singleton. Clearly $N \approx \{(\phi, \psi)\}$, where ϕ, ψ are the unique factorizations of the cocones ϕ_i, ψ_i, respectively, over the coproduct of domains of single identities.*
3. *As a consequence, every class naturally induced by a set of identities is single-induced.*

2.2.3 Conversion Theorem

Our aim is to prove that naturally induced classes and equational classes coincide on a locally small base category. At first we show that every single-based equational class is naturally induced. Then, conversely, we prove that every class induced by a single natural identity is equational.

Lemma 2.2.2 *Every single-based equational class is naturally single-induced class.*

Proof: Let $\mathcal{S}$ be a single-based equational class of F-algebras defined by an equation arrow e being a regular epimorphism $F_nX \to E$ such that (E, e) is a coequalizer of some $\phi_0, \psi_0 : Q \rightrightarrows F_nX$. We define a mapping $\theta_{\phi,A}$: $\hom(X, A) \to \hom(Q, F_nA)$. For every $f : X \to A$ let $\theta_{\phi,A}(f) = F_nf \circ \phi_0$: $Q \to F_nA$. Now let

$$G = G_{Q,X} = (- \bullet Q) \circ \hom(X, -),$$

$$\phi_A = \widetilde{\theta_{\phi,A}} : \hom(X, A) \bullet Q \to F_nA.$$

Clearly, ϕ_A is a component of a natural transformation $\phi : G \to F_n$. Observe that, due to Remark B.1.7, for every $f : X \to A$,

$$\phi_A \circ u_f = \theta_{\phi,A}(f) = F_nf \circ \phi_0.$$

Analogously, we define natural transformations $\theta_{\psi,-} : G \to F_n$ and $\psi : G \to F_n$ satisfying $\psi_A \circ u_f = F_n f \circ \psi_0$. Now we have the functor G and G-identity (ϕ, ψ). It remains to show that it induces exactly the equational class $\mathcal{S}$.

Let (A, α) satisfy the equation arrow e. Then, for every $f : X \to A$, there is an $h : E \to A$ such that $\epsilon_{n,(A,\alpha)} \circ F_n f = h \circ e$. Then we have

$$\begin{aligned} \epsilon_{n,(A,\alpha)} \circ \phi_A \circ u_f &= \epsilon_{n,(A,\alpha)} \circ F_n f \circ \phi_0 \\ &= h \circ e \circ \phi_0 = h \circ e \circ \psi_0 \end{aligned}$$

and, by symmetry, we get $\epsilon_{n,(A,\alpha)} \circ \phi_A \circ u_f = \epsilon_{n,(A,\alpha)} \circ \psi_A \circ u_f$. Since f was chosen arbitrarily and the injections u_f form a colimit cocone, we have $\epsilon_{n,(A,\alpha)} \circ \phi_A = \epsilon_{n,(A,\alpha)} \circ \psi_A$, i.e., (A, α) satisfies the G-identity (ϕ, ψ).

Now let (B, β) be an F-algebra in the class induced by the G-identity (ϕ, ψ). Let $g : X \to B$ be a morphism in $\mathcal{C}$. Then we have

$$\begin{aligned} \epsilon_{n,(B,\beta)} \circ F_n g \circ \phi_0 &= \epsilon_{n,(B,\beta)} \circ \phi_B \circ u_g \\ &= \epsilon_{n,(B,\beta)} \circ \psi_B \circ u_g \end{aligned}$$

and, again by symmetry, we get $\epsilon_{n,(B,\beta)} \circ F_n g \circ \phi_0 = \epsilon_{n,(B,\beta)} \circ F_n g \circ \psi_0$, hence $\epsilon_{n,(B,\beta)} \circ F_n g$ coequalizes the pair (ϕ_0, ψ_0) and there is a unique $h : E \to B$ such that $\epsilon_{n,(B,\beta)} \circ F_n g = h \circ e$. Thus (B, β) satisfies the equation arrow e. $\square$

Remark 2.2.3 *Within the proof of Lemma 2.2.2, we have proved the algebraic equivalence of the n-ary equation arrow with variable-object X and a n-ary natural G-identity.*

The functor G is in fact equal to $G_{Q,X}$ defined in Remark B.2.8. As observed by J. Velebil, the natural identity (ϕ, ψ) can be easily derived from the pair of morphisms ϕ_0, ψ_0 by isomorphism (B.8) since $G_{Q,X}$ is the left Kan extension of C_Q along C_X.

The assignment $(X, e, E) \mapsto (G, \phi, \psi)$ may be uniquely determined if the category $\mathcal{C}$ satisfies some additional requirements.

Lemma 2.2.4 *Every naturally single-induced class is equational.*

Proof: Let G be a $\mathcal{C}$-endofunctor. Let $\mathcal{N}$ be a class induced by a G-identity (ϕ, ψ). Due to Remark 2.2.1 we may assume that ϕ and ψ have the same arity, say n. Therefore, both are the natural transformations $G \to F_n$. Let (E, e) be the coequalizer of ϕ and ψ. Then, for every object X of $\mathcal{C}$, we have a morphism $e_X : F_n X \to EX$. Let $\mathcal{E} = \{e_X | X \in \mathrm{Ob}\mathcal{C}\}$. We will prove $\mathcal{N} = \mathbf{Alg}\,(F, \mathcal{E})$.

Let (A, α) satisfy (ϕ, ψ). Then, for every $X \in \mathrm{Ob}\mathcal{C}$ and $f : X \to A$, we have

$$\begin{aligned} \epsilon_{n,(A,\alpha)} \circ F_n f \circ \phi_X &= \epsilon_{n,(A,\alpha)} \circ \phi_A \circ Gf \\ &= \epsilon_{n,(A,\alpha)} \circ \psi_A \circ Gf \\ &= \epsilon_{n,(A,\alpha)} \circ F_n f \circ \psi_X. \end{aligned}$$

Therefore, we have a coequalizing morphism $\epsilon_{n,(A,\alpha)} \circ F_n f$ for (ϕ_X, ψ_X). Since the colimits of functors are calculated componentwise, e_X is a coequalizer of (ϕ_X, ψ_X), which means that there is a unique $h : EX \to A$ such that $\epsilon_{n,(A,\alpha)} \circ F_n f = h \circ e_X$.

Given an F-algebra (B, β) satisfying all equation arrows from $\mathcal{E}$, it satisfies the arrow $e_B : F_n B \to EB$ and there is $h : EB \to B$ (chosen for $\mathrm{id}_B : B \to B$) such that $\epsilon_{n,(B,\beta)} = h \circ e_B$. Thus, the property is satisfied since e_B coequalizes the pair (ϕ_B, ψ_B). □

Theorem 2.2.5 (Conversion theorem) *Let F be an endofunctor on a cocomplete category $\mathcal{C}$. Then the equational classes of F-algebras coincide with the naturally induced classes of F-algebras.*

Proof: Every equational class $\mathcal{S}$ is a (possibly large) intersection of single-based ones and these are, by the Lemma 2.2.2, naturally induced, more precisely single-induced. Hence, $\mathcal{S}$ is naturally induced by the class of corresponding natural identities. Conversely, the naturally induced class $\mathcal{N}$ is a (possibly large) intersection of the ones induced by a single natural identity, which, due to Lemma 2.2.4, are equational classes induced by a class of equation arrows. The union of these classes defines the class of all equation arrows defining the class $\mathcal{N}$ as an equational class. □

Definition 2.2.3 *A class of algebras induced by equations or natural identities is called a* variety*.*

2.2.4 Varieties are L-algebraic

Let F be an endofunctor on a cocomplete category $\mathcal{C}$. We show that every variety of F-algebras is an l-algebraic category. And as we will see in next section, every variety is algebraic. In fact, since $\mathcal{C}$ is cocomplete, due to Lemma 1.3.23 it is also l-algebraic. But the proof in this section seems to be much more transparent and constructive, so we present it, too.

Lemma 2.2.6 *For every ordinal n, the assignment $(A, \alpha) \mapsto (A, \epsilon_{n,(A,\alpha)})$ defines a functor $E_n : \mathbf{Alg}\, F \to \mathbf{Alg}\, F_n$.*

Proof: Let U_F be the forgetful functor for $\mathbf{Alg}\, F$ and $f : (A, \alpha) \to (B, \beta)$ be a morphism of F-algebras. We will show by induction that the diagram

$$\begin{tikzcd} F_n A \arrow[r, "\epsilon_{n,(A,\alpha)}"] \arrow[d, "Ff"] & A \arrow[d, "f"] \\ F_n B \arrow[r, "\epsilon_{n,(B,\beta)}"] & B \end{tikzcd}$$

commutes for every ordinal n:

Initial step: Since $F_0 = \mathrm{Id}_{\mathcal{C}}$ and $\epsilon_{0,-} = \mathrm{id}\_U_F$, the statement is obvious.

Isolated step: Let the diagram commute for ordinal m. Then we have the commutative diagram

$$\begin{tikzcd}
FF_mA \arrow[rrrr, "F\epsilon_{m,(A,\alpha)}"] \arrow[drr, "q_{m,A}"'] \arrow[dd, "FF_mf"'] & & & & FA \arrow[drr, "\alpha"] \arrow[dd, "Ff"'] & & & & \\
& & F_{m+1}A \arrow[rrrr, "\epsilon_{m+1,(A,\alpha)}"] \arrow[dd, "F_{m+1}f"] & & & & A \arrow[dd, "f"] & & \\
FF_mB \arrow[rrrr, "F\epsilon_{m,(B,\beta)}"] \arrow[drr, "q_{m,B}"'] & & & & FB \arrow[drr, "\beta"] & & & & A \arrow[ull, "\mathrm{id}_A"'] \arrow[ullllll, "\iota_{m+1,A}"] \arrow[dd, "f"] \\
& & F_{m+1}B \arrow[rrrr, "\epsilon_{m+1,(B,\beta)}"] & & & & B & & \\
& & & & & & & & B \arrow[ull, "\mathrm{id}_B"'] \arrow[ullllll, "\iota_{m+1,B}"]
\end{tikzcd}$$

which makes the property $f \circ \epsilon_{m+1,(A,\alpha)} = \epsilon_{m+1,(B,\beta)} \circ F_m f$ valid.

Limit step: Let the diagram above commute for every ordinal m smaller then a limit ordinal l. Then we have for every $m < l$:

$$\begin{aligned}
f \circ \epsilon_{l,(A,\alpha)} \circ w_{m,l,A} &= f \circ \epsilon_{m,(A,\alpha)} \\
&= \epsilon_{m,(B,\beta)} \circ F_m f \\
&= \epsilon_{l,(B,\beta)} \circ w_{m,l,B} \circ F_m f \\
&= \epsilon_{l,(B,\beta)} \circ F_l f \circ w_{m,l,A}.
\end{aligned}$$

Since $\epsilon_{l,(A,\alpha)} = \operatorname*{colim}_{m<l} \epsilon_{m,(A,\alpha)}$, we get $f \circ \epsilon_{l,(A,\alpha)} = \epsilon_{l,(B,\beta)} \circ F_l f$.

Hence the assignments send a morphism onto a morphism. Since they now commute with forgetful functors, they preserve identities and compositions of morphisms. Thus these assignments define functors. □

Lemma 2.2.7 *Every single-induced variety is an l-algebraic category.*

Proof: Consider an n-ary natural G-identity (ϕ, ψ). Since $\phi, \psi : G \to F_n$ are natural transformations, we may use their **Alg** -image and compose it with E_n to get the diagram:

$$\begin{tikzcd}
\mathbf{Alg}\, F \arrow[r, "E_n"] & \mathbf{Alg}\, F_n \arrow[r, shift left, "\mathbf{Alg}\, \phi"] \arrow[r, shift right, "\mathbf{Alg}\, \psi"'] & \mathbf{Alg}\, G
\end{tikzcd}$$

Then $\mathbf{Alg}\,(F, (\phi, \psi))$ is clearly the equalizer of $\mathbf{Alg}\, \phi \circ E_n$ and $\mathbf{Alg}\, \psi \circ E_n$. □

Theorem 2.2.8 *Every variety is an l-algebraic category.*

Proof: The statement clearly follows from the Lemma 2.2.7 and Metacategorical remark 1.2.5. □

Remark 2.2.9 *Observe that, by duality of the construction above, one gets the covarieties to be also limits of concrete categories, namely of Coalg-categories. Hence it makes sense to consider* co-l-algebraic category *to be dual of l-algebraic category over the dual base category.*

2.2.5 Varieties vs. Polymeric Varieties

Polymeric Varieties are Varieties

The notion of polymeric variety can replace variety if the category does not have colimits. For the case of cocomplete category $\mathcal{C}$ the relations of polymeric varieties to varieties will be clarified in the following text. In the section Examples we show examples of both varieties and polymeric varieties and one can see that many of natural examples of varieties of functor-algebras are in fact polymeric. Therefore it makes sense to consider polymeric presentation even if $\mathcal{C}$ is cocomplete. Finally, every single-based variety can be seen as limit of polymeric categories, as we show further on.

Definition 2.2.4 *Natural transformation $r_n : F^n \to F_n$, for $n \in \omega$, is defined by induction:*

Initial step: $r_0 = \mathrm{id}_{\mathrm{Id}_\mathcal{C}}$,

Inductive step: *for $n \in \mathbb{N}$ let $r_{n+1} = q_{n,A} \circ F r_n$.*

Given an n-ary polymeric G-term ϕ, we define polymeric natural term *G-term*

$$\widehat{\phi} = r_n \circ \phi.$$

The following lemma converts evaluations of finitary terms to the polymers.

Lemma 2.2.10 *Let (A, α) be an F-algebra, $n \in \omega$. Then $\epsilon_{n,(A,\alpha)} \circ r_n = \alpha^{(n)}$.*

Proof: By induction we have:

Initial step: For $n = 0$ we have $\epsilon_{0,(A,\alpha)} \circ r_0 = \mathrm{id}_A \circ \mathrm{id}_A = \alpha^{(0)}$

Inductive step: Let $n \in \mathbb{N}$ and assume $\epsilon_{n,(A,\alpha)} \circ r_n = \alpha^{(n)}$, then we have:

$$\begin{aligned}
\alpha^{(n+1)} &= \alpha \circ F\alpha^{(n)} \\
&= \alpha \circ F\epsilon_{n,(A,\alpha)} \circ Fr_n \\
&= \epsilon_{n+1,(A,\alpha)} \circ q_{n+1,A} \circ Fr_n \\
&= \epsilon_{n+1,(A,\alpha)} \circ r_{n+1}
\end{aligned}$$

$\square$

As an immediate consequence we get compatibility of polymeric varieties with varieties of F-algebras on a cocomplete category:

Lemma 2.2.11 *Let G be a $\mathcal{C}$-endofunctor, $m, n \in \omega$, ϕ and ψ be m-ary and n-ary polymeric G-terms, respectively. Then for every F-algebra (A, α)*

$$(A, \alpha) \models (\phi, \psi)_p \Leftrightarrow (A, \alpha) \models (\widehat{\phi}, \widehat{\psi}).$$

Corollary 2.2.12 *Every polymeric variety of F-algebras is a variety.*

Induction of a Variety by Polymeric Identities

We will show a converse connection between varieties and polymeric varieties. In fact, every variety presentable by a set of natural identities can be treated as a polymeric, hence algebraic, category, as we prove bellow.

In our investigation we derive the term-functors from various functors and but all the natural transformations between the endofunctors on $\mathcal{C}$ are derived from the functor F, i.e., e.g., $w_{m,n} : F_m \to F_n$, even when we use it to constrain F_n-algebras.

Let $n \in \mathbb{N}$. Consider an $(0,1)$-ary polymeric $\mathrm{Id}_{\mathcal{C}}$-identity $h_{n,0} = (\mathrm{id}_{\mathrm{Id}_{\mathcal{C}}}, \iota_n)_p$ and, for every $k < n$, $(1,2)$-ary polymeric FF_k-identities $h_{k,n} = (w_{k+1,n} \circ q_k, y_n * w_{k,n})$ in category of F_n-algebras as shown bellow.

$$h_{n,0}: \quad \begin{tikzcd}
F_n \arrow[r, equal] & (F_n)^1 \\
\mathrm{Id}_{\mathcal{C}} \arrow[u, "\iota_n"] \arrow[r, equal] & (F_n)^0
\end{tikzcd}
\qquad
h_{k,n}: \quad \begin{tikzcd}
FF_n \arrow[r, "y_n F_n"] & F_nF_n \arrow[r, equal] & (F_n)^2 \\
FF_k \arrow[u, "Fw_{k,n}"] \arrow[r, "y_n F_k"] \arrow[ur, "y_n * w_{k,n}"] \arrow[d, "q_k"] & F_nF_k \arrow[u, "F_n w_{k,n}"'] & \\
F_{k+1} \arrow[r, "w_{k+1,n}"] & F_n \arrow[r, equal] & (F_n)^1
\end{tikzcd}$$

Lemma 2.2.13 *For every F_n-algebra (A, α) and $m < n$,*

$$(A, \alpha) \models h_{m,n} \Rightarrow (\forall k \leq m)(A, \alpha) \models h_{k,n}.$$

Proof: Clearly $y_n F_n \circ F w_{k,n} = y_n F_n \circ F w_{m,n} \circ F w_{k,m}$ and since

$$\begin{aligned} w_{k+1,n} \circ q_k &= w_{m+1,n} \circ w_{k+1,m+1} \circ q_k \\ &= w_{m+1,n} \circ q_m \circ F w_{k,m}, \end{aligned}$$

the satisfaction of $h_{m,n}$ implies satisfaction of $h_{k,n}$. $\square$

Note 5 *We will use the notation* $\mathcal{P}_{n+1} = \{h_{n+1,0}, h_{n,n+1}\}$ *for* $n \in \mathrm{Ord}$.

Lemma 2.2.14 *For every ordinal* n*, the assignment* $(A, \alpha) \mapsto (A, \epsilon_{n+1,(A,\alpha)})$ *defines a concrete isomorphism* $I_{n+1} : \mathbf{Alg}\, F \to \mathbf{Alg}\, (F_{n+1}, \mathcal{P}_{n+1}\})$.

Proof: Since $E_{n+1} : \mathbf{Alg}\, F \to \mathbf{Alg}\, F_{n+1}$ is the functor defined in Lemma 2.2.6, we need to show that it is an embedding with the image $\mathbf{Alg}(F_{n+1}, \mathcal{P}_{n+1})$. To prove the former, consider the functor $E'_{n+1} : \mathbf{Alg}\, F_{n+1} \to \mathbf{Alg}\, F$ given by $(A, \beta) \mapsto (A, \beta \circ y_{n+1,A})$. The property (2.5) in Remark 2.1.2 yields E'_{n+1} being the left inverse for E_{n+1}. Hence E_{n+1} is an embedding and we will prove that $\mathrm{Im}(E_{n+1}) = \mathbf{Alg}\, (F_n, \{h_{n+1,0}, h_{n,n+1}\})$.

$\subseteq$: Let (A, α) be an F-algebra. Let $\beta = \epsilon_{n,(A,\alpha)}$. At first we verify satisfaction of $h_{n+1,0}$:

$$\begin{aligned} \beta \circ \iota_{n+1,A} &= \epsilon_{n+1,(A,\alpha)} \circ \iota_{n+1,A} \\ &\overset{(2.3)}{=} \mathrm{id}_A. \end{aligned}$$

By composition of the properties (2.2), (2.4) and (2.5) together, one can get

$$\beta \circ w_{n+1,n+1} \circ q_n \circ \mathrm{id}_A = \beta \circ F_n \beta \circ (y_{n+1} F_n \circ F w_{n,n+1})_A$$

hence $(A, \beta) \models h_{n,n+1}$.

$\supseteq$: Let $(A, \beta) \models h_{n+1,0}, h_{n,n+1}$. Then, according to Lemma 2.2.13 we have, for every $k < n$, the property $(A, \beta) \models h_{k,n+1}$. To prove $(A, \beta) \in \mathrm{Im}(E_{n+1})$, we show $(A, \beta) = E_{n+1} E'_{n+1}(A, \beta)$. Let $\alpha = \beta \circ y_{n+1,A}$, then $E_{n+1} E'_{n+1}(A, \beta) = (A, \epsilon_{n+1,(A,\alpha)})$. Now we prove by induction that

$$\epsilon_{k,(A,\alpha)} = \beta \circ w_{k,n+1,A}.$$

<u>Initial step</u> $\epsilon_{0,(A,\alpha)} = \mathrm{id}_A = \beta \circ \iota_{n+1,A} = \beta \circ w_{0,n+1,A}$ by $h_{n+1,0}$.
<u>Isolated step</u> Assume the hypothesis $\epsilon_{\kappa,(A,\alpha)} = \beta \circ w_{\kappa,n+1,A}$ is satisfied for an ordinal κ.

Then $\beta \circ w_{\kappa+1,n+1,A} \circ \iota_{\kappa+1,A} = \mathrm{id}_A = \epsilon_{\kappa+1,(A,\alpha)} \circ \iota_{\kappa+1,A}$ and we have

$$\begin{aligned}
\beta \circ w_{\kappa+1,n+1,A} \circ q_{\kappa,A} &\overset{(h_{\kappa,n+1})}{=} \beta \circ F_{n+1}\beta \circ y_{n+1,F_{n+1}A} \circ F w_{\kappa,n+1,A} \\
&= \beta \circ y_{n+1,A} \circ F\beta \circ F w_{\kappa,n+1,A} \\
&\overset{\text{(induction)}}{=} \alpha \circ F\epsilon_{n,(A,\alpha)} \\
&= \epsilon_{\kappa+1,(A,\alpha)} \circ q_{\kappa,A},
\end{aligned}$$

thus $\beta \circ w_{\kappa+1,n+1,A} = \epsilon_{\kappa+1,(A,\alpha)}$.

Limit step Let $\lambda \leq n+1$ be a limit ordinal and $\epsilon_{\kappa,(A,\alpha)} = \beta \circ w_{\kappa,n+1,A}$ for every $\kappa < \lambda$. Since $w_{\kappa,n+1,A} = w_{\lambda,n+1,A} \circ w_{\kappa,\lambda,A}$ and $\epsilon_{\kappa,(A,\alpha)} = \epsilon_{\lambda,(A,\alpha)} \circ w_{\kappa,\lambda,A}$ for every $\kappa < \lambda$, we have the equality of λ-chains and the uniqueness of its factorization over the colimit $\epsilon_{\lambda,(A,\alpha)}$ yields $\epsilon_{\lambda,(A,\alpha)} = \beta \circ w_{\lambda,n+1,A}$.

Now, we get the required property by setting $k = n+1$ into the received equation.

□

Another consequence gives us the relation between natural identities in **Alg** F and **Alg** F_n:

Corollary 2.2.15 *Every natural identity (ϕ, ψ) satisfies:*

$$(A, \alpha) \models (\phi, \psi) \quad \Leftrightarrow \quad I_{n+1}(A, \alpha) \models \{(\phi, \psi)_p, h_{n+1,0}, h_{n,n+1}\}.$$

Theorem 2.2.16 *Let $\mathcal{N}$ be a class of natural identities of arites smaller then some ordinal n. Then there is a class of polymeric identities $\mathcal{Q}$ and a concrete isomorphism:*

$$\mathbf{Alg}\,(F, \mathcal{N}) \cong \mathbf{Alg}\,(F_{n+1}, \mathcal{Q}).$$

Proof: Let n be an ordinal, $\mathcal{I}$ be a class and $\mathcal{N} = \{(\phi_i, \psi_i) | i \in \mathcal{I}\}$ be a class of natural identities where each (ϕ_i, ψ_i) is a natural G_i-identity of arity k_i smaller then n. Due to Remark 2.2.1,(1), we can substitute some algebraically equivalent identity (ϕ'_i, ψ'_i) of arity $n+1$ for every identity (ϕ_i, ψ_i) in $\mathcal{N}$. Let

$$\mathbf{poly}_{n+1}\mathcal{N} = \{(\phi'_i, \psi'_i)_p | i \in \mathcal{I}\}, \text{ and } \mathcal{Q} = \mathcal{P}_{n+1} \cup \mathbf{poly}_{n+1}\mathcal{N}.$$

Due to the previous corollary, we have, for every i,

$$(A, \alpha) \models (\phi'_i, \psi'_i) \quad \Leftrightarrow \quad I_{n+1}(A, \alpha) \models \{(\phi'_i, \psi'_i)_p, h_{n+1,0}, h_{n,n+1}\},$$

hence

$$(A, \alpha) \models \{(\phi'_i, \psi'_i) | i \in \mathcal{I}\} \quad \Leftrightarrow \quad I_{n+1}(A, \alpha) \models \bigcup_{i \in \mathcal{I}} \{(\phi'_i, \psi'_i)_p, h_{n+1,0}, h_{n,n+1}\} = \mathcal{Q}.$$

Therefore $\mathbf{Alg}\,(F, \mathcal{N}) \cong \mathbf{Alg}\,(F_{n+1}, \mathcal{Q})$. □

As a direct consequence we get the following.

Corollary 2.2.17 *Every single-induced variety is a polymeric category.*

As we will see in the Chapter 4.2 (Remark 4.2.2), the results 2.2.14 and 2.2.16 are in full correspondence with the well known result on f-algebraic categories with free algebras which states that such categories are monadic.

2.2.6 Varieties are Algebraic

Let $F : \mathcal{C} \to \mathcal{C}$ be a functor and $\mathcal{V} = \mathbf{Alg}\,(F, \mathcal{N})$ be a variety. Then $\mathcal{N} = \bigcup_{i \in \mathrm{Ord}} \mathcal{N}^i$, where $\mathcal{N}^i$ is the subclass of $\mathcal{N}$ containing the natural identities of arity smaller then i (this class will be called is the *i-th restriction* of $\mathbf{Alg}\, F$). Then

$$\mathcal{V} = \bigcap_{i \in \mathrm{Ord}} \mathbf{Alg}\,(F, \mathcal{N}^i) = \bigcap_{i \in \mathrm{Ord}} \mathbf{Alg}\,(F, \mathcal{N}^{i+1}),$$

hence each $\mathbf{Alg}\,(F, \mathcal{N}^i)$ is the i-th approximation of $\mathcal{V}$ and $\mathbf{Alg}\,(F, \mathcal{N}^{i+1})$ is, due to the Remark 2.2.1, a single-induced variety. Then from Corollary 2.2.17 we have each $\mathbf{Alg}\,(F, \mathcal{N}^{i+1})$ polymeric.

We will use the following notational system for the expressions of approximation of $\mathcal{V}$ of a given restriction and given term functors in whose language the restrictions are expressed. The notation follows a general rule:

superscript	-	the size of restriction
subscript	-	the arity of the "language" term functor.

To make the system complete, we write o instead of a blank space such as $F_o = F$. Then we have the embedding

$$U_o^{n,m} : \mathbf{Alg}\,(F, \mathcal{N}^n) \to \mathbf{Alg}\,(F, \mathcal{N}^m)$$

for every $n \geq m$. Let $\mathcal{P}_{j+1}^o = \mathcal{P}_{j+1}$ and, for given $m \leq j+1$, we define $\mathcal{P}_{j+1}^m = \mathcal{P}_{j+1}^o \cup \mathbf{poly}_{j+1}\mathcal{N}_{j+1}^m$ where $\mathbf{poly}_{j+1}\mathcal{N}_{j+1}^m$ is the class of polymeric identities in category of F_{j+1}-algebras obtained by procedure described in Theorem 2.2.16 from the class $\mathcal{N}_{j+1}^m$ of m-ary natural identities expressed in terms of $j+1$-ary identities as in Remark 2.2.1. Then we have an obvious embedding $U_{j+1}^{n,m} : \mathbf{Alg}\,(F_{j+1}, \mathcal{P}_{j+1}^n) \to \mathbf{Alg}\,(F_{j+1}, \mathcal{P}_{j+1}^m)$ for $n \geq m$, since clearly $\mathcal{P}_{j+1}^n \supseteq \mathcal{P}_{j+1}^m$. Analogously, we define $U_o^{n,o}$ and $U_{j+1}^{n,o}$.

Moreover we use the isomorphism I_{i+1} from Lemma 2.2.14 to define

$$I_{j+1,i+1}^o = I_{i+1}^{-1} \circ I_{j+1} : \mathbf{Alg}\,(F_{j+1}, \mathcal{P}_{j+1}^o) \to \mathbf{Alg}\,(F_{i+1}, \mathcal{P}_{i+1}^o)$$

for $i \leq j$.

We will describe the variety $\mathcal{V}$ as a concrete limit of a reversed ordinal chain of polymeric categories. Clearly, $\mathcal{V}$ is the limit of the chain

$$\mathcal{V} \dashrightarrow \mathbf{Alg}\,(F, \mathcal{N}^{j+1}) \xrightarrow{U_o^{j+1,i+1}} \mathbf{Alg}\,(F, \mathcal{N}^{i+1}) \xrightarrow{U_o^{i+1,o}} \mathbf{Alg}\, F$$

indexed by all ordinals $j \geq i$. Due to the Theorem 2.2.16, for each pair if ordinals $j \geq i$ there is an isomorphism $I^{i+1}_{j+1,o} : \mathbf{Alg}\,(F_{j+1}, \mathcal{P}^{i+1}_{j+1}) \to \mathbf{Alg}\,(F, \mathcal{N}^{i+1})$. Let $I^{m}_{j+1,i+1} = I^{m}_{j+1,o} \circ (I^{m}_{j+1,o})^{-1}$ for every $j \geq i$, $i+1 \geq m$.

Observation 2.2.18 *The above diagram may be extended as follows:*

$$\begin{tikzcd}
\mathcal{V} \arrow[r, dashed] & \mathbf{Alg}\,(F, \mathcal{N}^{j+1}) \arrow[r, "U_o^{j+1,i+1}"] & \mathbf{Alg}\,(F, \mathcal{N}^{i+1}) \arrow[r, "U_o^{i+1,o}"] & \mathbf{Alg}\,F \\
 & & \mathbf{Alg}\,(F_{i+1}, \mathcal{P}^{i+1}_{i+1}) \arrow[u, "I^{i+1}_{i+1,o}"] \arrow[r, "U^{i+1,o}_{i+1}"] \arrow[ur] & \mathbf{Alg}\,(F_{i+1}, \mathcal{P}^{o}_{i+1}) \arrow[u, "I^{o}_{i+1,o}"'] \\
 & \mathbf{Alg}\,(F_{j+1}, \mathcal{P}^{j+1}_{j+1}) \arrow[uu, "I^{j+1}_{j+1,o}"] \arrow[r, "U^{j+1,i+1}_{j+1}"] \arrow[ur] & \mathbf{Alg}\,(F_{j+1}, \mathcal{P}^{i+1}_{j+1}) \arrow[u, "I^{i+1}_{j+1,i+1}"] \arrow[r, "U^{j+1,i+1}_{j+1}"] & \mathbf{Alg}\,(F_{j+1}, \mathcal{P}^{o}_{j+1}) \arrow[u, "I^{o}_{j+1,i+1}"'] \\
\mathit{Limit} \arrow[uuu, "iso"] \arrow[ur, dashed] & & &
\end{tikzcd}$$

and $\mathcal{V}$ is the limit of the reversed ordinal chain $S : \mathrm{Ord}^{op} \to \mathbf{Con}\,\mathcal{C}$ given by

$$S_i = \mathbf{Alg}\,(F_{i+1}, \mathcal{P}^{i+1}_{i+1}),$$

$$S_{i,j} = I^{i+1}_{j+1,i+1} \circ U^{j+1,i+1}_{j+1}.$$

This property can be rewritten as follows.

Proposition 2.2.19 *Every variety is the limit of a concrete reversed ordinal chain of polymeric varieties.*

Since every polymeric variety is algebraic, every variety is a limit of algebraic categories. However, we will prove even more.

Lemma 2.2.20 *Every variety is the limit of an algebraically concrete reversed ordinal chain.*

Proof: Consider a variety $\mathcal{V} = \mathbf{Alg}\,(F, \mathcal{N})$ and the diagram $S : \mathrm{Ord}^{op} \to \mathbf{Con}\,\mathcal{C}$ from Observation 2.2.18; we have $\mathcal{V} = \lim S$. Due to the Corollary 1.3.20 it suffices to find an ordinal chain D of functorial theories such that $S = -\mathbf{alg}^f \circ D$. We will find it via functorial Reiterman conversion which, for $i \in \mathrm{Ord}$ yields the isomorphism

$$R_{i+1} : \mathbf{Alg}\,(F_{i+1}, \mathcal{P}^{i+1}_{i+1}) \to (\mathcal{Q}_{F_{i+1}}, T_{F_{i+1}})/\mathcal{J}^{i+1}_{i+1} - \mathbf{alg}$$

with $\mathcal{J}^{i+1}_{i+1}$ described further on.

Let $i \leq j$, $m = i + 1$, $n = j + 1$ and (A, β) be an F_n-algebra satisfying the polymeric identities in $\mathcal{P}_n$ and let $\alpha = \beta \circ y_{n,A}$.

We have

$$I^m_{n,m}(A, \beta) = I^m_{n,o} \circ (I^n_{n,o})^{-1}(A, \beta) = I^m_{n,o}(A, \beta \circ y_{n,A}) = (A, \epsilon_{m,(A,\alpha)})$$

As shown in the proof of the Lemma 2.2.14, $\epsilon_{m,(A,\alpha)} = \beta \circ w_{m,n,A}$, hence

$$I^m_{n,m}(A, \beta) = (A, \beta \circ w_{m,n,A}).$$

Hence the functor $I^m_{n,m}$ is given by the same assignment as $\mathbf{Alg}\, w_{m,n}$.

Consider the diagram

$$\begin{tikzcd}
\mathbf{Alg}\, F_n \arrow[r, "\mathbf{Alg}\, w_{m,n}"] & \mathbf{Alg}\, F_m \\
\mathbf{Alg}\,(F_n, \mathcal{P}^m_n) \arrow[u, "U^{m,o}_n"] \arrow[r, "I^m_{n,m}"] & \mathbf{Alg}\,(F_m, \mathcal{P}^m_m). \arrow[u, "U^{m,o}_m"']
\end{tikzcd}$$

Let $x \in \{m, n\}$. The class $\mathcal{P}^m_x$ is generated by identities from $\mathcal{P}_x$ and by all unary polymeric identities $(\phi, \psi)_p$ in category of F_x-algebras obtained from natural identities $(\phi, \psi) \in \mathcal{N}^m$. This means that $\mathcal{P}^m_m$, $\mathcal{P}^m_n$ consist of all $(\phi, \psi)_p$, $(w_{m,n} \circ \phi, w_{m,n} \circ \psi)_p$, respectively, with $(\phi, \psi) \in \mathcal{N}^m_m$.

The Reiterman conversion sends the embedding of polymeric categories $U^{m,o}_n : \mathbf{Alg}(F_x, \mathcal{P}^m_x) \to \mathbf{Alg} F_x$ on the enlargement of theories id : $(t_{F_x}, \mathcal{I}_{F_x}) \to (t_{F_x}, \mathcal{J}^m_x)$ with $\mathcal{J}^m_x$ described bellow. We will use the notation, $t_x = t_{F_x}$, $\mathcal{I}_x = \mathcal{I}_{F_x}$, $\sigma^x = \sigma^{F_x}$ and the arity of the the polymeric t_x-term $\tau^{(2)}_X$ will be denoted in the superscript such as $\tau^{m,(2)}_X$.

The class $\mathcal{J}^m_x$ is the closure of $\mathcal{I}_x \cup \mathcal{B}_x \cup \mathcal{H}_x$. Here, $\mathcal{B}_x$ is the set of two kinds of polymeric equations obtained by Reiterman isomorphism from polymeric identities in $\mathcal{P}_x$ (as shown in Corollary 1.3.29) and $\mathcal{H}_x$ corresponds to $\mathcal{P}^m_x$. Hence $\mathcal{H}_x = \{(\overline{\phi_X} \cdot \sigma^x_X, \overline{\psi_X} \cdot \sigma^x_X) | X \in \mathrm{Ob}\mathcal{C}, (\phi, \psi)_p \in \mathcal{P}^m_x\}$ and $\mathcal{B}_x$ consists of identities:

1. $(\overline{\mathrm{id}_X}, \overline{\iota_{x,X}} \cdot \sigma^x_X)$ for every object X in $\mathcal{C}$,

2. $(\overline{q_{i,X}} \cdot \sigma^x_X, \overline{(y_x * w_{i,x})_X} \cdot \tau^{x,(2)}_X)$ for every object X in $\mathcal{C}$.

The functorial counterparts for the theories $(t_{F_x}, \mathcal{I}_x)$, $(t_{F_x}, \mathcal{J}^m_x)$ are the theories $(\mathcal{Q}_{F_x}, T_{F_x})$, $(\mathcal{Q}_{F_x}, T_{F_x})/\mathcal{J}^m_x = (\mathcal{Q}_{F_x}/\mathcal{J}^m_x, T_{F_x}/\mathcal{J}^m_x)$, respectively, where $T_{F_x}/\mathcal{J}^m_x$ is T prolonged into the factorized category.

The functor $\mathbf{Alg} w_{m,n}$ is algebraically concrete due to Lemma 1.3.31. The transformation $w_{m,n}$ induces, via functorial Reiterman conversion, the functor $\mathcal{Q}_{w_{m,n}} : \mathcal{Q}_{F_m} \Rightarrow \mathcal{Q}_{F_n}$ such that $\mathbf{Alg} w_{m,n} \cong \mathcal{Q}_{w_{m,n}}$–$\mathbf{alg}$. We will show, that there is a morphism of theories $W_{m,n} : (\mathcal{Q}_{F_m}, T_{F_m})/\mathcal{J}_m^m \to (\mathcal{Q}_{F_n}, T_{F_n})/\mathcal{J}_n^m$ and a commutative diagram

$$\begin{tikzcd}
\mathcal{Q}_{F_m} \arrow[r, "\mathcal{Q}_{w_{m,n}}"] \arrow[d, "{[\ \ ]_{\mathcal{J}_m^m}}"] & \mathcal{Q}_{F_m} \arrow[d, "{[\ \ ]_{\mathcal{J}_n^m}}"] \\
\mathcal{Q}_{F_m}/\mathcal{J}_m^m \arrow[r, "W_{i,j}"] & \mathcal{Q}_{F_n}/\mathcal{J}_n^m .
\end{tikzcd}$$

Here the surjections are given by enlargements of Reiterman theories. It is clear that it suffices to prove that the derived mapping $\underline{P_{w_{m,n}}} : \mathcal{T}(t_m) \to \mathcal{T}(t_n)$ sends an equation from $\mathcal{J}_m^m$ into $\mathcal{J}_n^m$. Here, $P_{w_{m,n}}$ is defined by the assignment

$$\sigma_X^m \mapsto \overline{w_{m,n,X}} \cdot \sigma_X^n$$

as in the proof of Lemma 1.3.31.

Let X be an object in $\mathcal{C}$. In the following, the equations in $\mathcal{J}_n^m$ will be denoted by $\sim$. Given a polymeric identity $(\phi, \psi)_p \in \mathcal{P}_m^m$, then $\underline{P_{w_{m,n}}}$ maps $(\overline{\phi_X} \cdot \sigma_X^m, \overline{\psi_X} \cdot \sigma_X^m)$ on $(\overline{\phi_X} \cdot \overline{w_{m,n,X}} \cdot \sigma_X^m, \overline{\psi_X} \cdot \overline{w_{m,n,X}} \cdot \sigma_X^m)$. But

$$\begin{array}{rcl}
\overline{\phi_X} \cdot \overline{w_{m,n,X}} \cdot \sigma_X^m & \sim & \overline{(w_{m,n} \circ \phi)_X} \cdot \sigma_X^m \\
& \sim & \overline{(w_{m,n} \circ \phi)_X} \cdot \sigma_X^n \\
& \sim & \overline{\psi_X} \cdot \overline{w_{m,n,X}} \cdot \sigma_X^m
\end{array}$$

hence $(\overline{\phi_X} \cdot \overline{w_{m,n,X}} \cdot \sigma_X^m, \overline{\psi_X} \cdot \overline{w_{m,n,X}} \cdot \sigma_X^m) \in \mathcal{J}_n^m$. Moreover, $\underline{P_{w_{m,n}}}$ sends the X-instance of the identity (1) on $\gamma = (\overline{\mathrm{id}_X}, \overline{\iota_{m,X}} \cdot \overline{w_{m,n,X}} \cdot \sigma_X^n)$. We have:

$$\overline{\mathrm{id}_X} \sim \overline{\iota_{n,X}} \cdot \sigma_X^n \sim \overline{(w_{m,n,X} \circ \iota_{m,X})} \cdot \sigma_X^n \sim \overline{\iota_{m,X}} \cdot \overline{w_{m,n,X}} \cdot \sigma_X^n,$$

hence $\gamma \in \mathcal{J}_n$. Recall from the Definition 1.3.13 that $\tau_X^{m,(2)} = \sigma_{F_mX}^m \cdot \sigma_X^m$, hence the X-instance of identity (2) is mapped on $\delta = (\overline{q_{i,X}} \cdot \overline{w_{m,n,X}} \cdot \sigma_X^n, \overline{(y_m * w_{i,m})_X} \cdot \overline{w_{m,n,F_mX}} \cdot \sigma_{F_mX}^n \cdot \overline{w_{m,n,X}} \cdot \sigma_X^n)$. Then we have

$$\begin{array}{rcl}
\overline{q_{i,X}} \cdot \overline{w_{m,n,X}} \cdot \sigma_X^n & \sim & \overline{Fw_{i,j,X}} \cdot \overline{q_{j,X}} \cdot \sigma_X^n \\
& \sim & \overline{Fw_{i,j,X}} \cdot \overline{(y_n * w_{j,n})_X} \cdot \sigma_{F_nX}^n \cdot \sigma_X^n \\
& \sim & \overline{(y_n * w_{j,n} \circ Fw_{i,j})_X} \cdot \sigma_{F_nX}^n \cdot \sigma_X^n
\end{array}$$

and due to properties of Godement product - see Remark B.0.3 - we have

$$\begin{array}{rcl}
(y_n * w_{j,n}) \circ Fw_{i,j} & = & y_n * (w_{j,n} \circ w_{i,j}) \\
& = & (w_{m,n} \circ y_m) * (w_{m,n} \circ w_{i,m}) \\
& = & (w_{m,n} * w_{m,n}) \circ (y_m * w_{i,m}) \\
& = & F_n w_{m,n} \circ w_{m,n} F_m \circ (y_m * w_{i,m})
\end{array}$$

hence

$$\begin{aligned}\overline{(y_n * w_{j,n} \circ Fw_{i,j})_X} \cdot \sigma^n_{F_nX} \cdot \sigma^n_X &\sim \overline{(y_m * w_{i,m})_X} \cdot \overline{w_{m,n,F_m,X}} \cdot \overline{F_n w_{m,n,X}} \cdot \sigma^n_{F_nX} \cdot \sigma^n_X \\ &\sim \overline{(y_m * w_{i,m})_X} \cdot \overline{w_{m,n,F_m,X}} \cdot \sigma^n_{F_mX} \cdot \overline{w_{m,n,X}} \cdot \sigma^n_X\end{aligned}$$

therefore $\delta \in \mathcal{J}_n$, hence $\underline{P_{w_{m,n}}}$ preserves identities and $W_{i,j} : \mathcal{Q}_m \to \mathcal{Q}_n$ is the corresponding factorization of $\mathcal{Q}_{w_{m,n}}$.

Finally, since $\mathcal{P}^n_n \supseteq \mathcal{P}^m_n$, the functor $U^{n,m}_n : \mathbf{Alg}\,(F_n, \mathcal{P}^n_n) \to \mathbf{Alg}\,(F_n, \mathcal{P}^m_n)$ induces an enlargement of equational theories id : $(t_{F_n}, \mathcal{J}^m_{F_n}) \to (t_{F_n}, \mathcal{J}^n_{F_n})$. This, converted to functorial theories, clearly yields a morphism

$$Z_{i,j} : (\mathcal{Q}_{F_m}, T_{F_m})/\mathcal{J}^m_n \to (\mathcal{Q}_{F_n}, T_{F_n})/\mathcal{J}^n_n$$

of theories such that $Z_{i,j}-\mathbf{alg} = U^{n,m}_n$.

Hence, we have found a collection of morphisms

$$D_{i,j} = R^{-1}_{i+1} \circ Z_{i,j} \circ W_{i,j} \circ R_{j+1} : \mathbf{Alg}\,(F_{j+1}, \mathcal{P}^{j+1}_{j+1}) \to \mathbf{Alg}\,(F_{i+1}, \mathcal{P}^{i+1}_{i+1})$$

such that $D_{i,j} - \mathbf{alg} = S_{i,j}$ for every $i \leq j \in \mathrm{Ord}$. Thus, the diagram $S = -\mathbf{alg}^f \circ D$ is algebraically concrete. □

As a consequence, we get the final theorem which connects two categorical approaches to variety-like classes.

Theorem 2.2.21 *Every variety is an algebraic category.*

Proof: Due to the previous lemma, for every variety $\mathcal{V}$ there is an ordinal chain D of functorial theories, such that $\mathcal{V}$ is the limit of $D_i-\mathbf{alg}$. But, due to the Metacategorical remark 1.3.8, the colimit K of D_i exists. And since the contravariant functor $-\mathbf{alg}^f$ turns colimits into limits (Lemma 1.3.13), $K-\mathbf{alg}$ is the limit of $D_i-\mathbf{alg}$, hence

$$\mathcal{V} \cong_{\mathcal{C}} K-\mathbf{alg}.$$

□

Chapter 3

Examples

3.1 Algebras and Coalgebras

Many examples of algebras and coalgebras can be found in [7], [9] and [11]. As representatives we show the following:

Example 3.1.1 *Let $\mathcal{C} = \mathcal{S}et$, $\Sigma = \{\times, \star\}$ be single-sorted signature with $ar(\times) = 2, ar(\star) = 0$.*

- *The Σ-algebras are groupoids with a fixed point, i.e., the sets equipped with the binary operation (the realization of $\times$) and with one chosen element (the realization of $\star$).*

- *The Σ-coalgebras are the dynamic systems with two inputs and a set of final states, i.e., the sets equipped with a transition function assigning, to every element, a pair of other elements - the transition along both inputs (the "corealization" of $\times$) or nothing - final state (the "corealization" of $\star$).*

Example 3.1.2 *Let P be a powerset functor.*

- *A P-algebra is a sets A equipped with function $\mathsf{P}A \to A$. The category* **Alg** *P contains the category of all join-complete semilattices as the subcategory (Example 3.2.3).*

- *The P-coalgebras can be seen as directed graphs (the coalgebra function $f : A \to \mathsf{P}A$ yields the binary relation $EDGE$ given by*

$$(a, b) \in EDGE \Leftrightarrow a \in f(b).$$

  However the P-coalgebra morphisms are the graph homomorphisms reflecting the incidence of edges, i.e., the image of an edge e is incident *with an edge d iff d itself is an image of some edge d' incident with the edge e.*

Here we show less standard examples of algebras and coalgebras.

Example 3.1.3 *Given a functor $F : \mathcal{C} \to \mathcal{C}$, then there is an endofunctor F_{alg} on $\mathbf{Alg}\,F$ given by the assignment $(A, \alpha) \mapsto (FA, F\alpha)$. Then the categories $\mathbf{Alg}_{\mathbf{Alg}\,F} F_{alg}$ and $\mathbf{Coalg}_{\mathbf{Alg}\,F} F_{alg}$ consist of all triples (A, α, β) making the following squares commute:*

Alg **Coalg**

$$\begin{tikzcd} F^2A \arrow[r, "F\alpha"] \arrow[d, "F\beta"'] & FA \arrow[d, "\beta"] \\ FA \arrow[r, "\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "F\beta"'] & A \arrow[d, "\beta"] \\ F^2A \arrow[r, "F\alpha"] & FA, \end{tikzcd}$$

i.e., satisfying equations $\alpha \circ F\beta = \beta \circ F\alpha$ and $F\alpha \circ F\beta = \beta \circ \alpha$, respectively. Let us note, that here α always stands for the structure arrow of the underlying algebra. Dually, we may consider an endofunctor F_{coalg} on $\mathbf{Coalg}F$ given by the notationally same assignment as F_{alg}. Then the categories $\mathbf{Alg}_{\mathbf{Coalg}F} F_{coalg}$ and $\mathbf{Coalg}_{\mathbf{Coalg}F} F_{coalg}$ are, in sense of the previous discussion, given by the equations $\alpha \circ \beta = F\beta \circ F\alpha$ and $F\alpha \circ \beta = F\beta \circ \alpha$, respectively, where β stands for the structure arrow of the underlying coalgebra. Namely, if $F = \mathrm{Id}_{\mathcal{C}}$, in all four cases we get the pairs (α, β) of endomorphisms on some base object A satisfying $\alpha \circ \beta = \beta \circ \alpha$.

Example 3.1.4 *Let $\mathcal{C} = \mathbf{Fld}$ be category of fields. Then $\mathbf{Alg}_{\mathcal{C}}\mathrm{Id}$, and $\mathbf{Alg}_{\mathcal{C}}\mathrm{Id} \times_{\mathcal{C}} \mathbf{Alg}_{\mathcal{C}}\mathrm{Id}$ are categories of fields with an endomorphism, a pair of endomorphisms, respectively. Since $\mathbf{Fld}$ does not have colimits, the latter l-algebraic category is neither a variety nor a polymeric variety over $\mathbf{Fld}$.*

If P is a field and $F = \mathcal{C}_P$ is the corresponding constant functor, then $\mathbf{Alg}_{\mathcal{C}}F$ is the category of extensions of P.

3.2 Varieties and Covarieties

As shown in [9], every variety of algebras in classical sense is equational class. Hence, due to the Conversion theorem, it is a naturally induced class. An explicit correspondence is shown in the following.

3.2.1 Classical Varieties as Naturally Induced Classes

Let $\mathcal{C} = \mathbf{Set}$, Σ be a single-sorted signature with arity function $ar : \Sigma \to \mathrm{Ord}$. Let $F = \coprod_{\sigma \in \Sigma} \hom(ar(\sigma), -)$ and $u_\sigma : \hom(ar(\sigma), -) \to F$ be the canonical inclusion for every $\sigma \in \Sigma$. Then, as discussed in Remark 1.1.3, the category of Σ-algebras is isomorphic to $\mathbf{Alg}\, F$. For each Σ-term τ, let X_τ be the set of variables occurring in τ and let $d(\tau)$ be the depth of τ (supremum of ordinals corresponding to the chains of the proper subterms of τ ordered by the subterm-relation).

We define a $d(\tau)$-ary G_τ-term ϕ_τ where $G_\tau = \hom(X_\tau, -)$ for a given term τ. The transformation ϕ_τ is defined inductively: if τ is a variable x, then $\phi_\tau : \hom(\{x\}, -) \to F_0$ is an obvious isomorphism. If $\tau = \sigma(\rho_i; i \in ar(\sigma))$ and we have $\phi_{\rho_i} : \hom(ar(\rho_i), -) \to F_{d(\rho_i)}$ for each $i \in ar(\sigma)$, we can extend all transformations ϕ_{ρ_i} to $\phi'_i : \hom(ar(\rho_i), -) \to F_n$ where $n = \sup\{d(\rho_i) | i \in ar(\sigma)\}$. We define ϕ_τ in the following way. Since $X_{\rho_i} \subseteq X_\tau$ for every i, we have $p_i : \hom(X_\tau, -) \to \hom(X_{\rho_i}, -)$, hence the factorization over the limit cone yields a unique $r : \hom(X_\tau, -) \to \prod_{i \in ar(\sigma)} \hom(X_{\rho_i}, -)$. We define ϕ_τ as the following composition:

$$\begin{tikzcd}
\hom(X_\tau, -) \arrow[r, "r"] \arrow[d, "\phi_\tau"] & \prod\limits_{i \in ar(\sigma)} \hom(X_{\rho_i}, -) \arrow[r, "\Pi \phi'_i"] & \prod\limits_{i \in ar(\sigma)} F_n \arrow[d, equal, "iso"] \\
F_{n+1} & F F_n \arrow[l, "q_n"'] & \hom(ar(\sigma), -) \circ F_n \arrow[l, "u_\sigma F_n"']
\end{tikzcd}$$

Observe that $n + 1 = d(\tau)$.

To each Σ-term, we have assigned a natural term. Now, to an identity (τ_1, τ_2) consisting of two Σ-terms with variables in X, we assign a pair of corresponding natural $\hom(X, -)$-terms. It is easy to see, that we get an identity which induces exactly the variety given by (τ_1, τ_2). Monoids, for example, are objects of $\mathbf{Alg}\,((\hom(2, -) + \hom(0, -), \{i, j, k\})$, where i is a binary identity with domain $\hom(3, -)$ and stands for associativity while j, k are unary with domain Id and correspond to left and right neutrality of 1.

A polymeric term in $\mathbf{Alg}\Sigma$ for a signature Σ corresponds to a Σ-tree given by the following restriction: all the branches having the variables in leaves have the same length and the others (i.e. the branches with constant symbols in leaves) are not longer. Hence commutative groupoids form a polymeric variety while semigroups do not.

3.2.2 Other Examples

The concept above may be used to define naturally induced classes of algebras even on some illegitimate categories. The following example reformulates the well-known property that each monad on a category $\mathcal{A}$ is defined as a monoid in the strict monoidal category $\mathbf{End}\mathcal{A}$ (see [24]).

Example 3.2.1 *Let $\mathcal{C} = \mathbf{End}\mathcal{A}$ be the metacategory of endofunctors on some cocomplete category $\mathcal{A}$. The composability of objects of $\mathcal{C}$ yields, for every $k \in \omega$, the existence of the composition power functor $S_k : \mathcal{C} \to \mathcal{C}$ given by composition on functors and by Godement product on natural transformations (see Definition B.0.2). We can define analogies for universal algebras - all we need to do is to substitute composition of functors for the products of sets and S_k for each* $\hom(k, -)$ *in the description above. As analogy of monoids, we get the category* $\mathbf{Monad}\,\mathcal{A}$ *of monads on $\mathcal{A}$. Namely,* $\mathbf{Monad}\,\mathcal{A} = \mathbf{Alg}\,((S_2 + S_0), \{i, j, k\})$*, where domains of identities i, j, k are S_3, S_1, S_1, respectively. Each operation $\pi : (S_2 + S_0)(P) \to P$ decomposes into $\mu : S_2(P) = PP \to P$ and $\eta : S_0 = \mathrm{Id} \to P$ and the satisfaction of identities fully corresponds to usual condition claimed on μ and η.*

Another example is quite general, frequently used by G. M. Kelly in [20].

Example 3.2.2 *Let $F : \mathcal{C} \to \mathcal{C}$ be* pointed functor*, i.e., a functor equipped with a natural transformation $t : \mathrm{Id} \to F$. Then (A, α) is a* pointed F-algebra *iff $\alpha \circ t_A = \mathrm{id}_A$, i.e.*

$$(A, \alpha) \models (\mathrm{id}, t)_p.$$

Theorem 3.6 in [9] describes the equational presentation of category of algebras for a monad. The Lemma 1.2.11 shows their presentation by polymeric identities. Here we depict its instance for powerset monad.

Example 3.2.3 *Consider the power-set monad on $\mathcal{S}et$ defined by power-set functor P and transformations $\eta : \mathrm{Id}_{\mathbf{Set}} \to \mathsf{P}$, $\mu : \mathsf{P}^2 \to \mathsf{P}$ given by assignments $\eta_X(x) = \{x\}$, $\mu_X(\{X_i | i \in I\}) = \bigcup_{i \in I} X_i$. As a concrete instance of the Eilenberg-Moore category for power-set monad (P, η, μ) we get the category of join-complete semilattices* **JCSlat**. *Hence, due to Lemma 1.2.11, this class is presentable by a pair of polymeric identities - compare with presentation by a proper class of equation arrows (see [9], Example 3.3 - we need equation arrows $e_X : F_3X \to E_X$ for every set X).*

Example 3.2.4 *Given a functor $F : \mathcal{C} \to \mathcal{C}$, then there is a natural transformation $h : F_{alg} \to \mathrm{Id}_{\mathbf{Alg}\,F}$ given by $h_{(A,\alpha)} = \alpha$. Moreover if $\mathcal{C}$ has (*$\mathrm{Epi}, \mathrm{Mono}$*)-factorizations, then there is an* $\mathbf{Alg}\,F$*-endofunctor*

$$\mathrm{Im}_{alg} : (A, \alpha) \mapsto (\mathrm{Im}(\alpha), \alpha_{\mathrm{Epi}} \circ F\alpha_{\mathrm{Mono}})$$

(determined uniquely up to isomorphism) and a natural transformation

$$d : \mathrm{Im}_{alg} \to \mathrm{Id}_{\mathbf{Alg}\,F}$$

with $d_{(A,\alpha)} = \alpha_{\mathrm{Mono}}$.

The class of F*-algebras with the structure arrow being an isomorphism (respectively epimorphism) forms a polymeric covariety in* $\mathbf{Coalg}F_{alg}$ *(resp. in* $\mathbf{Coalg}\mathrm{Im}_{alg}$*) induced by* $(\mathrm{id}, h)_p$ *(resp.* $(\mathrm{id}, d)_p$*). Since covarieties are classes which tend be coreflective, this coheres the result of J. Lambek from [23], which proves reflectivity of dually defined class of coalgebras (the category of fixpoints).*

Part II

Universality and Freeness

Chapter 4

Free Algebras

Free algebras play an important role in universal algebra. They enable us to study the properties of a family of algebras within a single algebra. On the other hand, in the theory of coalgebras one often asks for the existence of terminal (final) coalgebras, which are mathematical counterparts for universal automata in computer science. Both of these notions have a property in common: they are "best" in some sense - either universal or couniversal with respect to some properties, which are easily expressible for every concrete category. These objects are the matter of study for this part of the thesis. We will focus only on the free objects and the corresponding results for cofree objects follow from duality.

Our aim for this chapter is to prove the theorem, which states the sufficient condition for a variety having the free objects. It will be done in Section 4.2.3. Before that, we need to recall the useful facts. They include some well known results on free monads (4.1.1), Eilenberg-Moore categories (4.1.2), very important Kelly's theorem (4.1.3) and free-algebra chain construction (4.2). Most of the facts are taken from [24], [7] and [20].

4.1 Freeness and Monadicity

4.1.1 Adjunction: Free $\dashv$ Forgetful

Definition 4.1.1 *The endofunctor F on category $\mathcal{C}$ is called a* varietor *if the free F-algebra exists over every object of $\mathcal{C}$.*

Let F be a varietor on a category $\mathcal{C}$. Then by Remark B.1.8 the forgetful functor $U = U_{\mathbf{Alg}\,F} : \mathbf{Alg}\,F \to \mathcal{C}$ has a left adjoint $W : \mathcal{C} \to \mathbf{Alg}\,F$. Let $V = U \circ W$, then it is easy to see that there exists a natural transformation $\upsilon : F \circ V \to V$ such that $WA = (VA, \upsilon_A)$. The adjunction $W \dashv U$ has the unit η, where each universal arrow $\eta_A : A \to VA$ can be considered an "insertion of generators". The counit $\epsilon : UW \to \mathrm{Id}_{\mathbf{Alg}\,F}$ is given by $\epsilon_{(A,\alpha)} = \widetilde{\mathrm{id}_A} : UW(A,\alpha) \to (A,\alpha)$. Since every adjoint situation induces a monad, we have a monad $\underline{M} = (V,\eta,\mu)$, where $\mu_A = U\epsilon_{VA,\upsilon_A}$.

We recall two important properties. Although they are not crucial for the following results of this thesis, we present them to enable deeper understanding of the relation of the freeness on the level of algebras and on the level of monads. The proof of the following well known fact can be found e.g. in [7], Theorem 20.56. Let $Z : \mathbf{Monad}\mathcal{C} \to \mathbf{End}\mathcal{C}$ be the forgetful functor.

Proposition 4.1.1 (Free monad property) *The monad $\underline{M}_F = (U_{\mathbf{Alg}\,F}W,\eta,\mu)$ induced by adjunction $W \dashv U_{\mathbf{Alg}\,F}$ is the free monad over the functor F in the metacategory of monads. Namely, the monad $\underline{M}_F$ has a Z-universal arrow $y : F \to Z\underline{M}_F$ for the functor F such that*

$$y \;=\; \upsilon \circ F\eta, \tag{4.1}$$

where $\upsilon : FU_{\mathbf{Alg}\,F}W \to U_{\mathbf{Alg}\,F}W$ is the structure arrow of the free algebra.

It means that for each monad $\underline{M}' = (M,i,h)$ and a transformation $\rho : F \to M$ there is a unique monad transformation $\overline{\rho} : \underline{M}_F \to \underline{M}'$ such that $\overline{\rho} \circ y = \rho$. One can derive another auxiliary equality:

$$\upsilon \;=\; \mu \circ yV. \tag{4.2}$$

More detailed approach to the theory of monads can be found in [7], [15] and [20].

Now we use another varietor G and work with its algebras. The corresponding entities for each of the functors F, G will be distinguished by the notation in the subscript (for the functors) and superscript (for the transformations). The discussion above has the following consequences.

Proposition 4.1.2 (Extension property) *Let there be a transformation $\rho : G \to V_F$. Then there is a transformation $\sigma : V_G \to V_F$, subject to the conditions:*

1. *$\sigma = \overline{\rho}$ is given by the freeness of $\underline{M}_F$ as the unique monad transformation $\underline{M}_G \to \underline{M}_F$ corresponding to $\rho : G \to V_F$; thus*

$$\sigma \circ y^G = \rho.$$

2. *given a $\mathcal{C}$-object A, $\sigma_A = \widetilde{\eta^F_A}$ is given by the adjunction $W_G \dashv U_G$ as the unique G-algebra morphism $W_G(A) \to P(A)$ corresponding to*

$\eta^F_A : A \to V_F A = U_G P(A)$, where $P : \mathcal{C} \to \mathbf{Alg}G$ is a functor assigning, to an object A, a G-algebra $(V_F A, \beta_A)$ and $\beta_A = (\mu^F \circ \rho V_F)_A$; hence

$$\sigma \circ \eta^G = \eta^F.$$

3. $\sigma = U_G \epsilon^G P \circ V_G \eta^F$.

Proof: Since $V_F = Z(\underline{M}_F)$ then the Z-universal arrow y^G yields the existence of a monad transformation $\sigma = \overline{\rho} : \underline{M}_G \to \underline{M}_F$ such that $\overline{\rho} \circ y^G = \rho$. Consider the transformation $\beta = \mu^F \circ \rho V_F : GV_F \to V_F$ and the functor $P = (V_F, \beta) : \mathcal{C} \to \mathbf{Alg}\, G$ from the lemma. Due to (4.2) and $\overline{\rho}$ being a monad transformation, we have the following commutative diagram

$$\begin{tikzcd}
GV_G \arrow[rrrr, "G\overline{\rho}"] \arrow[dr, "y^G V_G"] \arrow[ddd, "\upsilon^G"'] & & & & GV_F \arrow[dll, "y^G V_F"'] \arrow[ddll, "\rho V_F"] \arrow[ddd, "\beta"] \\
& V_G^2 \arrow[r, "V_G\overline{\rho}"] \arrow[dr, "\overline{\rho} * \overline{\rho}"'] \arrow[ddl, "\mu^G"] & V_G V_F \arrow[d, "\overline{\rho} V_F"'] & & \\
& & V_F^2 \arrow[drr, "\mu^F"] & & \\
V_G \arrow[rrrr, "\overline{\rho}"] & & & & V_F
\end{tikzcd}$$

where $*$ is the Godement product (see Appendix A). The diagram yields, for each $\mathcal{C}$-object A, a G-algebra morphism $\overline{\rho}_A : (V_G, \upsilon^G_A) \to (V_F A, \beta_A)$. Since $\overline{\rho}$ is a monad transformation, it satisfies $\overline{\rho} \circ \eta^G = \eta^F$. From the uniqueness of factorization we have $\overline{\rho}_A = \widetilde{\eta^F_A}$ for each A. Therefore σ satisfies the first two properties. Since it can be seen as $U_G \widetilde{\eta^F}$ and since we have the adjunction $W_G \dashv U$, it follows that $\sigma = U_G \epsilon^G P \circ U_G W_G \eta^F = U_G \epsilon^G P \circ V_G \eta^F$. □

4.1.2 Eilenberg-Moore Category

We recall some facts on free algebras for a monad. Most of the results and their proofs can be found in [7].

Given a monad $\underline{M} = (M, \eta, \mu)$, then the Eilenberg-Moore category $\underline{M}$–**alg** has free objects. The corresponding free functor $W_{\underline{M}\text{–}\mathbf{alg}}$ assigns, to an object A, the algebra (MA, μ_A). It is indeed an object in $\underline{M} - \mathbf{alg}$ and it is easy to prove, that this functor is left adjoint to $U_{\underline{M}\text{–}\mathbf{alg}}$. Then, it yields the adjunction $W_{\underline{M}\text{–}\mathbf{alg}} \dashv U_{\underline{M}\text{–}\mathbf{alg}}$ with the associated monad equal to $\underline{M}$. Since the adjunction yields a codensity monad (Proposition B.2.10), the forgetful functor $U_{\underline{M}\text{–}\mathbf{alg}}$ has codensity monad - it is $\underline{M}$ itself.

The following theorem characterizes all monadic categories:

Theorem 4.1.3 (Beck's theorem)
Let $\mathcal{A}$ be a concrete category. Then

$$\mathcal{A} \text{ is monadic} \Leftrightarrow \mathcal{A} \text{ is a Beck category with free objects.}$$

Remark 4.1.4 *Given a concrete category $(\mathcal{A}, U_{\mathcal{A}})$ with free objects, consider the adjunction $(\eta, \epsilon) : W_{\mathcal{A}} \dashv U_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}$ with the associated monad $\underline{M} = (M, \eta, \mu)$ and the Eilenberg-Moore category $(\underline{M}-\mathbf{alg}, U_{\underline{M}\text{–}\mathbf{alg}})$ with the free functor $W_{\underline{M}\text{–}\mathbf{alg}}$. Then there is a unique* comparison functor $I : \mathcal{A} \to \underline{M}-\mathbf{alg}$ *satisfying:*

- *I is concrete, i.e., $U_{\underline{M}\text{–}\mathbf{alg}} \circ I = U_{\mathcal{A}}$*
- *I preserves freeness, i.e., $I \circ W_{\mathcal{A}} = W_{\underline{M}\text{–}\mathbf{alg}}$*

The functor I assigns, to an $\mathcal{A}$-object α, the M-algebra $(U_{\mathcal{A}}(\alpha), \epsilon_\alpha)$.

The following fact will be useful:

Proposition 4.1.5 *The comparison functor for a monadic category is an isomorphism.*

4.1.3 Kelly's Theorem

In the paper [20], chapter VIII., G. M. Kelly asked about the existence of algebraic colimit of the diagram of monads as defined in 1.2.2. It came out to be equivalent to existence of the free objects in the category D–**alg** of algebras for a diagram D of monads. He proved the existence in his Theorem 27.1 in [20] under the general assumptions of existence of suitable factorization systems and some smallness requirements for the monads. Using the trivial factorization system $(\mathrm{Iso}, \mathrm{Mor})$ and preservation of colimits of λ-chains, we get this theorem in the following form:

Theorem 4.1.6 (Kelly's theorem) *Let the underlying functor of each $D(x)$ preserve the colimits of λ-chains. Then $D-$**alg** has free objects.*

This statement will be crucial for the existence of free algebras in a variety - see Section 4.2.3.

4.2 Free Algebras via Chain Construction

We recall here more of the results obtained from the free-algebra construction. Namely, we show the description of the free algebra as was done in [5] but we use it also in relation to varieties. It enables us to convert the problem of free algebras in a variety to the problem of free algebras for a diagram of monads which was solved out by Kelly in [20]. This is how we reach one of the main results of the thesis (Theorem 4.2.5). Author is grateful to his supervisor Jiří Rosický for his advise to connect the matter of varieties with concept of Kelly.

4.2.1 Free Algebras in an F-algebraic Category

This section summarizes some results from [5], [9] and [8].

Definition 4.2.1 *Let λ be an infinite limit ordinal. Given a category $\mathcal{C}$, a diagram $\lambda \to \mathcal{C}$ will be called* λ-chain.

We will work with $\mathcal{C}$-endofunctors preserving the colimits of λ-chains where λ is an infinite limit ordinal; let the class containing these functors be denoted by $\mathbf{End}_\lambda\mathcal{C}$. Since the colimits commute with colimits we get the following property (see also [20], 2.4.).

Proposition 4.2.1 *The class $\mathbf{End}_\lambda\mathcal{C}$ is closed under colimits and compositions.*

Let F preserve the colimits of λ-chains. Recall the free-algebra chain construction:

$$A \xrightarrow[w_{0,1,A}]{\iota_{A+FA}} A + FA \xrightarrow[w_{1,2,A}]{\mathrm{id}_A + Fw_{0,1,A}} A + F(A + FA) \xrightarrow[q_{2,3,A}]{\mathrm{id}_A + Fq_{1,2,A}} \cdots$$

As proved in [5], this situation yields that F is a varietor. We will explain briefly how the construction works. Since F preserves the colimits of λ-chains, FF_λ is a colimit of chain $\{FF_n | n < \lambda\}$. Hence one can see that $w_{\lambda,\lambda+1}$ is an isomorphism. In such a case we say that *the free F-algebra construction stops after λ steps* and the functor F is called *constructive varietor*. Let $\upsilon = \operatorname*{colim}_{n<\lambda} q_n$. Given an object A in $\mathcal{C}$, we get the free F-algebra over A in the form

$$W_F A = (F_\lambda A, \upsilon_A).$$

The property of being constructive varietor is generally stronger then of being just a varietor - see [10] for the counterexample.

Dually, we get the construction of cofree coalgebra. If it stops, i.e., some of the connecting morphisms is an isomorphism, then cofree coalgebra exists.

Proposition 4.2.2 *For every λ-accessible set functor both free-algebra and cofree-coalgebra constructions stop.*

Proof: The situation for algebras is discussed above. For the coalgebras we need the theorem of J. Worrell [39] which proves that if $F \in \mathbf{End}_\lambda \mathcal{C}$, then the chain stops after $\lambda + \lambda$ steps. $\square$

Corollary 4.2.3 *For every signature Σ on the base category $\mathcal{S}et$, the categories $\mathbf{Alg}\,\Sigma$ and $\mathbf{Coalg}\Sigma$ have free, cofree, respectively, objects over every object.*

4.2.2 Comparison Functors for F-algebraic Categories

Out of the main line of research, we show an alternative description of f-algebraic categories and Eilenberg-Moore categories for a free monad.

Let $F \in \mathbf{End}_\kappa \mathcal{C}$ for some infinite limit ordinal κ and let $\lambda = \kappa + 1$. The adjunction given by the freeness on $\mathbf{Alg}\,F$ yields the free monad $\underline{M}_F = (V_F, \eta, \mu)$ over F. But since the free algebra $W_F(A)$ over an object A is in the form $(F_\kappa A, \upsilon_A)$ and $F_\kappa \cong F_\lambda$ we may assume $V_F = F_\lambda$. The unit of the adjunction is clearly $\eta = \iota_\lambda$, while one can see that the counit is given by the evaluation of terms on the level of λ, i.e., $\mu = \epsilon_{\lambda,(F_\lambda,q_\lambda)}$. Then $\mathbf{Alg}\,F \cong \underline{M}_F - \mathbf{alg}$ and that is, due to Lemma 1.2.11, a polymeric variety induced by polymeric identities $h_{\lambda,0} = (\iota_\lambda, \mathrm{id}_{\mathrm{Id}})_p$ and $h'_\lambda = (\epsilon_{\lambda,(F_\lambda,q_\lambda)}, \mathrm{id}_{F^2_\lambda})_p$. The Lemma 2.2.14 yields the isomorphism $I_\lambda : \mathbf{Alg}F \cong \mathbf{Alg}(F_\lambda, \{h_{\lambda,0}, h_{\kappa,\lambda}\})$, where $h_{\kappa,\lambda}$ can be written as the polymeric identity $(q_\lambda, y_\lambda F_\lambda)_p$, because $w_{\kappa,\lambda}$ is an isomorphism. The assignment $(A,\alpha) \mapsto (A, \epsilon_{(A,\alpha)})$ defines both the functor I_λ and the comparison functor I and since they are isomorphisms (Lemma 2.2.14 and Proposition 4.1.5), the polymeric presentations of $\mathbf{Alg}\,F$ in $\mathbf{Alg}\,F_\lambda$ are equal:

$$\mathbf{Alg}\,F \cong_{\mathcal{C}} \mathbf{Alg}\,(F_\lambda, \{h_{\lambda,0}, h_{\kappa,\lambda}\}) = \mathbf{Alg}\,(F_\lambda, \{h_{\lambda,0} h'_\lambda\}).$$

Hence, $I_\lambda = I$ and the isomorphisms $I_{n+1} : \mathbf{Alg}F \to \mathbf{Alg}(F_{n+1}, \{h_{\lambda,0}, h_{n,n+1}\})$ for $n < \lambda$ are sort of "lower instances" of the comparison functor.

It also implies the algebraic equivalence of $\{h_{\lambda,0}, h_{\kappa,\lambda}\}$ and $\{h_{\lambda,0}, h'_\lambda\}$. However, since $q_\lambda = \epsilon_{\lambda,(F_\lambda,q_\lambda)} \circ y_\lambda F_\lambda$, the former is satisfied by an F_λ-algebra (A,α) iff the following conditions hold:

$$\alpha \circ \eta_A = \mathrm{Id}_A, \tag{4.3}$$
$$\alpha \circ F_\lambda \alpha \circ (y_\lambda F_\lambda)_A = \alpha \circ \mu_A \circ (y_\lambda F_\lambda)_A\,. \tag{4.4}$$

We compare them with Eilenberg-Moore identities $\{h_{\lambda,0}, h'_\lambda\}$ and since h'_λ can be rewritten as $\alpha \circ F_\lambda \alpha = \alpha \circ \mu_A$, clearly $h'_\lambda \Rightarrow h_{\kappa,\lambda}$ but the converse does not hold generally. However, their conjunctions with $h_{\lambda,0}$ are equivalent.

4.2.3 Free Algebras in a Variety

Our aim is to find the free algebras in varieties. To prove the existence theorem, we will need the restriction on the following cases.

Definition 4.2.2 *Let G be an endofunctor on $\mathcal{C}$. The natural G-identity is called* accessible *if G preserves the colimits of λ-chains for some infinite limit ordinal λ.*

Connection to the Concept of Kelly

We will use Kelly's theorem 4.1.6 to prove that, in a variety induced by accessible identities, the free objects exist. Let F be a functor in $\mathbf{End}_\kappa\mathcal{C}$ for some infinite limit ordinal κ and consider the variety of F-algebras induced by a set of accessible natural identities. Since the free F-algebra construction stops after κ steps, we may consider arity of each natural term to be less or equal to κ. Then, due to the Remark 2.2.1, a single identity (ϕ, ψ) can be substituted for the whole set of natural identities. Its domain, denoted by G, is the coproduct of domains of single identities, hence, due to 4.2.1, it preserves colimits of ν-chains for some limit ordinal ν large enough. Let $\lambda = \max\{\kappa, \nu\}$, then $F, G \in \mathbf{End}_\lambda\mathcal{C}$. Hence the arity of (ϕ, ψ) can be set to λ.

Lemma 4.2.4 *For the λ-ary G-identity (ϕ, ψ) there exists a diagram D of monads such that:*

$$\mathbf{Alg}\,(F, (\phi, \psi)) \cong_{\mathcal{C}} D{-}\mathbf{alg}.$$

Proof: Since both F and G are varietors, both comparison functors $I_F : \mathbf{Alg}\,F \to \underline{M}_F{-}\mathbf{alg}$ and $I_G : \mathbf{Alg}\,G \to \underline{M}_G{-}\mathbf{alg}$ are isomorphisms. Using the diagram from the Lemma 2.2.7 we may get $\mathbf{Alg}\,(F, (\phi, \psi))$ as an equalizer of

$$\underline{M}_F{-}\mathbf{alg} \xrightarrow{I_F^{-1}} \mathbf{Alg}\,F \xrightarrow{E_n} \mathbf{Alg}\,F_n \underset{\mathbf{Alg}\,\psi}{\overset{\mathbf{Alg}\,\phi}{\rightrightarrows}} \mathbf{Alg}\,G \xrightarrow{I_G} \underline{M}_G{-}\mathbf{alg}.$$

Since every concrete functor between monadic categories is an $-\mathbf{alg}$-image of a monad transformation (see Lemma 1.1.8), we have $\mathbf{Alg}(F, (\phi, \psi))$ isomorphic to the limit of $-\mathbf{alg} \circ D$ for a suitable diagram D of monads.

□

Now we can use Kelly's theorem to conclude our investigation:

Theorem 4.2.5 *Let F preserve the colimits of λ-chains for some limit ordinal λ. Then the free algebra exists in every variety induced by a set of accessible identities.*

To express the consequence for the varieties presented by equation arrows, recall the notion of presentability of an object (see [12]):

Definition 4.2.3 *Let λ be a regular cardinal. An object A of a category is called* λ-presentable *provided that its hom-functor* $\hom(A, -)$ *preserves λ-directed colimits. An object is called* presentable *if it is λ-presentable for some λ.*

Theorem 4.2.6 *Let F preserve the colimits of λ-chains for some limit ordinal λ. Then the free algebra exists in every variety induced by a set of equation arrows with presentable variable-objects.*

Proof: As stated in Remark 2.2.3, an equation arrow $e : F_n X \to E$ converts to a natural identity with the domain $G_{Q,X} = (- \bullet Q) \circ \hom(X, -)$ for some $Q \in \mathrm{Ob}\mathcal{C}$. If the variable-object X is presentable, $\hom(X, -)$ preserves κ-directed colimits for some κ and, since $(- \bullet Q)$ is left adjoint, G preserves κ-directed colimits, too. Therefore the colimits of κ-chains are preserved and due to Theorem 4.2.5 the corresponding variety has free objects. The rest is obvious. $\square$

Chapter 5

Universality and Codensity Monads

In order to solve some problems involving categories of algebras (coalgebras) on category $\mathcal{C}$ such as existence of free (cofree) objects, it might be useful to know their properties on the level of objects of metacategory $\mathbf{Con}\,\mathcal{C}$. Their status can be expressed in terms of properties of contravariant functor $\mathbf{Alg}$ for algebras and functor $\mathbf{Coalg}$ for coalgebras. Since both concepts are mutually dual, it suffices to study only one of them. Namely we focus on the property of categories having universal arrows for these functors. To simplify the proofs, we express the concept of universality in the language of Kan extensions and codensity monads. The main results characterize free-objects existence in l-algebraic categories and all Beck categories and provide an alternative characterization of monadic categories.

The author acknowledges the advises from H. E. Porst, J. Rosický and J. Velebil concerning the connection of universality and the concept of Kan extensions.

All the topic in this chapter deals with the category $\mathcal{C}$, generally without any additional assumptions.

5.1 Universality and Kan Extensions

Remark 5.1.1 *Let F be an endofunctor on $\mathcal{C}$, $U_F : \mathbf{Alg}\,F \to \mathcal{C}$ be the forgetful functor and $\mathcal{A}$ be a category. Then each functor $H : \mathcal{A} \to \mathbf{Alg}\,F$ can be seen as $H = (K, \kappa)$ where $K = U_F \circ H : \mathcal{A} \to \mathcal{C}$ and $\kappa : FK \to K$ is a natural transformation such that $HA = (KA, \kappa_A)$ for every object A in $\mathcal{A}$. Moreover, if $(\mathcal{A}, U_\mathcal{A})$ is a $\mathcal{C}$-concrete category and H is concrete, then $K = U_\mathcal{A}$. This property can be easily expressed in terms of isomorphism* *between two (illegitimate) contravariant functors*

$$\hom_{\mathbf{Con}\,\mathcal{C}}(\mathcal{A},-)\circ\mathbf{Alg} \quad\cong\quad \hom_{\mathbf{End}\mathcal{C}}(-,U_{\mathcal{A}})\circ\hom_{\mathbf{CAT}}(U_{\mathcal{A}},\mathcal{C}), \quad (5.1)$$

where $\hom_{\mathbf{CAT}}(U_{\mathcal{A}},\mathcal{C})$ *is given by the assignment* $F \mapsto F\circ U_{\mathcal{A}}$ *(see Metacategorical remark B.2.1).*

Recall the concept of universal arrows (see Appendix B). We will apply it on the contravariant functor $\mathbf{Alg}$ between the metacategories $\mathbf{End}\mathcal{C} \to \mathbf{Con}\,\mathcal{C}$ as follows (see Remark B.1.2):
Given a $\mathcal{C}$-concrete category $\mathcal{A}$, a $\mathcal{C}$-endofunctor F with a concrete functor $H : \mathcal{A} \to \mathbf{Alg}\,F$, then (F,H) is an $\mathbf{Alg}$-universal arrow for $\mathcal{A}$ iff for every $\mathcal{C}$-endofunctor G with a concrete functor $J : \mathcal{A} \to \mathbf{Alg}\,G$ there exists a unique transformation $\widetilde{J} : G \to F$ such that $J = \mathbf{Alg}\,\widetilde{J}\circ H$.

Our aim is to find a relation between the existence of an $\mathbf{Alg}$-universal arrow and the existence of free objects.

Since the functor $\mathbf{Alg}$ is contravariant, we may use the property (B.6) from the Remark B.1.2 to express the existence of an $\mathbf{Alg}$-universal arrow for $\mathcal{A}$ with the base object (which is a functor, in fact) F by

$$\hom_{\mathbf{Con}\,\mathcal{C}}(\mathcal{A},-)\circ\mathbf{Alg} \quad\cong\quad \hom_{\mathbf{End}\mathcal{C}}(-,F). \quad (5.2)$$

The following observation is due to J. Rosický:

Lemma 5.1.2 *Let* $\mathcal{A}$ *be a concrete category. Then*

$\mathcal{A}$ *has an* $\mathbf{Alg}$ *-universal arrow* $\Leftrightarrow$ $\mathcal{A}$ *has a codensity monad.*

Proof: Since the existence of a codensity monad is just a different way of saying that $\underline{\mathrm{Ran}_U} U$ exists, we need to show that $\mathbf{Alg}$-universal arrow exists for $\mathcal{A}$ iff $\underline{\mathrm{Ran}_U} U$ does. But the natural isomorphisms (5.1) and (5.2) together with (B.7) yield exactly what we need: a functor F is a base of an $\mathbf{Alg}$-universal arrow for $\mathcal{A}$ iff $F = \mathrm{Ran}_U U$ since

$$\hom(-,F) \overset{(5.2)}{\cong} \hom_{\mathbf{Con}\,\mathcal{C}}(\mathcal{A},-)\circ\mathbf{Alg} \overset{(5.1)}{\cong}$$
$$\overset{(5.1)}{\cong} \hom_{\mathbf{End}\mathcal{C}}(-,U_{\mathcal{A}})\circ\hom_{\mathbf{CAT}}(U_{\mathcal{A}},\mathcal{C}) \overset{(B.7)}{\cong} \hom(-,F)$$

□

This fact enables us to investigate the universality in terms of Kan extensions and codensity monads. In the following sections, we will use this language to characterize the free-objects existence for two significant families of concrete categories.

Remark 5.1.3 *By duality, the* $\mathbf{Coalg}$*-universality is equivalent to the existence of a density comonad.*

5.2 Beck Categories with Codensity Monad

5.2.1 Categories with Pointwise Codensity Monad

Lemma 5.2.1 *Let $U : \mathcal{A} \to \mathcal{C}$ be a functor which creates all limits and has a pointwise codensity monad. Then U has a left adjoint.*

Proof: Let (M, η) be the pointwise codensity monad for U. Then, for every $A \in \mathcal{C}$, $MA = \lim(U \circ Q_A)$ where Q is the projection functor $A \downarrow U \to \mathcal{A}$. Since U creates the limits, there exists an object $L(A) = \lim Q_A$ in $\mathcal{A}$ such that $\lim(U \circ Q_A) = U \circ \lim Q_A$ (see Appendix B.2). Moreover, the whole limit cone can be lifted to $\mathcal{A}$, i.e., for every morphism $f : A \to UB$ there is a unique morphism $\widetilde{f} : L(A) \to B$ such that $U\widetilde{f}$ is $\langle f \rangle$, i.e., the f-labeled component of the limit cone for $U \circ Q_A$. This implies the equality $f = U\widetilde{f} \circ \eta_A$ for every f. Hence $\eta_A : A \to MA = UL(A)$ is the U-universal arrow for A. Therefore the assignment $A \mapsto L(A)$ can be extended to a functor $L : \mathcal{C} \to \mathcal{A}$, which is left adjoint to U. □

As a direct consequence we have the following.

Corollary 5.2.2 *Every Beck category with a pointwise codensity monad is monadic.*

This, in fact, implies a stronger version of Beck's theorem:

Theorem 5.2.3 *Let $\mathcal{A}$ be a concrete category. Then*

$$\mathcal{A} \textit{ is monadic} \Leftrightarrow \mathcal{A} \textit{ is a Beck category with a pointwise codensity monad.}$$

5.2.2 L-algebraic Categories with Codensity Monad

In the text bellow we show a seemingly weaker result, that every l-algebraic category with a codensity monad has free objects. However, this framework is independent of pointwiseness of a given codensity monad and cannot be derived from the above theorem, as shown at the end of this chapter.

Lemma 5.2.4 *Let $(\mathcal{A}, U)$ be a $\mathcal{C}$-concrete category with the right Kan extension of U along itself. Then for every $\mathcal{C}$-endofunctor F and a concrete functor $J : \mathcal{A} \to \mathbf{Alg}\, F$ there exists $\underline{\mathrm{Ran}}_U J = (V, e)$. This Kan extension is preserved by*

1. *the forgetful functor $U_F : \mathbf{Alg}\, F \to \mathcal{C}$,*
2. *every concrete functor $T : \mathbf{Alg}\, F \to \mathbf{Alg}\, G$ (for every $G : \mathcal{C} \to \mathcal{C}$).*

Proof: Let $J : \mathcal{A} \to \mathbf{Alg}\, F$ be a concrete functor and $\underline{\mathrm{Ran}}_U U = (M, \epsilon)$. Due to the Remark 5.1.1, $J = (U, \iota)$ for some natural transformation $\iota : FU \to U$. Then $\epsilon : MU \to U$ yields the natural transformation $\iota \circ F\epsilon : FMU \to U$. Hence there is a transformation $\upsilon : FM \to M$ such that the diagram

$$\begin{tikzcd} MU \arrow[r, "\epsilon"] & U \\ FMU \arrow[u, "\upsilon U"] \arrow[r, "F\epsilon"] & FU \arrow[u, "\iota"'] \end{tikzcd}$$

commutes. We have gained a functor $V = (M, \upsilon) : \mathcal{C} \to \mathbf{Alg}\, F$. The diagram above implies that the transformation ϵ can be extended to a transformation $\zeta : VU \to J$ such that $U_F \zeta = \epsilon$.

Let G be another $\mathcal{C}$-endofunctor and let $U_G : \mathbf{Alg}\, G \to \mathcal{C}$ be the forgetful functor. Let $T = (U_F, \tau) : \mathbf{Alg}\, F \to \mathbf{Alg}\, G$ be a concrete functor. We will show, that $(TV, T\zeta)$ is the right Kan extension of $TJ = (U, \tau J)$ along U.

Consider a functor $K = (U_G K, \kappa) : \mathcal{C} \to \mathbf{Alg}\, G$ and a transformation $\lambda : KU \to TJ$. Then $U_G \lambda : U_G KU \to U$ is a natural transformation, hence there is $\rho : U_G K \to M$ such that $U_G \lambda = \epsilon \circ \rho U$. Since λ is the transformation $(U_G KU, \kappa U) \to (U_G TJ, \tau J)$, we have $U_G \lambda \circ \kappa U = \tau J \circ G U_G \lambda$. The functor T turns the diagram above into:

$$\begin{tikzcd} MU \arrow[r, "\epsilon"] & U \\ GMU \arrow[u, "\tau VU"] \arrow[r, "G\epsilon"] & GU, \arrow[u, "\tau J"'] \end{tikzcd}$$

hence $\tau J \circ G\epsilon = \epsilon \circ \tau VU$. We prove that ρ underlies a transformation $K \to TV$.

$$\begin{aligned} \epsilon \circ \rho U \circ \kappa U &= U_G \lambda \circ \kappa U \\ &= \tau J \circ G U_G \lambda \\ &= \tau J \circ G\epsilon \circ G\rho U \\ &= \epsilon \circ \tau VU \circ G\rho U. \end{aligned}$$

Now from (M, ϵ) being the right Kan extension of U along U, the uniqueness of factorization of a transformation $GU_G KU \to U$ over ϵ yields $\rho \circ \kappa = \upsilon \circ F\rho U$. Hence there is a transformation $\gamma : K \to TV$ such that $T\zeta \circ \gamma U = \lambda$ and $(TV, T\zeta)$ is the right Kan extension of TJ along U.

Since the procedure holds for every G and T, by choice of $G = F$ and $T = \mathrm{Id}_{\mathbf{Alg}\, F}$ we get $(V, \zeta) = \underline{\mathrm{Ran}}_U J$. For every other choice of G, T we have $T\underline{\mathrm{Ran}}_U J = T(V, \zeta) = (TV, T\zeta) = \underline{\mathrm{Ran}}_U(TJ)$, hence T preserves $\underline{\mathrm{Ran}}_U J$.

Moreover $U_F \underline{\mathrm{Ran}}_U J = U_F(V, \zeta) = (U_F V, U_F \zeta) = (M, \epsilon) = \underline{\mathrm{Ran}}_U U$, hence even U_F preserves $\underline{\mathrm{Ran}}_U J$. $\square$

Lemma 5.2.5 *Let $(\mathcal{A}, U)$ be the limit of a diagram $D : \mathcal{D} \to \mathbf{Con}\,\mathcal{C}$, where Dd is an f-algebraic category for each object d of $\mathcal{D}$. If the codensity monad for U exists, then $\underline{\mathrm{Ran}}_U \mathrm{Id}_{\mathcal{A}}$ exists and is preserved by each component of the limit cone $L_d : \mathcal{A} \to Dd$ ($d \in \mathrm{Ob}\mathcal{D}$).*

Proof: If $\underline{\mathrm{Ran}}_U U$ exists, then, for every $\mathcal{D}$-morphism $\phi : d \to d'$, the functor $D\phi : Dd \to Dd'$ is concrete between f-algebraic categories, thus, due to Lemma 5.2.4, $\underline{\mathrm{Ran}}_U L_d$ and $\underline{\mathrm{Ran}}_U L_{d'}$ exist and $\underline{\mathrm{Ran}}_U L_{d'} = D(\phi)\underline{\mathrm{Ran}}_U L_d$. Let $\underline{\mathrm{Ran}}_U L_d = (V_d, \zeta_d)$ for every $d \in \mathrm{Ob}\mathcal{D}$. Then the functors $V_d : C \to D_d$ form a cone for D, hence there is a unique $H : \mathcal{C} \to \mathcal{A}$ such that $V_d = L_d \circ H$ for every d. Now each transformation $\zeta_d : V_d U \to L_d$ has the domain $L_d HU$. We will find a morphism γ in $\mathcal{A}$ such that $L_d(\gamma) = \zeta_d$ for each d.

Since $D\phi(\zeta_d) = \zeta_{d'}$ for every $\mathcal{D}$-morphism $\phi : d \to d'$, we have a D-compatible cone $K_d : \mathbf{2} \to Dd$ given by $K_d(\iota) = \zeta_d$. Then there is a unique $T : \mathbf{2} \to \mathcal{A}$ such that $L_d \circ T = K_d$. Therefore, we may set $\gamma = T\iota : HU \to \mathrm{Id}_{\mathcal{C}}$ and the property is satisfied. The universal property of each ζ_d and the unique lifting property yields the universal property of γ and (H, γ) becomes $\underline{\mathrm{Ran}}_U \mathrm{Id}_{\mathcal{A}}$ and $L_d \underline{\mathrm{Ran}}_U \mathrm{Id}_{\mathcal{A}} = \underline{\mathrm{Ran}}_U L_d$. □

Lemma 5.2.6 *The forgetful functor U of an l-algebraic category in $\mathbf{Con}\,\mathcal{C}$ with codensity monad creates the right Kan extension of identity along U.*

Proof: If $(\mathcal{A}, U)$ is a limit of an empty diagram, then it is the terminal object of $\mathbf{Con}\,\mathcal{C}$, hence $(\mathcal{A}, U) \cong (\mathcal{C}, \mathrm{Id}_C)$ and the situation is trivial.

Suppose $(\mathcal{A}, U)$ is a limit of a nonempty diagram and $\underline{\mathrm{Ran}}_U U$ exists. The limit cone is formed by f-algebraic categories $(\mathcal{B}_d, U_d)$ and concrete functors $L_d : \mathcal{A} \to \mathcal{B}_d$. Then according to the previous lemma, $\underline{\mathrm{Ran}}_U Id$ exists and each functor L_d preserves it. Moreover, due to Lemma 5.2.4 each U_d and the limit cone L_d preserves $\underline{\mathrm{Ran}}_U L_d$, hence their composition, which is equal to U, preserves $\underline{\mathrm{Ran}}_U Id$. Indeed, $U\underline{\mathrm{Ran}}_U \mathrm{Id} = U_d L_d \underline{\mathrm{Ran}}_U \mathrm{Id} \overset{5.2.5}{=} U_d \underline{\mathrm{Ran}}_U L_d \overset{5.2.4}{=} \underline{\mathrm{Ran}}_U U_d L_d = \underline{\mathrm{Ran}}_U U$. □

As a direct consequence of this lemma and Proposition B.2.6 we get the main result:

Theorem 5.2.7 *Let $\mathcal{A}$ be an l-algebraic category. Then*

$$\mathcal{A} \textit{ has a codensity monad} \Leftrightarrow \mathcal{A} \textit{ has free objects.}$$

Since every monadic category is an l-algebraic category (see Lemma 1.2.11), we get an alternative characterization of monadic categories:

Theorem 5.2.8 *Let $\mathcal{A}$ be a concrete category. Then*

$$\mathcal{A} \textit{ is monadic} \Leftrightarrow \mathcal{A} \textit{ is an l-algebraic category with a codensity monad.}$$

The statement cannot be extended to all Beck categories unless the codensity monad is pointwise, as shown in the following example.

Example 5.2.1 *Consider the category* $\mathcal{C} = \mathbf{2}+\mathbf{2}$ *consisting of objects* $0, 1, 0', 1'$ *and morphisms* $\iota : 0 \to 1$, $\iota' : 0' \to 1'$ *and identities. Let* $\mathcal{A} = \mathbf{1} + \mathbf{1}$ *and* $U : \mathcal{A} \to \mathcal{C}$ *be the inclusion of* $\{0, 0'\}$*. Then the following holds:*

1. $(\mathcal{A}, U)$ *has a codensity monad (the trivial monad).*

2. U *does not have an adjoint (1 does not have an universal arrow).*

3. $(\mathcal{A}, U)$ *is algebraic.*

Algebraicity: consider the type $t : \{\rho, \sigma\} \to (\mathrm{Ob}\mathcal{C})^2$, $t(\rho) = (1, 0')$, $t(\sigma) = (1', 0)$*. Then* $t-\mathbf{alg}$ *consists of* $(0, \alpha)$, $(0', \alpha')$ *with* $\alpha(\rho) = \varnothing$, $\alpha(\sigma) = \varnothing_{\{\mathrm{id}_{0'}\}}$ *since* $\hom(1, 0) = \hom(1', 0) = \hom(1, 0') = \emptyset$*. There are no* t*-algebras on* 1 *and on* $1'$ *since there are no maps* $\hom(1, 1) \to \hom(0', 1)$ *and* $\{\mathrm{id}_{1'}\} \to \emptyset$*.*

As a consequence, we see that there exists a codensity monad which is not pointwise and an algebraic, hence Beck, category that is not l-algebraic.

5.3 An Overview of the Obtained Results

We have proved the propositions which, together with Beck's theorem, may be collected to the following theorem.

Theorem 5.3.1 *Let* $\mathcal{C}$ *be a category and* $(\mathcal{A}, U)$ *be concrete category over* $\mathcal{C}$*. The following statements are equivalent:*

1. $\mathcal{A}$ *is monadic.*

2. $\mathcal{A}$ *is Beck and* U *has a left adjoint.*

3. $\mathcal{A}$ *is Beck and* U *has a pointwise codensity monad.*

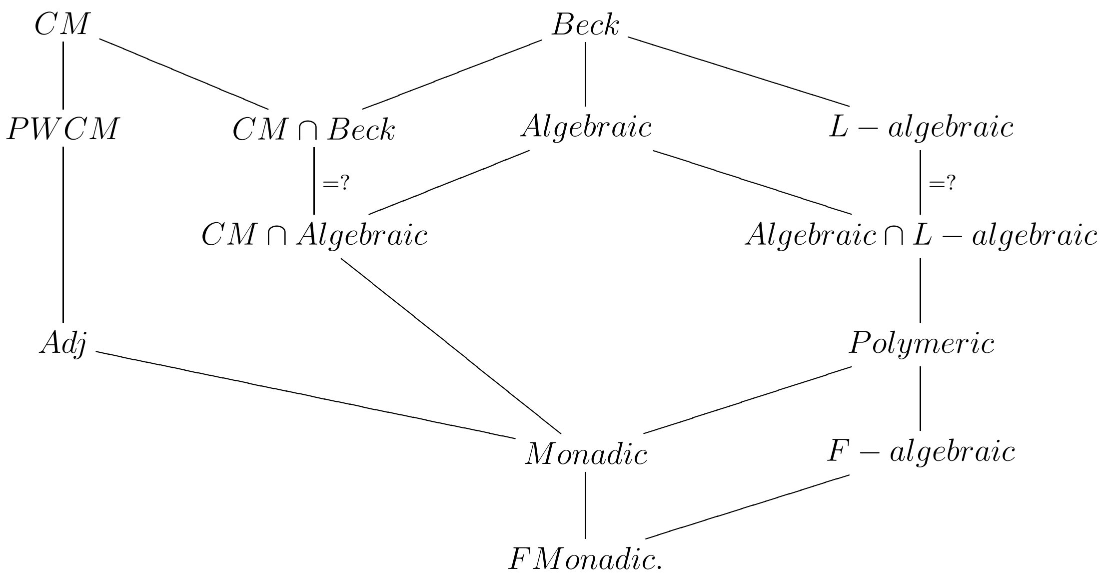
4. $\mathcal{A}$ *is l-algebraic and* U *has a codensity monad.*

5. $\mathcal{A}$ *is an l-algebraic category with an* **Alg** *-universal arrow.*

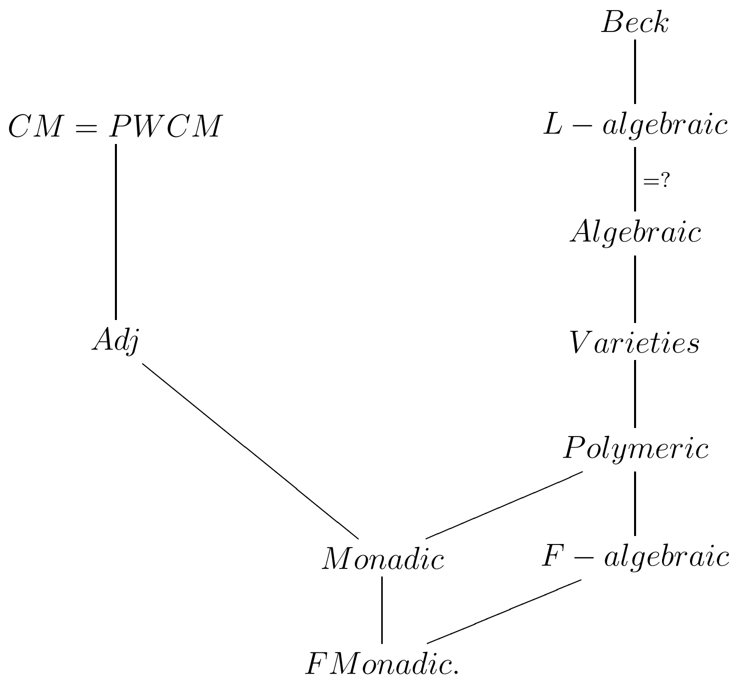
Remark 5.3.2 *If* $\mathcal{C}$ *has copowers, then, due to Proposition B.2.7, we have even stronger result:*

$\mathcal{A}$ *is a monadic category* $\Leftrightarrow$ $\mathcal{A}$ *is a Beck category with a codensity monad* $\Leftrightarrow$ $\mathcal{A}$ *is a Beck category with an* **Alg** *-universal arrow.*

The relations between metaclasses of concrete categories, namely the inclusions and intersections, are shown on the Hasse diagrams bellow. Here $CM, PWCM, Adj$ stand for metaclasses of concrete categories whose forgetful functor has codensity monad, pointwise codensity monad, left adjoint, respectively and $FMonadic$ denotes the monadic categories for a free monad.



For a cocomplete base category, the diagram also includes an item *Varieties*:



The questions, whether the inclusions labeled by =? are strict, **remain open**.

Appendix A

Concrete Categories

A.1 Basic Concept of Concreteness

Definition A.1.1 *Let $\mathcal{C}, \mathcal{D}$ be categories and $U : \mathcal{D} \to \mathcal{C}$ a faithful functor. We say that $(\mathcal{D}, U)$ is a* concrete category over $\mathcal{C}$*, or $\mathcal{C}$-concrete category in short. U and C are called* forgetful functor *and* base category*, respectively.*

Note 6 *The pair $(\mathcal{D}, U)$ is often identified with $\mathcal{D}$ if we do not need to emphasize the name of the forgetful functor. If the choice of this functor is obvious, the forgetful functor is usually denoted by $U_{\mathcal{D}}$.*

Definition A.1.2 *Let $(\mathcal{D}, U)$ be a concrete category over $\mathcal{C}$.*

- *For an object B in $\mathcal{D}$, $U(B)$ is called the* underlying object *or* carrier *of B.*
- *The class* **Fib** *$\mathcal{D}(A)$ of $\mathcal{D}$ of all $\mathcal{D}$-objects with carrier A is called the* fibre *of A.*
- *$\mathcal{D}$ is called* fibre-small *if fibres of all objects in $\mathcal{C}$ are sets.*

Definition A.1.3 *Let $(\mathcal{A}, U_{\mathcal{A}}), (\mathcal{B}, U_{\mathcal{B}})$ be concrete categories over $\mathcal{C}$ and let $F : \mathcal{A} \to \mathcal{B}$ be a functor. We say that F is* concrete over $\mathcal{C}$ *(or $\mathcal{C}$-*concrete*) if $U_{\mathcal{B}} \circ F = U_{\mathcal{A}}$. Moreover, if F is an isomorphism, $\mathcal{A}$, $\mathcal{B}$ are called* concretely isomorphic *and we write $\mathcal{A} \cong_{\mathcal{C}} \mathcal{B}$.*

Metacategorical remark A.1.1 *The metacategory of all $\mathcal{C}$-concrete categories and $\mathcal{C}$-concrete functors is denoted by* **Con**$\mathcal{C}$*. Then there is an obvious "forgetful functor"* **Con** $\mathcal{C} \to$ **CAT** *given by the assignment $(\mathcal{A}, U_{\mathcal{A}}) \mapsto \mathcal{A}$.*

Definition A.1.4 *Let $\mathcal{C}$ be a category. An* absolute colimit *in $\mathcal{C}$ is a colimit in $\mathcal{C}$, which is preserved by every functor $\mathcal{C} \to \mathcal{C}'$ for every $\mathcal{C}'$.*

Let $(\mathcal{A}, U)$ be a concrete category over $\mathcal{C}$. We say that $(\mathcal{A}, U)$ is a Beck category *if U creates all limits and absolute coequalizers.*

A.2 Comma-Categories

Definition A.2.1 *Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}$ be categories and $F_1 : \mathcal{C}_1 \to \mathcal{D}$, $F_2 : \mathcal{C}_2 \to \mathcal{D}$ functors. A* comma-category *$F_1 \downarrow F_2$ is the category with objects (A, f, B), where A and B are objects of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively and $f : F_1A \to F_2B$ is a morphism in $\mathcal{D}$ called a* structure arrow. *We say that there is a morphism $(\phi, \psi) : (A, f, B) \to (A', f', B')$ if $\phi : A \to A'$ and $\psi : B \to B'$ are the morphisms in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, such that the diagram bellow commutes:*

$$\begin{tikzcd} F_1A \arrow[r, "f"] \arrow[d, "F_1\phi"] & F_2B \arrow[d, "F_2\psi"] \\ F_1A' \arrow[r, "f'"] & F_2B'. \end{tikzcd}$$

The identity and the composition of morphisms are given componentwise.

Let $\mathcal{A} = \mathcal{C}_1 \times \mathcal{C}_2$, then $F_1 \downarrow F_2$ is a concrete category over $\mathcal{A}$ with the forgetful functor given by $(A, f, B) \mapsto (A, B)$.

Remark A.2.1 *If any of the functors F_1, F_2 is Id_D, it is replaced by $\mathcal{D}$ in the notation of comma-category. On the other hand, if any of F_1, F_2 is a constant functor C_X with the domain category $\mathbf{1}$ for some object X in $\mathcal{D}$, we substitute X for the label of the appropriate functor.*

Let $F_2 = C_X$. Then we have a comma category $F_1 \downarrow C_X$ with the forgetful functor $F_1 \downarrow C_X \to \mathcal{C}_1 \times \mathbf{1} \cong C_1$, hence it makes sense to assume such a category to be concrete over C_1.

*Namely, for $F_1 = \mathrm{Id}_{\mathcal{C}}$, $F_2 = C_X$ we get an X-*slice category, *usually denoted by $\mathcal{C}/X$, with its objects being the pairs (A, f), where $f : A \to X$ is a morphism in $\mathcal{C}$.*

*We define an X-*coslice category *$X\backslash\mathcal{C}$ by duality on $\mathcal{C}$.*

Remark A.2.2 *As proved in [16], if $\mathcal{C}$ is a complete category and X is its object, then $\mathcal{C}/X$ is complete as well. In fact, the limit of a diagram $Q : \mathcal{D} \to \mathcal{C}/X$ is of the form (L, λ) where, under the notation of Note 1, L is the limit object of the diagram $Q_1 : \mathcal{D}^* \to \mathcal{C}$ where Q_1 is defined as Q on $\mathcal{D}$ and $Q_1(1) = X$ and, for each object $d \in \mathrm{Ob}\mathcal{D}$, the terminal arrow $t_d : d \to 1$ in $\mathcal{D}^*$ is mapped on the structure arrow of $Q(d)$. The morphism $\lambda : L \to X$ is the component of the limit cone labeled by 1.*

Appendix B

Universal Constructions in Categories

The following text summarizes the main definitions and facts of two involved concepts of category theory needed for this thesis: an adjunction and Kan extensions. At first we set the notation for the natural transformations.

Notation of Natural Transformations

Definition B.0.2 *Let there be categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, functors $P, Q : \mathcal{A} \to \mathcal{B}$ and $R, S : \mathcal{B} \to \mathcal{C}$ and natural transformations $\phi : P \to Q$ and $\chi : R \to S$. Then the induced natural transformations are denoted by $R\phi : RP \to RQ$, $\chi Q : RP \to SQ$, $S\phi : SP \to SQ$ and $\chi P : RP \to SP$. Moreover, there is a transformation $\chi * \phi : RP \to SQ$ called* Godement product *given by compositions*

$$
\begin{tikzcd}
 & SQ \arrow[dr, "\chi Q"] & \\
RP \arrow[ur, "R\phi"] \arrow[rr, "\chi * \phi"] \arrow[dr, "\chi P"'] & & SQ \\
 & SP. \arrow[ur, "S\phi"'] &
\end{tikzcd}
$$

Remark B.0.3 *The properties of the Godement product (see [16]):*

$$
\begin{aligned}
(\chi_2 * \chi_1) \circ (\phi_2 * \phi_1) &= (\chi_2 \circ \phi_2) * (\chi_1 \circ * \phi_1) && \text{(B.1)} \\
\mathrm{id}_R * \phi_1 &= R\phi_1 && \text{(B.2)} \\
\chi_1 * \mathrm{id}_P &= \chi_1 P && \text{(B.3)}
\end{aligned}
$$

for every $N, P, Q : \mathcal{A} \to \mathcal{B}$, $R, S, T : \mathcal{B} \to \mathcal{C}$, $N \overset{\phi_1}{\to} P \overset{\phi_2}{\to} Q$, $R \overset{\chi_1}{\to} S \overset{\chi_2}{\to} T$.

B.1 Adjunction and Monads

B.1.1 Universality and Adjunction

In this section we recall the basic categorial concept of universal arrows and of adjunction. Proofs of the facts can be found in [7], [24].

Definition B.1.1 *Given a functor $F : \mathcal{C} \to \mathcal{D}$ and object A in $\mathcal{D}$. Let B be an object in $\mathcal{C}$ and $f : A \to RB$ a morphism in $\mathcal{D}$. We say (B, f) is an* R-universal arrow *for A if for every $C \in \mathrm{Ob}\mathcal{C}$ and $g : A \to RC$ there is a unique $\widetilde{g} : B \to C$ such that $R\widetilde{g} \circ f = g$. Object B is called a* base object *of the universal arrow. The dual notion is a* couniversal arrow.

Lemma B.1.1 *Given a functor $R : \mathcal{C} \to \mathcal{D}$ and object $A \in \mathrm{Ob}\mathcal{D}$, $B \in \mathrm{Ob}\mathcal{C}$, then there exists an R-universal arrow with the base object B iff*

$$\hom_{\mathcal{D}}(A, -) \circ R \cong \hom_{\mathcal{C}}(B, -). \tag{B.4}$$

Proof: It is easy to see that the assignment $g \mapsto \widetilde{g}$ for $g : A \to RC$ with $C \in \mathcal{C}$ defines an isomorphism $\iota_C : \hom_{\mathcal{D}}(A, RC) \cong \hom_{\mathcal{C}}(B, C)$. The universality, moreover, yields the naturalness of such a collection of isomorphisms.

Conversely, given an isomorphism $\iota : \hom_{\mathcal{D}}(A, -) \circ R \cong \hom_{\mathcal{C}}(B, -)$, then $f = \iota_B^{-1} : A \to RB$ yields the R-universal arrow (B, f). □

Remark B.1.2 *By analogy, under the above assumptions, the existence of an R-couniversal arrow over A with the base object B is equivalent to the isomorphism*

$$\hom_{\mathcal{D}}(-, A) \circ R \cong \hom_{\mathcal{C}}(-, B). \tag{B.5}$$

Moreover, if $S : \mathcal{C} \to \mathcal{D}$ is a contravariant functor, then the existence of a S-universal arrow over A with the base object B is equivalent to the isomorphism

$$\hom_{\mathcal{D}}(A, -) \circ S \cong \hom_{\mathcal{C}}(-, B). \tag{B.6}$$

Proposition B.1.3 *Given functors $R : \mathcal{C} \to \mathcal{D}$ and $L : \mathcal{D} \to \mathcal{C}$, then the following statements are equivalent:*

1. *Every $\mathcal{D}$-object has an R-universal arrow.*
2. *Every $\mathcal{C}$-object has an L-couniversal arrow.*

3. *There is a natural transformation $\eta : \mathrm{Id}_{\mathcal{D}} \to R \circ L$ such that, for every $\mathcal{D}$-object A, $\mathcal{C}$-object B and a mapping $f : A \to RB$, there is a unique $\widetilde{f} : LA \to B$ satisfying $R\widetilde{f} \circ \eta_A = f$.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"] & RB \\ RLA \arrow[ur, "R\widetilde{f}"'] & \end{tikzcd}$$

4. *There is a natural transformation $\epsilon : L \circ R \to \mathrm{Id}_{\mathcal{C}}$ such that, for every $\mathcal{C}$-object B, $\mathcal{D}$-object A and a mapping $g : LA \to B$, there is a unique $\widetilde{g} : A \to RB$ satisfying $\epsilon_B \circ L\widetilde{g} = g$.*

$$\begin{tikzcd} LA \arrow[r, "g"] \arrow[dr, "L\widetilde{g}"] & B \\ & LRB \arrow[u, "\epsilon_B"'] \end{tikzcd}$$

Definition B.1.2 *If the conditions stated above are satisfied, we say that R is* right adjoint functor *for L, L is* left adjoint functor *for R, and we deal with an* adjoint situation, *shortly* adjunction, *denoted by $(\eta, \epsilon) : L \dashv R : \mathcal{C} \to \mathcal{D}$ and η and ϵ are called a* unit *and a* counit *of the adjunction, respectively.*

Remark B.1.4 *Given an adjunction $L \dashv R : \mathcal{C} \to \mathcal{D}$ with morphisms $f : RA \to RB$, $g : RC \to RC$ for some $\mathcal{C}$-objects A, B, C, then the morphism $LRA \to C$ induced by the composition $g \circ f$ is*

$$\widetilde{g \circ f} = \widetilde{g} \circ Lf.$$

We recall the following well-known property:

Proposition B.1.5 *Let there be an adjunction $L \dashv R : \mathcal{C} \to \mathcal{D}$. Then L preserves colimits and R preserves limits.*

The following property can be found as an exercise in [3].

Proposition B.1.6 *If the functor $R : \mathcal{C} \to \mathcal{S}et$ is right adjoint, then it is isomorphic to* $\hom(K, -)$ *for some $K \in \mathcal{C}$.*

Remark B.1.7 *Let $\mathcal{C}$ have copowers and Q be an object in $\mathcal{C}$. Then there is a* copower functor $- \bullet Q : \mathbf{Set} \to \mathcal{C}$ *which is left adjoint to* $\hom(Q, -) : \mathcal{C} \to$ **Set**. *To a set M, it assigns the coproduct of M copies of Q (the "M-th" copower of Q) and for a mapping $h : M \to N$ we define $h \bullet Q$ as the unique* *factorization of cocone* $u_{h(m)} : Q \to \coprod_{j\in N} Q, m \in M$*, over a colimit cocone* $u_m : Q \to \coprod_{j\in M} Q$.

We get an adjunction $(\eta, \varepsilon) : (- \bullet Q) \dashv \hom(Q, -) : \mathcal{C} \to \mathbf{Set}$ *with the unit morphism* $\eta_X : X \to \hom(Q, X \bullet Q)$*, for a set* X *and* $x \in X$*, defined by* $\eta_X(x) = u_x : Q \to X \bullet Q$*, i.e. the* x*-labeled canonical injection into the coproduct. Moreover, for an object* A *of* $\mathcal{C}$*, the counit* $\varepsilon : \hom(Q, A) \bullet Q \to A$ *is defined as the unique factorization of a cocone* $\{\phi | \phi : Q \to A\}$ *over the colimit.*

Free Objects

Definition B.1.3 *Let* $\mathcal{C}$ *be a category and* $\mathcal{A}$ *be a* $\mathcal{C}$*-concrete category with the forgetful functor* $U : \mathcal{A} \to \mathcal{C}$ *and let* A *be an object in* $\mathcal{C}$*. We say that the object* B *of* $\mathcal{A}$ *is* free *over* A *if there is a* U*-universal arrow* $\eta : A \to UB$*.*

The theory of universality and adjunction applied on the situation described in the definition yields the following:

Remark B.1.8 *Let* $(\mathcal{A}, U)$ *be a* $\mathcal{C}$*-concrete category.*

- *A free object over a* $\mathcal{C}$*-object* A*, if any, is determined uniquely up to an isomorphism.*
- *Let the free objects exist over every* A*. Then these can be selected functorially by a* free functor $W : \mathcal{C} \to \mathcal{A}$ *which is left adjoint to the forgetful functor* U*.*
- *The existence of free objects yields the preservation of limits by the forgetful functor.*
- *The free object in* $\mathcal{A}$ *over an initial object in the base category is the initial object in* $\mathcal{A}$*.*

B.1.2 Monads and Comonads

Definition B.1.4 *Let* M *be an endofunctor on a category* $\mathcal{C}$ *and let there be natural transformations* $\eta : \mathrm{Id}_{\mathcal{C}} \to M$*,* $\mu : M \circ M \to M$*. The triple* $\underline{M} = (M, \eta, \mu)$ *is called a* monad *on* $\mathcal{C}$ *if the following conditions are satisfied:*

$$\mu \circ \eta M = \mu \circ M\eta = \mathrm{id}_M,$$

$$\mu \circ M\mu = \mu \circ \mu M.$$

Then η *and* μ *are called* unit *and* multiplication *of monad* $\underline{M}$*.*

Dually we define a *comonad.*

Proposition B.1.9 *Let* $(\eta, \epsilon) : L \dashv R : \mathcal{C} \to \mathcal{D}$ *be an adjunction. Let* $M = R \circ L$, $N = L \circ R$, $\mu = R\epsilon L$, $\nu = L\eta R$. *Then there is an* associated monad (M, η, μ) *and an* associated comonad (N, ϵ, ν) *for the adjunction.*

Definition B.1.5 *Let* $\underline{M}_1 = (M_1, \eta_1, \mu_1)$ *and* $\underline{M}_2 = (M_2, \eta_2, \mu_2)$ *be monads. A natural transformation* $\phi : M_1 \to M_2$ *is a* monad transformation *if the diagrams*

$$\begin{tikzcd} \mathrm{Id} \arrow[r, "\eta_1"] \arrow[dr, "\eta_2"'] & M_1 \arrow[d, "\phi"] \\ & M_2 \end{tikzcd} \qquad \begin{tikzcd} M_1^2 \arrow[r, "\mu_1"] \arrow[d, "\phi * \phi"] & M_1 \arrow[d, "\phi"] \\ M_2^2 \arrow[r, "\mu_2"] & M_2 \end{tikzcd}$$

commute.

B.2 Kan Extensions

B.2.1 Basic Concept

The concept of Kan extensions, widely elaborated in [24], is another topic involving the universality. In fact, there are two dual notions - left and right Kan extension - related by the replacement of universality by couniversality in the definition. However, there are still two different definitions of the right Kan extension, which are not equivalent in general. The weaker one, which is called just *right Kan extension*, can be seen as an instance of couniversality, while the stronger one, *pointwise right Kan extension*, is given by a limit construction. For the weaker case we show two versions of definition, a direct one and the one involving the couniversality. The latter will be especially useful.

Definition B.2.1 (Kan extension - direct version) *Let* $\mathcal{A}$,$\mathcal{B}$ *and* $\mathcal{C}$ *be categories and* $S : \mathcal{A} \to \mathcal{B}$, $U : \mathcal{A} \to \mathcal{C}$ *be functors. A* right Kan extension of S along U *is a pair* $\underline{\mathrm{Ran}}_U S = (T, e)$ *consisting of a functor* $T : \mathcal{C} \to \mathcal{B}$ *and a natural transformation* $e : TU \to S$ *satisfying the following universal property: given a functor* $T' : \mathcal{C} \to \mathcal{B}$ *and a transformation* $e' : T'U \to S$, *then there is a unique transformation* $t : T' \to T$ *such that* $e' = e \circ tS$. *Then we write* $T = \mathrm{Ran}_U S$.

By duality on categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *we define left Kan extension* $\underline{\mathrm{Lan}}_U S = (\mathrm{Lan}_U S, h)$.

The following remark involves some strongly illegitimate notions such as the collection of all metacategories and a functor between these hypercategories. These are, however, used to enable a formal explanation of notation, rather then to build a theory upon.

Metacategorical remark B.2.1 *Let $\mathcal{A}$ and $\mathcal{C}$ be categories and $U : \mathcal{A} \to \mathcal{C}$ be a functor. Then there are "functors"*

$$\hom_{\mathbf{CAT}}(\mathcal{A}, -), \hom_{\mathbf{CAT}}(\mathcal{C}, -) : \mathbf{CAT} \to \mathbf{METACAT}$$

together with the "transformation"

$$\hom_{\mathbf{CAT}}(U, -) : \hom_{\mathbf{CAT}}(\mathcal{C}, -) \to \hom_{\mathbf{CAT}}(\mathcal{A}, -)$$

with the component on a category $\mathcal{D}$ given, for every functor $T : \mathcal{C} \to \mathcal{D}$, by

$$\hom_{\mathbf{CAT}}(U, \mathcal{D})(T) = T \circ U.$$

Therefore, under the above assumptions, we have the functor $\hom_{\mathbf{CAT}}(U, \mathcal{D}) : \hom_{\mathbf{CAT}}(\mathcal{C}, \mathcal{D}) \to \hom_{\mathbf{CAT}}(\mathcal{A}, \mathcal{D})$ for each category $\mathcal{D}$. To simplify the notation, it will be denoted by $[- \circ U]$.

Definition B.2.2 (Kan extension - unversality version) *Let $\mathcal{A}$,$\mathcal{B}$ and $\mathcal{C}$ be categories and $U : \mathcal{A} \to \mathcal{C}$ be a functor. A $[- \circ U]$-couniversal arrow (T, e) over a functor $S : \mathcal{A} \to \mathcal{B}$ is called* right Kan extension of S along U.

Dually, $[- \circ U]$-universal arrow defines a left Kan extension.

Proposition B.2.2 *Definitions B.2.1 and B.2.2 are equivalent.*

Proof: The proof is straightforward. □

Remark B.2.3 *Under the assumptions of the definitions, we get from (B.5) that the existence of $\underline{\mathrm{Ran}_U} S$ is equivalent to the isomorphism*

$$\hom_{\mathbf{End}\mathcal{C}}(-, S) \circ [- \circ U] \ \cong \ \hom_{\mathbf{End}\mathcal{C}}(-, \mathrm{Ran}_U S). \qquad \text{(B.7)}$$

Although the main definitions are taken from [24], we choose a different definition for the stronger version of Kan extension and the definition from [24] will be stated as a proposition.

Let $\mathcal{A}$ and $\mathcal{C}$ be categories with a functor $U : \mathcal{A} \to \mathcal{C}$ and let A be an object in $\mathcal{C}$. Then $Q_A : A \downarrow U \to \mathcal{A}$ will denote the forgetful functor for the comma category (see Remark A.2.1).

Proposition B.2.4 *Let $\mathcal{A}$,$\mathcal{B}$ and $\mathcal{C}$ be categories and $S : \mathcal{A} \to \mathcal{B}$, $U : \mathcal{A} \to \mathcal{C}$ be functors. Consider the functors $S \circ Q_A$ for every $A \in \mathrm{Ob}\mathcal{C}$. If each of these functors, seen as a large diagram, has a limit $T_A \in \mathrm{Ob}\mathcal{B}$, then there is a functor $T : \mathcal{C} \to \mathcal{B}$ and a natural transformation $e : T \circ U \to S$ such that $(T, e) = \underline{\mathrm{Ran}}_U S$.*

Definition B.2.3 (Pointwise Kan extension) *Let $\mathcal{A}$,$\mathcal{B}$ and $\mathcal{C}$ be categories and $S : \mathcal{A} \to \mathcal{B}$, $U : \mathcal{A} \to \mathcal{C}$ be functors. If, for each $A \in \mathrm{Ob}\mathcal{C}$, the functor SQ_A has a limit, then the induced right Kan extension of S along U is called* pointwise.

To say more about the relation between the stronger and weaker Kan extensions, we will recall the definition of the preservation of Kan extensions.

Note 7 *Given functors $P, Q : \mathcal{C} \to \mathcal{B}$, $R : \mathcal{B} \to \mathcal{D}$ and a transformation $p : P \to Q$, then we apply the functor R on the pair (P, p) by*

$$R(P, p) = (RP, Rp).$$

Definition B.2.4 *Let $\mathcal{A}$,$\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be categories, $S : \mathcal{A} \to \mathcal{B}$, $U : \mathcal{A} \to \mathcal{C}$, $R : \mathcal{B} \to \mathcal{D}$ be functors. Then we say that:*

- R preserves $\underline{\mathrm{Ran}}_U S$ *iff*

$$\exists \underline{\mathrm{Ran}}_U S \Rightarrow (\exists \underline{\mathrm{Ran}}_U (RS) \wedge R\underline{\mathrm{Ran}}_U S = \underline{\mathrm{Ran}}_U RS),$$

- R creates $\underline{\mathrm{Ran}}_U S$ *iff*

$$\exists \underline{\mathrm{Ran}}_U (RS) \Rightarrow (\exists \underline{\mathrm{Ran}}_U S \wedge R\underline{\mathrm{Ran}}_U S = \underline{\mathrm{Ran}}_U RS).$$

Now the relation between both kinds of right Kan extensions is captured within the following theorem (see [24], Definition on p. 240).

For functors $P, Q : \mathcal{A} \to \mathcal{B}$, by $\mathbf{Nat}(P, Q)$ we denote the class of natural transformations $P \to Q$. Using the illegitimate formalism we have

$$\mathbf{Nat}(P, Q) = \hom_{\hom_{\mathbf{CAT}}(\mathcal{A},\mathcal{B})}(P, Q).$$

Theorem B.2.5 *Let $\mathcal{A}$,$\mathcal{B}$ and $\mathcal{C}$ be categories and $S : \mathcal{A} \to \mathcal{B}$, $U : \mathcal{A} \to \mathcal{C}$ be functors such that $\underline{\mathrm{Ran}}_U S = (T, e)$ exists. Then the following statements are equivalent:*

1. $\underline{\mathrm{Ran}}_U S$ *is pointwise.*
2. $\underline{\mathrm{Ran}}_U S$ *is preserved by* $\hom(C, -)$ *for every object C in $\mathcal{C}$.*

3. *For all objects $B \in \mathcal{B}$ and $C \in \mathcal{C}$ there is a natural isomorphism*
$$\hom_{\mathcal{B}}(B, T(C)) \cong \mathbf{Nat}(\hom_{\mathcal{C}}(C, -) \circ U, \hom_{\mathcal{B}}(B, -) \circ S)$$
given by the assignment $(g : B \to TC) \mapsto \phi^g$ where ϕ^g is the natural transformation with the component $\phi^g_A : \hom_{\mathcal{C}}(C, UA) \to \hom_{\mathcal{B}}(B, SA)$ on an object A in $\mathcal{A}$ given by $\phi^g_A(f) = e_A \circ Tf \circ g$ for a morphism $f : C \to UA$ in $\mathcal{C}$.

Proof: For the proof see [24], Chapter X., Theorems 5.1 and 7.2. $\square$

The situation, where both $\underline{\mathrm{Ran}}_U S$ and $\underline{\mathrm{Ran}}_U RS$ exist and the equality $R\underline{\mathrm{Ran}}_U S = \underline{\mathrm{Ran}}_U RS$ holds, can be expressed equivalently by both

- $\underline{\mathrm{Ran}}_U S$ exists and R preserves $\underline{\mathrm{Ran}}_U S$,
- $\underline{\mathrm{Ran}}_U RS$ exists and R creates $\underline{\mathrm{Ran}}_U S$.

Its instance occurs in the relation between Kan extensions and adjunction (see [24]):

Proposition B.2.6 *Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $F : \mathcal{A} \to \mathcal{B}$ be a functor. Then the following statements are equivalent:*

1. *F has a left adjoint.*
2. *$\underline{\mathrm{Ran}}_F \mathrm{Id}_{\mathcal{A}}$ exists and F preserves it.*
3. *$\underline{\mathrm{Ran}}_F \mathrm{Id}_{\mathcal{A}}$ exists and F preserves all right Kan extensions with the values in $\mathcal{A}$.*

The implication (1) $\Rightarrow$ (3), together with Remark B.1.7 and Theorem B.2.5, yields an important result:

Proposition B.2.7 *A right Kan extension with the values in a category with copowers is pointwise.*

The copower functors from Remark B.1.7 have another application in context of Kan extensions.

Remark B.2.8 *Let $\mathcal{C}$ be a category with copowers and X, Y be its objects. We define its endofunctor*
$$G_{X,Y} = (- \bullet X) \circ \hom(Y, -).$$
Consider functors $C_X, C_Y : \mathbf{1} \to \mathcal{C}$ with values X, Y, respectively. As observed by J. Velebil, it is straightforward to prove that $G_{X,Y} = \mathrm{Lan}_{C_Y} C_X$. As a consequence, there is an isomorphism
$$\hom_{\mathbf{End}\mathcal{C}}(C_Y, -) \circ [- \circ C_X] \;\cong\; \hom_{\mathbf{End}\mathcal{C}}(G_{X,Y}, -). \tag{B.8}$$

B.2.2 Codensity Monads

Proposition B.2.9 *Given a functor $U : \mathcal{A} \to \mathcal{C}$ with the right Kan extension $\underline{\mathrm{Ran}}_U U = (M, e)$, then there exists a monad $\underline{M} = (M, \eta, \mu)$ with the transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to M$ and $\mu : M^2 \to M$ induced by $\mathrm{id}_U : U \to U$ and $e \circ Me : M^2U \to U$, respectively.*

Definition B.2.5 *The monad $\underline{M} = (M, \eta, \mu)$ together with the transformation e introduced above is called* codensity monad *for U. If the corresponding right Kan extension is pointwise, we say $(\underline{M}, e)$ is* pointwise codensity monad.

If $(\mathcal{A}, U)$ is the concrete category, we say $\underline{M}$ is its codensity monad.

The following consequence of B.2.6 states that the existence of a codensity monad is a weaker condition for the existence of an adjoint:

Proposition B.2.10 *If a functor $U : \mathcal{A} \to \mathcal{C}$ has a left adjoint L with a counit ϵ, then the induced monad together with the transformation $e = L\epsilon$ is the pointwise codensity monad for U.*

Let $\underline{M} = (M, \eta, \mu)$ be a pointwise codensity monad for the forgetful functor U of a concrete category $(\mathcal{A}, U)$ over a category $\mathcal{C}$. Let A be an object in $\mathcal{C}$ and (f, X) an object in $A \downarrow U$, i.e., $f : A \to UX$ is a morphism in $\mathcal{C}$. Then by $\langle f \rangle$ we denote the corresponding morphism $MA \to UX$ such that $\langle f \rangle \circ \eta_A = f$.

The property 3. in Theorem B.2.5 for pointwise codensity monad yields the following:

Proposition B.2.11 *Under the above assumptions, for every $A, B \in \mathrm{Ob}\mathcal{C}$, there is an isomorphism*

$$\Upsilon : \mathbf{Nat}(\hom(B, -) \circ U, \hom(A, -) \circ U) \to \hom(A, M(B)).$$

Given a natural transformation $\phi : \hom(B, -) \circ U \to \hom(A, -) \circ U$, $\Upsilon(\phi) : A \to MB$ is the unique morphism such that, for every object X in $\mathcal{A}$, every morphism $f : B \to UX$ satisfies

$$\phi_X(f) = \langle f \rangle \circ \Upsilon(\phi).$$

Bibliography

[1] P. Aczel: *Non-Well-founded Sets*, CSLI Lecture Notes, Stanford University. (1988)

[2] P. Aczel , N. P. Mendler: *A Final Coalgebra Theorem*, Category Theory and Computer Science, p.357-365, September 05-08, 1989

[3] J. Adámek, *Matematické struktury a kategorie*, SNTL (1982)

[4] J. Adámek, *Algebra* $\cap$ *Coalgebra =Presheaves*, Lecture Notes in Computer Science 3629, Proceedings CALCO (2005), 67-82

[5] J. Adámek, *Free algebras and automata realizations in the language of categories*, Commentationes Mathematicae Universitatis Carolinae 015 (1974), no. 4, 589-602.

[6] J. Adámek, H. Herrlich, L. Sousa: *A Logic of Injectivity*, Journal of Homotopy and Related Topics 2 (2007)

[7] J. Adámek, H. Herrlich, G. Strecker: *Abstract and Concrete categories*, Free Software Foundation (1990/2006)

[8] J. Adámek, S. Milius, J. Velebil: *On Coalgebra Based on Classes*, Theoretical Computer Science, v.316 n.1-3, p.3-23, (2004)

[9] J. Adámek, H. Porst: *From Varieties of Algebras to Covarieties of Coalgebras*, Electronic Notes in Theoretical Computer Science (2001)

[10] J. Adámek, H. Porst: *On Varieties and Covarieties in a Category* Mathematical Structures in Computer Science (2003)

[11] J. Adámek , H.-E. Porst: *On tree coalgebras and coalgebra presentations*, Theoretical Computer Science, v.311 n.1-3, p.257-283, 23 January 2004

[12] J. Adámek, J. Rosický: *Locally Presentable and Accessible Categories*, Cambridge University Press, (1994)

[13] J. Adámek, M. Sobral, L. Sousa: *Morita equivalence of many-sorted algebraic theories*, Journal of Algebra (2006)

[14] J. Adámek, V. Trnková: *Birkhoff's Variety Theorem With And Without Free Algebras*, Theory and Applications of Categories, Vol. 14, No. 18, pp 424-450 (2005)

[15] M. Barr: *Coequaliezers and Free Triples*, Mathematische Zeitschrift 116, 307-322, Springer-Verlag (1970)

[16] F. Borceux: *Handbook of Categorical Algebra 1.*, Encyclopedia of Mathematics an Its Applications 50, Cambridge University Press (1994)

[17] F. Borceux: *Handbook of Categorical Algebra 2.*, Encyclopedia of Mathematics an Its Applications 51, Cambridge University Press (1994)

[18] A. Carboni, P. T. Johnstone: *Connected limits, familial representability and Artin glueing*

[19] P. Johnstone, J. Power, T. Tsujishita, H. Watanabe, J. Worrell: *On the structure of categories of coalgebras*, Theoretical Computer Science, Volume 260 , Issue 1-2, Pages: 87 - 117, (2001), ISSN:0304-3975

[20] G. M. Kelly: *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bulletin of the Australian Mathematical Society, 22 (1980)

[21] V. Kůrková–Pohlová, V. Koubek: *When a generalized algebraic category is monadic*, Comment.Math.Univ.Carolinae 15(1974) 577–587.

[22] A. Kurz, J. Rosický: *Modal Predicates and Coequations*, Electronic Notes in Theoretical Computer Science, Vol. 65, 156-175 (2002)

[23] J. Lambek: *A fixpoint theorem for complete categories*. Math. Zeitschrift 103 151-161. (1968)

[24] S. Mac Lane: *Categories for the Working Mathematician*, Springer-Verlag (1971)

[25] F.E.J. Linton: *An outline of functorial semantics*, Lecture Notes in Math. 80, Springer (1969), 7 -52.

[26] J. van Oosten: *Basic Category Theory*, Basic Research in Computer Science, Dept. of Comuter Sciences, University of Aarhus (1995)

[27] D. Pattinson: *An Introduction to the Theory of Coalgebras*, Notes accompanying the a course at NASSLLI, Institut fr Matematik, LMU Mnchen (2003)

[28] J. Pavlík: *Free Algebras in Varieties*, Archivum Mathematicum (2009), accepted, [arXiv:0904.1678v1]

[29] J. Pavlík: *Varieties defined without colimits*, 7th Panhellenic Logic Symposium - proceedings (2009)

[30] J. Reiterman: *One More Categorical Model of Universal Algebra*, Mathematische Zeitschrift, Springer-Verlag (1978)

[31] J. Reiterman: *On locally small based algebraic theories*, Commenatationes Mathematicae Unversitatis Carolinae (1986)

[32] J. Rosický: *Concrete Categories and Infinitary Languages*, Journal of Pure and Applied Algebra 22 (1981)

[33] J. Rosický: *On algebraic categories*, Colloquia Mathematica Societatis János Bolai(1977)

[34] J. Rosický:*Equational categories.* Cahiers de Topologie et Gomtrie Diffrentielle Catgoriques, 22 no. 1 (1981), p. 85-95

[35] J. Rosický: *A Note on algebraic categories*, Archivum Mathematicum, Vol. 18 (1982), No. 3, 163167

[36] V. Trnková: *Some properties of set functors*, Commenatationes Mathematicae Unversitatis Carolinae 10, 2(1969)

[37] V. Trnková: *On descriptive classification of set-functors*, I. Commenatationes Mathematicae Unversitatis Carolinae 12(1971) 143–174.

[38] V. Trnková: *Topological Representations of Algebraic Systems*, Encyclopedia of General Topology, Elsevier (2004), 409-414

[39] J. Worrell: *Terminal sequences for accessible functors*, Elsevier Science (1999)

Index