MASARYK UNIVERSITY, FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS

Spectral theory of difference operators

Ph.D. Thesis

BRNO 2010

PETR HASIL

Bibliographic entry

Author	Petr Hasil
Title	Spectral theory of difference operators
Study programme	Mathematics
Study field	Mathematical analysis
Supervisor	Prof. RNDr. Ondřej Došlý, DrSc.
Year	2010
Keywords	Friedrichs extension; linear Hamiltonian difference system; matrix operator; p-critical operator; positive solution; recessive system of solutions; Sturm-Liouville difference equation; Sturm-Liouville difference operator; sub/supercritical operator; three term difference equation.

Acknowledgements

I would like to sincerely thank my supervisor Prof. Ondřej Došlý for his advice, willingness, and great patience.

I also thank Prof. Zuzana Došlá for her helpful advice and for reading the manuscript of the thesis.

I am obliged to Prof. Werner Kratz for preparing a perfect background during my stay at Ulm University in 2009.

Last but not least, I thank my family and friends for their support.

Abstract

In this thesis, we solve three problems. The first one (in Chapter 2) deals with positive solutions of symmetric three-term $2n$-order difference equation

$$a_k y_k + b_{k+n} y_{k+n} + a_{k+n} y_{k+2n} = 0, \quad k \in \mathbb{Z}.$$

We associate this equation with an operator given in the natural way by infinite symmetric matrix and using this matrix and the technique of diagonal minors (Section 2.2) we show that this equation possesses a positive solution for $k \in \mathbb{Z}$ if and only if the infinite symmetric matrix associated with this equation is non-negative definite.

In Chapters 3 and 4, we study the Sturm-Liouville equation of the $2n$-order

$$\sum_{\nu=0}^{n} (-\Delta)^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k-\nu} \right) = 0, \quad k \in \mathbb{Z}, \qquad (*)$$

and associated matrix operator given by an infinite symmetric banded matrix (see Section 3.3). Using the relationship of equation $(*)$ to the so-called linear Hamiltonian systems and the concept of the recessive system of solutions, in Chapter 3 we describe the domain of the Friedrichs extension of the matrix operator associated to $(*)$ and in Chapter 4 we introduce the p-critical operators (Definition 4.1) and we show that arbitrarily small (in a certain sense) negative perturbation of a non-negative critical operator leads to an operator which is no longer non-negative.

2010 Mathematics Subject Classification: 39A10; 39A21; 39A70; 47B25; 47B36; 47B39.

Contents

Chapter 1

Introduction

1.1 Organization of the thesis

This thesis consists of three papers. Each of them includes its own original introduction, where the necessary background is mentioned, but there is one very important theorem, that is used in Chapters 3 and 4, which is usually taken for granted in spite of the fact, that it is (in general discrete case) quite new result given by M. Bohner [5] in 1996. This theorem is Reid Roundabout Theorem. We will recall this theorem in Section 1.2 in this short introductory chapter.

The aim of the Chapter 2, which is basically the paper [21], was to prove that the symmetric three-term $2n$-order difference equation

$$a_k y_k + b_{k+n} y_{k+n} + a_{k+n} y_{k+2n} = 0, \quad k \in \mathbb{Z} \tag{1.1}$$

possesses a positive solution for $k \in \mathbb{Z}$ if and only if the infinite symmetric matrix associated in the natural way with this equation is non-negative definite. We have proved this by introducing the *diagonal minors* (Section 2.2). Behind this technique stands a simple idea, that equation (1.1) may be rewritten, in a certain sense, as n equations of second order

$$\tilde{a}_k^{[i]} \tilde{y}_k^{[i]} + \tilde{b}_{k+1}^{[i]} \tilde{y}_{k+1}^{[i]} + \tilde{a}_{k+1}^{[i]} \tilde{y}_{k+2}^{[i]} = 0, \quad k \in \mathbb{Z},\ i = 1, \dots, n.$$

While the equation from Chapter 2 is associated with the symmetric banded matrix, which is "almost empty" in the sense that there are only three nonzero diagonals, in Chapters 3 and 4 we deal with the general $2n$-order Sturm-Liouville difference equation

$$\sum_{\nu=0}^{n} (-\Delta)^{\nu} \left(r_k^{[\nu]} \Delta^{\nu} y_{k-\nu} \right) = 0, \quad k \in \mathbb{Z}, \tag{1.2}$$

which is also associated with a symmetric banded matrix. This connection have been explicitly described by W. Kratz [25] in 2001 and is recalled in Section 3.3. It is clear, that the approach from Chapter 2 is not suitable anymore. Therefore we convert (1.2) into the linear Hamiltonian difference system

$$\Delta x_k = Ax_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A^T u_k, \tag{1.3}$$

as it is further described, and using the concept of the recessive system of solutions, in Chapter 3 (paper [13]), we describe the domain of the Friedrichs extension of the matrix operator associated to (1.2). In the last chapter (paper [14]) we use the relationship of equation (1.2) and system (1.3) and the concept of the recessive system of solutions to introduce the p-critical operators (Definition 4.1) and we show that arbitrarily small (in a certain sense) negative perturbation of a non-negative critical operator leads to an operator which is no longer non-negative. Then, in Section 4.4, we find an explicit condition for criticality of the one-term difference equation

$$(-\Delta)^2(r_k \Delta^2 y_k) = 0, \quad r_k > 0,\ k \in \mathbb{Z}.$$

Note that the notation $[\cdot,\cdot], (\cdot,\cdot)$ is used for discrete intervals, e.g., $[a, b) = \{a, a+1, \dots, b-2, b-1\}, a, b \in \mathbb{Z}$.

1.2 Sturm-Liouville equations, Hamiltonian systems, and Reid Roundabout Theorem

The aim of this section is to recall some facts about equivalency of $2n$-order Sturm-Liouville difference equations

$$\sum_{\nu=0}^{n} (-\Delta)^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k-\nu} \right) = 0, \quad r_k^{[n]} \neq 0, \forall k, \tag{1.4}$$

and linear Hamiltonian difference systems

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k, \tag{1.5}$$

where A_k, B_k, and C_k are $n \times n$ matrices, B_k and C_k are symmetric, and $I - A_k$ is invertible (where I stands for the identity matrix of the proper dimension). The aim of this section is to formulate the Reid Roundabout Theorem for the linear Hamiltonian difference system equivalent to equation (1.4). Full background, more general statements, and all proofs can be found in [4, 5, 24].

Now consider equation (1.4). Using the substitution

$$x_k^{[y]} = \begin{pmatrix} y_{k-1} \\ \Delta y_{k-2} \\ \vdots \\ \Delta^{n-1} y_{k-n} \end{pmatrix}, \quad u_k^{[y]} = \begin{pmatrix} \sum_{\nu=1}^{n} (-\Delta)^{\nu-1} \left(r_k^{[\nu]} \Delta^\nu y_{k-\nu} \right) \\ \vdots \\ -\Delta(r_k^{[n]} \Delta^n y_{k-n}) + r_k^{[n-1]} \Delta^{n-1} y_{k-n+1} \\ r_k^{[n]} \Delta^n y_{k-n} \end{pmatrix},$$

we can rewrite equation (1.4) to linear Hamiltonian system (1.5) with the $n \times n$ matrices A_k, B_k, and C_k given by the formulas

$$\begin{gathered} A_k = A := (a_{ij})_{i,j=1}^n, \quad a_{ij} = \begin{cases} 1 & \text{if } j = i+1,\ i = 1, \dots, n-1, \\ 0 & \text{elsewhere}, \end{cases} \\ B_k = \operatorname{diag} \left\{ 0, \dots, 0, \frac{1}{r_k^{[n]}} \right\}, \quad C_k = \operatorname{diag} \left\{ r_k^{[0]}, \dots, r_k^{[n-1]} \right\}. \end{gathered} \tag{1.6}$$

Therefore we have

$$\tilde{A} := (I - A_k)^{-1} = (\tilde{a}_{ij})_{i,j=1}^n, \quad \tilde{a}_{ij} = \begin{cases} 1 & \text{if } j \geq i, \\ 0 & \text{elsewhere}. \end{cases}$$

Then we say that the solution (x, u) of (1.5) is generated by the solution y of (1.4).

Let us consider the matrix linear Hamiltonian system

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \quad \Delta U_k = C_k X_{k+1} - A_k^T U_k, \tag{1.7}$$

where the matrices A_k, B_k, and C_k are given by (1.6). We say that the solution (X, U) of (1.7) is generated by the solutions $y^{[1]}, \dots, y^{[n]}$ if and only if its columns are generated by $y^{[1]}, \dots, y^{[n]}$, respectively. On the other hand, if we have the solution (X, U) of (1.7), the elements from the first line of the matrix X are exactly the solutions $y^{[1]}, \dots, y^{[n]}$ of (1.4). Now, we can define the oscillatory properties of (1.4) via the corresponding properties of the associated Hamiltonian system (1.5) with matrices A_k, B_k, and C_k given by (1.6). E.g., equation (1.4) is disconjugate if and only if the associated system (1.5) is disconjugate, the system of solutions $y^{[1]} \dots, y^{[n]}$ is said to be recessive if and only if it generates the recessive solution X of (1.7), etc. So, we define the following properties just for linear Hamiltonian systems.

Let (X, U) and $(\tilde{X}, \tilde{U})$ be two solutions of (1.7). Then

$$X_k^T \tilde{U}_k - U_k^T \tilde{X}_k \equiv W \tag{1.8}$$

holds with a constant matrix W. (This is an analog of the continuous *Wronskian identity*.) We say that the solution (X, U) of (1.7) is a *conjoined basis* if

$$X_k^T U_k \equiv U_k^T X_k \quad \text{and} \quad \operatorname{rank}\begin{pmatrix} X \\ U \end{pmatrix} = n.$$

Two conjoined bases $(X, U), (\tilde{X}, \tilde{U})$ of (1.7) are called normalized conjoined bases of (1.7) if $W = I$ in (1.8).

System (1.7) is said to be *disconjugate* in an interval $[l, m]$, $l, m \in \mathbb{N}$, if the solution $\binom{X}{U}$ given by the initial condition $X_l = 0$, $U_l = I$ satisfies

$$\text{Ker}\, X_{k+1} \subseteq \text{Ker}\, X_k \quad \text{and} \quad X_k X_{k+1}^{\dagger}(I - A_k)^{-1} B_k \geq 0$$

for $k = l, \ldots, m-1$. Here Ker, †, and $\geq$ stand for the kernel, Moore-Penrose generalized inverse, and non-negative definiteness of a matrix indicated, respectively. System (1.7) is said to be *non-oscillatory* if there exists $l \in \mathbb{N}$ such that it is disconjugate on $[l, m]$ for every $m \in \mathbb{N}$, otherwise it is called *oscillatory*.

We call a conjoined basis $\binom{\tilde{X}}{\tilde{U}}$ of (1.7) the *recessive solution* at ∞ if the matrices $\tilde{X}_k$ are nonsingular, $\tilde{X}_k \tilde{X}_{k+1}^{-1}(I - A_k)^{-1} B_k \geq 0$, both for large k, and for any other conjoined basis $\binom{X}{U}$ for which the (constant) matrix $X^T \tilde{U} - U^T \tilde{X}$ is nonsingular, we have

$$\lim_{k \to \infty} X_k^{-1} \tilde{X}_k = 0.$$

The solution (X, U) is usually called *dominant* at ∞. The recessive solution at ∞ is determined uniquely up to a right multiple by a nonsingular constant matrix and exists whenever (1.7) is non-oscillatory and eventually controllable, see p. 25.

We say that a pair (x, u) is *admissible* for system (1.5) if and only if the first equation in (1.5) holds.

Now we introduce the energy functional of equation (1.4)

$$\mathcal{F}(y) := \sum_k \sum_{\nu=0}^{n} r_k^{[\nu]} (\Delta^\nu y_{k-\nu})^2.$$

For admissible (x, u) we have

$$\begin{aligned} \mathcal{F}(y) &= \sum_k \sum_{\nu=0}^{n} r_k^{[\nu]} (\Delta^\nu y_{k-\nu})^2 \\ &= \sum_k \left[\sum_{\nu=0}^{n-1} r_k^{[\nu]} (\Delta^\nu y_{k-\nu})^2 + \frac{1}{r_k^{[n]}} (r_k^{[n]} \Delta^n y_{k-n})^2 \right] \\ &= \sum_k \left[x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right] =: \mathcal{F}(x, u). \end{aligned}$$

Similarly as in the continuous case, using the substitution $Q_k = U_k X_k^{-1}$ in (1.7) we obtain the Riccati matrix difference equation

$$Q_{k+1} = C_k + (I - A_k^T) Q_k (I + B_k Q_k)^{-1} (I - A_k). \tag{1.9}$$

Now we are ready to formulate the Reid Roundabout Theorem.

Theorem 1.1 (Reid Roundabout Theorem, [5])**.** *Let* $n < N,\ N \in \mathbb{N}$. *Then the following statements are equivalent.*

(i) *System* (1.5) *is disconjugate in* $[0, N]$.

(ii) $\mathcal{F}(x, u) > 0$ *for every admissible* (x, u) *on* $[0, N]$ *with* $x_0 = 0 = x_{N+1}$ *and* $x_k \not\equiv 0$ *on* $[1, N-1]$.

(iii) *There exists a conjoined basis* (X, U) *of* (1.7) *such that* X_k *is invertible for* $k \in [0, N+1]$ *and*

$$X_k X_{k+1}^{-1} \tilde{A} B_k \geq 0, \quad k \in [0, N].$$

(iv) *There exists a (symmetric) solution* Q_k *of the Riccati matrix difference equation* (1.9) *such that* $(I + B_k Q_k)^{-1} B_k \geq 0$ *for all* $k \in [0, N]$.

Chapter 2

On positivity of the three term $2n$-order difference operators

2.1 Introduction

Let $a_k < 0, b_k > 0, k \in \mathbb{Z}$, be real-valued sequences and consider the symmetric $2n$-order recurrence relation

$$(\tau y)_{k+n} := a_k y_k + b_{k+n} y_{k+n} + a_{k+n} y_{k+2n}.$$

We use a standard notation

$$\ell^2(\mathbb{Z}) := \left\{ y = \{y_k\}_{k\in\mathbb{Z}}, \sum_{k\in\mathbb{Z}} |y_k|^2 < \infty \right\}.$$

We associate with the difference equation

$$\tau y = 0 \tag{2.1}$$

the operator $\mathcal{T} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ with the domain

$$\mathcal{D}(\mathcal{T}) = \ell_0^2(\mathbb{Z}) = \left\{ y = \{y_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}),\ \text{only finitely many } y_k \neq 0 \right\}$$

given by the formula $\mathcal{T}f = \tau f, f \in \ell_0^2(\mathbb{Z})$, and we show that (2.1) has a positive solution $y_k, k \in \mathbb{Z}$, if and only if the operator $\mathcal{T}$ is non-negative, i.e.,

$$\langle \mathcal{T}y, y\rangle \geq 0 \qquad \forall y \in \ell_0^2(\mathbb{Z}),$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\ell^2(\mathbb{Z})$. The statements presented in this chapter extend some results given in [19], where the case $n = 1$ is considered.

2.2 Preliminaries

Let T be the infinite symmetric matrix associated in the natural way with the operator $\mathcal{T}$ and let us denote for $\mu \le \nu, \mu, \nu \in \mathbb{Z}$, the truncations of T by

$$t_{\mu,\nu} := \begin{pmatrix} b_\mu & 0 & \cdots & 0 & a_\mu & \cdots & 0 \\ 0 & b_{\mu+1} & 0 & \cdots & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & & \vdots & 0 & a_{\nu-n} \\ 0 & \vdots & & \ddots & 0 & \vdots & 0 \\ a_\mu & 0 & \cdots & 0 & b_{\mu+n} & 0 & \vdots \\ 0 & \ddots & 0 & \cdots & 0 & \ddots & 0 \\ \cdots & 0 & a_{\nu-n} & 0 & \cdots & 0 & b_\nu \end{pmatrix},$$

and their determinants by

$$d_{\mu,\nu} := \det\bigl(t_{\mu,\nu}\bigr).$$

For $r, s \in \mathbb{Z}, r \equiv s \pmod n$, we introduce the diagonal minors $D_{r,s}$, which are determinants of the submatrix of $t_{\mu,\nu}$ consisting of rows and columns which contain diagonal elements $b_r, b_{r+n}, b_{r+2n}, \dots, b_{s-n}, b_s$.

The following statement gives us the relationship between determinant $d_{\mu,\nu}$ and its minors $D_{r,s}$.

Lemma 2.1. *It holds that*

$$\prod_{i=0}^{n-1} D_{\mu+i,s_i} = d_{\mu,\nu} = \prod_{i=0}^{n-1} D_{r_i,\nu-i},$$

where, for $i = 0, \dots, n-1$,

$$s_i = \max\{x \in \mathbb{Z} : x \equiv \mu + i \pmod n,\ x \le \nu\},$$
$$r_i = \min\{x \in \mathbb{Z} : x \equiv \nu - i \pmod n,\ x \ge \mu\}.$$

Proof. Without any change of the determinant, the matrix $t_{\mu,\nu}$ can be transformed into the block diagonal matrix $\tilde{t}_{\mu,\nu}$ with blocks $D_{\mu+i,s_i}$ or $D_{r_i,\nu-i}$, $i = 0, \dots, n-1$. □

Corollary 2.1. *It holds that*

$$\frac{D_{r,\nu}}{D_{r,\nu-n}} d_{\mu,\nu-1} = d_{\mu,\nu} = \frac{D_{\mu,s}}{D_{\mu+n,s}} d_{\mu+1,\nu},$$

where $r := \min\{x \in \mathbb{Z} : x \equiv \nu \pmod n,\ x \ge \mu\}$ *and* $s := \max\{x \in \mathbb{Z} : x \equiv \mu \pmod n,\ x \le \nu\}$.

Corollary 2.2. *It holds that*

$$\begin{aligned} D_{\mu,s_0} &= b_\mu\, D_{\mu+n,s_0} - a_\mu^2\, D_{\mu+2n,s_0}, \\ D_{r_0,\nu} &= b_\nu\, D_{r_0,\nu-n} - a_{\nu-n}^2\, D_{r_0,\nu-2n}. \end{aligned}$$

Proof. Computing $d_{\mu,\nu}$ by expanding it along its first row we obtain

$$d_{\mu,\nu} = b_\mu\, d_{\mu+1,\nu} - a_\mu^2 \left(\prod_{i=1}^{n-1} D_{\mu+i,s_i} \right) D_{\mu+2n,s_0}.$$

Next, using Lemma 2.1, we have

$$\prod_{i=0}^{n-1} D_{\mu+i,s_i} = b_\mu \left(\prod_{i=1}^{n-1} D_{\mu+i,s_i} \right) D_{\mu+n,s_0} - a_\mu^2 \left(\prod_{i=1}^{n-1} D_{\mu+i,s_i} \right) D_{\mu+2n,s_0},$$

which is the first equality. The second equality can be proved similarly. □

Lemma 2.2. *Let $\nu > \mu + n$ and suppose that $t_{\mu,\nu} \geq 0$ (i.e., the matrix $t_{\mu,\nu}$ is non-negative definite), then $d_{\mu,\nu-n} > 0, \dots, d_{\mu,\nu-1} > 0$.*

Proof. We prove this statement for $d_{\mu,\nu-n}$. The rest can be proved similarly.

The assumption $t_{\mu,\nu} \geq 0$ implies

(i) $d_{\mu,k} \geq 0, \forall k \in [\mu, \nu]$,

(ii) $D_{\mu+i,s_{k_i i}} \geq 0, \forall k_i \in [0, \nu - \mu - i]$, where

$$s_{k_i i} = \max\{x \in \mathbb{Z} : x \equiv \mu + i \pmod n,\ x \leq \nu - k_i\}.$$

Denote

$$\ell := \min\{x \in [\mu, \nu] : x \equiv \nu \pmod n,\ d_{\mu,x} = 0\}.$$

Because $d_{\mu,\mu} = b_\mu > 0, \dots, d_{\mu,\mu+n-1} = \prod_{j=0}^{n-1} b_{\mu+j} > 0$, we have $\ell \geq \mu + n$. Now, suppose by contradiction that $d_{\mu,\nu-n} = 0$, which implies that $\ell \leq \nu - n$, and compute

$$\begin{aligned} d_{\mu,\ell+n} &= b_{\ell+n}\, d_{\mu,\ell+n-1} - a_\ell^2 \left(\prod_{i=1}^{n-1} D_{r_i,\ell+1-i} \right) D_{r_0,\ell-n}, \\ D_{r_0,\ell+n} &= b_{\ell+n}\, D_{r_0,\ell} - a_\ell^2\, D_{r_0,\ell-n}. \end{aligned}$$

Using Corollary 2.1 we have $0 = d_{\mu,\ell} = D_{r_0,\ell}\, \frac{d_{\mu,\ell-1}}{D_{r_0,\ell-n}}$ which implies that $D_{r_0,\ell} = 0$ and using Lemma 2.1 we have $0 < d_{\mu,\ell-n} = K\, D_{r_0,\ell-n}$, where $K > 0$, which implies that $D_{r_0,\ell-n} > 0$. Altogether, we obtained that $D_{r_0,\ell+n} < 0$, which contradicts our assumption $t_{\mu,\nu} \geq 0$. □

The following statement can be proved easily using Lemma 2.2 (see [19, p. 74–75]).

Lemma 2.3. *The operator $\mathcal{T}$ is non-negative definite if and only if $d_{\mu,\nu} > 0$, for all $\mu, \nu \in \mathbb{Z}, \mu \leq \nu$.*

Now, we present two auxiliary statements, which give us the possibility to construct a solution of the difference equation (2.1).

Lemma 2.4. *Let $\mathcal{T} \geq 0$, and r, s be the same as in Corollary 2.1. Then*

(i)
$$\frac{1}{b_\mu} < \frac{D_{\mu+n,s}}{D_{\mu,s}} < \frac{b_{\mu-n}}{a_{\mu-n}^2};$$

(ii)
$$\frac{1}{b_\nu} < \frac{D_{r,\nu-n}}{D_{r,\nu}} < \frac{b_{\nu+n}}{a_\nu^2};$$

(iii) The sequence
$$c_i := \left\{ \frac{D_{\mu+n,s+in}}{D_{\mu,s+in}} \right\}$$
is increasing for $i \in \mathbb{N}_0$.

(iv) The sequence
$$d_i := \left\{ \frac{D_{r-in,\nu-n}}{D_{r-in,\nu}} \right\}$$
is increasing for $i \in \mathbb{N}_0$.

Proof. We prove parts (i) and (iii). Parts (ii) and (iv) can be proved similarly. To prove the first part, we expand $d_{\mu,\nu}$ along its first row and using Lemma 2.1 we obtain
$$D_{\mu,s_0} = b_\mu\, D_{\mu+n,s_0} - a_\mu^2\, D_{\mu+2n,s_0}.$$
Because $a_\mu^2\, D_{\mu+2n,s_0} > 0$, the left inequality in (i) holds. Now, we expand the determinant $d_{\mu-n,\nu}$ and we obtain
$$0 < D_{\mu-n,s_0} = b_{\mu-n}\, D_{\mu,s_0} - a_{\mu-n}^2\, D_{\mu+n,s_0}$$
which proves the right inequality in (i).

The part (iii) we prove by induction. Directly one can verify that
$$\frac{D_{\mu+n,\mu+n}}{D_{\mu,\mu+n}} < \frac{D_{\mu+n,\mu+2n}}{D_{\mu,\mu+2n}}.$$

Now, we multiply equalities

$$D_{\mu,s} = b_\mu\, D_{\mu+n,s} - a_\mu^2\, D_{\mu+2n,s},$$
$$D_{\mu,s+n} = b_\mu\, D_{\mu+n,s+n} - a_\mu^2\, D_{\mu+2n,s+n}$$

by $\frac{1}{D_{\mu+n,s}}$ and $\frac{1}{D_{\mu+n,s+n}}$, respectively, and subtract them. We obtain

$$\frac{D_{\mu,s}}{D_{\mu+n,s}} - \frac{D_{\mu,s+n}}{D_{\mu+n,s+n}} = a_\mu^2 \left(\frac{D_{\mu+2n,s+n}}{D_{\mu+n,s+n}} - \frac{D_{\mu+2n,s}}{D_{\mu+n,s}} \right),$$

which proves the statement. □

Lemma 2.5. *Let f, g be solutions of $\tau y = 0$. If $f_j = g_j, \dots, f_{j+n-1} = g_{j+n-1}$, $j \in \mathbb{Z}$, then*

$$f_{k_0} \underset{(=)}{>} g_{k_0}, \dots, f_{k_0+n-1} \underset{(=)}{>} g_{k_0+n-1} \text{ for some } k_0 \geq j+n \quad (k_0 \leq j-n) \tag{2.2}$$

implies

$$f_k \underset{(=)}{>} g_k \quad \forall k \geq j+n \quad (k \leq j-n).$$

Proof. We follow the idea introduced in [19, 30]. Let f and g be solutions of the linear system

$$t_{j+n,k_0-1} \begin{pmatrix} y_{j+n} \\ y_{j+n+1} \\ \vdots \\ y_{k_0-1} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ 0 \\ \vdots \\ \mathbf{w} \end{pmatrix},$$

where

$$\mathbf{v} = \begin{pmatrix} -a_j y_j \\ \vdots \\ -a_{j+n-1} y_{j+n-1} \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} -a_{k_0-n} y_{k_0} \\ \vdots \\ -a_{k_0-1} y_{k_0+n-1} \end{pmatrix}.$$

Then $f_k \underset{(=)}{>} g_k$ for $k \in [j+n, k_0+n-1]$. Now, the existence of a $K \geq k_0+n$ such that $f_k < g_k$ contradicts (2.2). □

If we suppose that $\mathcal{T} \geq 0$, we can (due to Lemma 2.4, $j \in [0, n-1]$) introduce the limits

$$C_+^{[\mu,j]} := \lim_{i\to\infty} \frac{D_{\mu+n,s_j+in}}{D_{\mu,s_j+in}}, \quad C_-^{[\nu,j]} := \lim_{i\to\infty} \frac{D_{r_j-in,\nu-n}}{D_{r_j-in,\nu}}$$

and a positive map $u : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ defined by

$$u(\mu, k) := \begin{cases} \prod_{i=1}^{\ell} \left(-a_{k-in}\, C_{+}^{[k-(i-1)n, k-\mu-\ell n]} \right), & k \geq \mu + n, \\ 1, & k \in [\mu - n + 1, \mu + n - 1], \\ \prod_{i=0}^{\ell-1} \left(-a_{k+in}\, C_{-}^{[k+in, \mu-k-\ell n]} \right), & k \leq \mu - n, \end{cases}$$

where $\ell = \left| \lfloor \frac{k-\mu}{n} \rfloor \right|$, and $\lfloor \cdot \rfloor$ denotes the floor function (greatest integer less than or equal to the number indicated) of a real number. Now, let us recall the definition of a minimal solution of (2.1).

Definition 2.1. We say that a solution u of (2.1) is *minimal* on $[\mu + n, \infty)$ if any linearly independent solution v of (2.1) with $v_\mu = u_\mu, \dots, v_{\mu+n-1} = u_{\mu+n-1}$ satisfies $v_k > u_k$ for $k \geq \mu + n$. The minimal solution on $(-\infty, \mu - n]$ is defined analogously.

Lemma 2.6. *Let $\mathcal{T} \geq 0$. Then $u(\mu, k)$, $\mu \in \mathbb{Z}$ being fixed, is the minimal positive solution of $\tau y = 0$ on $[\mu + n, \infty)$ and $(-\infty, \mu - n]$.*

Proof. We introduce

$$u_k^{[\mu,\nu]} := \begin{cases} 1, & k \in [\mu, \mu + n - 1] \\ (-a_{k-n})(-a_{k-2n}) \cdots (-a_{k-\ell n}) \frac{D_{k+n, s_i}}{D_{k-(\ell-1)n, s_i}}, & k \in [\mu + n, s_0 - 1], \\ (-a_{k-n})(-a_{k-2n}) \cdots (-a_{k-\ell n}) \frac{1}{D_{k-(\ell-1)n, s_i}}, & k \in [s_0, s_{n-1}] \\ 0, & k \in [s_{n-1} + 1, s_{n-1} + n], \end{cases}$$

where $i = k - \mu - \ell n$, $\ell = \left| \lfloor \frac{k-\mu}{n} \rfloor \right|$.

The sequence $u_k^{[\mu,\nu]}$ satisfies $\tau y = 0$ on $[\mu + n, s_{n-1}]$ and it holds that

$$\lim_{s_{n-1} \to \infty} u_k^{[\mu,\nu]} = u(\mu, \nu), \quad k \geq \mu,$$

so we can see that $\tau u = 0$ on $[\mu + n, \infty)$.

Let v be a positive solution of $\tau y = 0$ on $[\mu + n, \infty)$ such that $v_\mu = 1, \dots, v_{\mu+n-1} = 1$, which is linearly independent of $u(\mu, \cdot)$. Because $v_{s_{n-1}+1} > 0, \dots, v_{s_{n-1}+n} > 0$, we have $v_k > u_k^{[\mu, s_{n-1}]}$ on $[s_{n-1} + 1, s_{n-1} + n]$. Thus, by Lemma 2.5, $v_k > u_k^{[\mu, s_{n-1}]}$ for all $k \geq \mu + n$. Hence, $v_k \geq u(\mu, k)$ for all $k \geq \mu$. Assume, by contradiction, that there exists $k_0 \geq \mu + n$ such that $v_{k_0} = u(\mu, k_0)$. Then, again by Lemma 2.5, $v_k = u(\mu, k)$ for all $k \geq \mu$, which is a contradiction.

One can show analogously that u is the minimal positive solution of $\tau y = 0$ on $(-\infty, \mu - n]$ using

$$u_k^{[\mu,\nu]} := \begin{cases} 1, & k \in [\mu - n + 1, \mu], \\ (-a_k)\,(-a_{k+n}) \cdots (-a_{k+(\ell-1)n})\,\frac{D_{r_j,k-n}}{D_{r_j,k+(\ell-1)n}}, & k \in [r_0 + 1, \mu - n], \\ (-a_k)\,(-a_{k+n}) \cdots (-a_{k+(\ell-1)n})\,\frac{1}{D_{r_j,k+(\ell-1)n}}, & k \in [r_{n-1}, r_0], \\ 0, & k \in [r_{n-1} - 1, r_{n-1} - n], \end{cases}$$

where $j = \mu - k - \ell n$, $\ell = \left|\left\lfloor \frac{k-\mu}{n} \right\rfloor\right|$. □

2.3 Positive solutions of $\tau y = 0$

If $\mathcal{T} \geq 0$, as a consequence of the previous section, the minimal positive solutions of the equation $\tau y = 0$ on $[n, \infty)$ and $(-\infty, n]$ are

$$u_k^+ := \begin{cases} u(0,k), & k \in \mathbb{N}_0, \\ u(k-i,i)^{-1}, & k \equiv i \pmod{n}, -k \in \mathbb{N}, i \in [0, n-1], \end{cases}$$

$$u_k^- := \begin{cases} u(0,k), & -k \in \mathbb{N}_0, \\ u(k+i,-i)^{-1}, & k \equiv i \pmod{n}, k \in \mathbb{N}, i \in [0, n-1]. \end{cases}$$

respectively.

Lemma 2.7. *Let $\mathcal{T} \geq 0$. Then u_k^+, u_k^- are positive solutions of $\tau y = 0$ on $\mathbb{Z}$.*

Proof. We show that u_k^+ is a positive solution of $\tau y = 0$ on $\mathbb{Z}$, the statement for u_k^- can be proved similarly.

The sequence u^+ is a positive solution of $\tau y = 0$ on $[n, \infty)$. It follows from the definition of $u(\mu, k)$ that for

$$\forall k, \ell \in \mathbb{Z},\ \forall m \in [k, \ell],\ m \equiv k \pmod{n}$$

it holds that

$$u(k,\ell) = u(k,m)\,u(m,\ell).$$

Let $m \in (-\infty, n-1]$ be arbitrary and $M \in -\mathbb{N}$ such that $M \leq m - n$ be fixed. Then for some $i \in [0, n-1]$ we have $M \equiv m - i \pmod{n}$ and

$$u(M,i) = u(M, m-i)\,u(m-i,i).$$

Hence, by the definition of u^+, we obtain

$$u_m^+ = u(m-i,i)^{-1} = \frac{u(M,m-i)}{u(M,i)},$$

which implies that u^+ is a solution of $\tau y = 0$ on $[M+i,i]$. □

Theorem 2.1. *$\mathcal{T} \geq 0$ if and only if there exists a positive solution of $\tau y = 0$.*

Proof. The necessity follows from Lemma 2.7. To prove sufficiency we show at first positivity of all determinants $D_{\mu,\xi},\ \mu \leq \xi \leq \nu,\ \xi \equiv \mu \pmod n$. Assume that there exists a positive solution u of $\tau y = 0$. Then u solves the system

$$t_{\mu,\nu} \begin{pmatrix} u_\mu \\ u_{\mu+1} \\ \vdots \\ u_\nu \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ 0 \\ \vdots \\ 0 \\ \mathbf{w} \end{pmatrix},$$

where

$$\mathbf{v} = \begin{pmatrix} -a_{\mu-n}u_{\mu-n} \\ \vdots \\ -a_{\mu-1}u_{\mu-1} \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} -a_{\nu-n+1}u_{\nu+1} \\ \vdots \\ -a_\nu u_{\nu+n} \end{pmatrix}.$$

By Cramer's rule we obtain

$$u_\mu d_{\mu,\nu} =$$
$$\Big[-a_{\mu-n}u_{\mu-n}D_{\mu+n,s_0}+(-a_\sigma u_{\sigma+n})(-a_\mu)(-a_{\mu+n})\cdots(-a_{s_0-n})\Big]\left(\prod_{i=1}^{n-1} D_{\mu+i,s_i}\right),$$

where $\sigma \in \{\nu-n+1,\ldots,\nu\},\ \sigma \equiv \mu \pmod n$. Hence

$$D_{\mu,s_0}u_\mu = -a_{\mu-n}u_{\mu-n}D_{\mu+n,s_0} + (-a_\mu)(-a_{\mu+n})\cdots(-a_{s_0-n})(-a_\sigma)u_{\sigma+n}.$$

If $s_0 = \mu+n$ then $D_{\mu+n,s_0} = D_{\mu+n,\mu+n} = b_{\mu+n} > 0$ which implies that $D_{\mu,\mu+n} > 0$. Hence all determinants $D_{\mu,\xi}$ are positive. Similarly we can show positivity of the determinants $D_{\mu+i,\xi_i},\ \mu+i \leq \xi_i \leq \nu,\ \xi_i \equiv \mu+i \pmod n,\ i = 1,\ldots,n-1$, which gives positivity of the determinants $d_{\mu,\nu}, \mu \leq \nu$ and the statement follows from Lemma 2.3. □

In an analogous way as above we can generalize a number of statements of [19]. For example, the following result, which characterizes the minimal solutions of (2.1), can be proved in the same way as in [19] (see also [31]).

Theorem 2.2. *If $\mathcal{T} \geq 0$, then for all $\mu \in \mathbb{Z}$*

$$\sum_{i=\mu}^{\pm\infty} \left(-a_i u_i^{\pm} u_{i+n}^{\pm}\right)^{-1} = \infty.$$

We end up this chapter illustrating the previous theory by an example. For simplicity we use an equation with constant coefficients, which enables us to demonstrate both implications in Theorem 2.1.

Example. Let us consider the equation

$$y_{k+4} - 2y_{k+2} + y_k = 0, \quad k \in \mathbb{Z}. \tag{2.3}$$

By Theorem 2.1, the equation (2.3) possesses a positive solution on $\mathbb{Z}$ if and only if the operator $\mathcal{T}$ given by the matrix

$$\begin{pmatrix} \ddots & & \ddots & & \ddots & & & \\ 0 & -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 & 0 \\ & & & \ddots & & \ddots & & \ddots \end{pmatrix} \tag{2.4}$$

is non-negative definite.

(i) By a direct computation we obtain the solutions $1, (-1)^k, k, (-1)^k k$ of the equation (2.3). Hence, the matrix operator $\mathcal{T}$ is non-negative definite, because the equation (2.3) has the positive solution $y_k = 1$.

(ii) We show that the matrix (2.4) is non-negative definite. According Lemma 2.3 it suffices to show that all minors of the matrix (2.4) are positive, i.e., $d_{\mu,\nu} > 0, \mu, \nu \in \mathbb{Z}$. This we can show by induction using Lemma 2.1 and Corollary 2.1. We obtain two identical tridiagonal blocks

$$\begin{pmatrix} \ddots & \ddots & \ddots & & \\ 0 & -1 & 2 & -1 & 0 \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Now, it suffices to show that all main minors $D_i, i \in \mathbb{N}$, of these blocks are positive. (The index i stands for the dimension of the minor D_i.) We have $D_1 = 2, D_2 = 3, D_3 = 4, \ldots$. We suppose that $D_k = k + 1$ and calculate D_{k+1}. Expanding along its last row we obtain

$$D_{k+1} = 2D_k - D_{k-1} = 2(k+1) - k = k + 2.$$

So, the matrix operator $\mathcal{T}$ is non-negative definite. Therefore the equation (2.3) has a positive solution. This solution we can obtain using the formulas for u_k^+ and/or for u_k^-. Here, we obtain that $u_k^+ = u_k^- = 1$.

Chapter 3

Friedrichs extension of operators defined by symmetric banded matrices

3.1 Introduction

We consider the infinite symmetric banded matrix with the bandwidth $2n + 1$

$$T = (t_{\mu,\nu}), \quad t_{\mu,\nu} = t_{\nu,\mu} \in \mathbb{R},\ \mu, \nu \in \mathbb{N} \in \{0\}, \quad t_{\mu,\nu} = 0 \quad \text{for } |\mu - \nu| > n. \tag{3.1}$$

If we set $t_{\mu,\nu} = 0$ (and $y_\nu = 0$) for $\mu < 0$, $\nu < 0$, we associate with T the operator $\mathcal{T} : \ell^2 \to \ell^2$ defined for $y = \{y_k\}_{k=0}^{\infty} \in \ell^2$ such that $\mathcal{T}y \in \ell^2$ by

$$(\mathcal{T}y)_k = \sum_{j=k-n}^{k+n} t_{k,j} y_j, \quad k \in \mathbb{N}. \tag{3.2}$$

We are motivated by the papers [8] and [26], where the authors investigated the Friedrichs extension of operators defined by infinite Jacobi matrices and by singular $2n$-order differential expressions, respectively. It was shown there that the domain of the Friedrichs extension of these operators can be characterized by the so-called recessive and principal solutions of certain associated difference and differential equations.

Here we associate with (3.2) a $2n$-order Sturm-Liouville difference equation and using the concept of the recessive system of solutions of this Sturm-Liouville equation we characterize the domain of the Friedrichs extension of $\mathcal{T}$. The crucial role in our treatment is played by the results of [25], where the relationship between banded matrices, $2n$-order Sturm-Liouville difference operators, and linear Hamiltonian difference systems is established.

This chapter is organized as follows. In the next section we recall elements of the theory of symmetric operators in Hilbert spaces and their self-adjoint extensions. In Section 3 we discuss the relationship between banded symmetric matrices, Sturm-Liouville difference operators, and linear Hamiltonian difference systems. We also present elements of the spectral theory of symmetric difference operators in this section. The main result of this chapter is given in Section 3.4.

3.2 Friedrichs extension of a symmetric operator

First let us briefly recall the concept of the Friedrichs extension of a symmetric operator. Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and let $\mathcal{T}$ be a densely defined symmetric operator in H with the domain $\mathcal{D}(\mathcal{T})$. Suppose also that $\mathcal{T}$ is bounded below, i.e., there exists a real constant γ such that

$$\langle \mathcal{T}x, x\rangle \geq \gamma\langle x, x\rangle, \quad x \in \mathcal{D}(\mathcal{T}).$$

Friedrichs [17] showed that there exists a self-adjoint extension $\mathcal{T}_F$ of $\mathcal{T}$, later named *Friedrichs extension* of $\mathcal{T}$, which preserves the lower bound of $\mathcal{T}$. The domain $\mathcal{D}(\mathcal{T}_F)$ of this extension can be characterized as follows. The sesquilinear form

$$G(x,y) := \langle \mathcal{T}x, y\rangle - (\gamma-\varepsilon)\langle x, y\rangle, \quad \varepsilon > 0,$$

defines an inner product on H, denote by $\langle\cdot,\cdot\rangle_{\mathcal{T}}$ this inner product, and by $H_{\mathcal{T}}$ the completion of $\mathcal{D}(\mathcal{T})$ in this product. Then the domain of $\mathcal{T}_F$ is

$$\mathcal{D}(\mathcal{T}_F) = H_{\mathcal{T}} \cap \mathcal{D}(\mathcal{T}^*),$$

where $\mathcal{T}^*$ is the adjoint operator of $\mathcal{T}$. It can be shown (see, e.g., [23, p. 352]) that for any $x \in \mathcal{D}(\mathcal{T}_F)$ there exists a sequence $x_n \in \mathcal{D}(\mathcal{T})$ such that

$$\widetilde{G}(x - x_n, x - x_n) \to 0 \quad \text{as } n \to \infty,$$

where $\widetilde{G}$ denotes the closure of G. Another characterization of $\mathcal{D}(\mathcal{T}_F)$ comes from Freudenthal [16]:

$$\begin{aligned}\mathcal{D}(\mathcal{T}_F) =& \{x \in \mathcal{D}(\mathcal{T}) : \exists x_k \in \mathcal{D}(\mathcal{T}) \text{ such that } x_k \to x \text{ in } H \\ & \text{and } G(x_j - x_k, x_j - x_k) \to 0 \text{ as } j, k \to \infty\}.\end{aligned} \tag{3.3}$$

The construction of the sequence x_n in our particular case, when $\mathcal{T}$ is the operator defined by the infinite matrix in (3.1), is based on the so-called Reid's construction of the *recessive solution* of linear Hamiltonian difference systems (see, e.g., [1, 2]) and on the resulting concept of the *recessive system of solutions* of even-order Sturm-Liouville equations introduced in [11]. The concept of the recessive solution of difference equations is the discrete version of the concept of the *principal solution* of differential equations and systems.

To explain the role of these concepts in the theory of Friedrichs extensions of differential and difference operators, let us start with the *regular* Sturm-Liouville differential operator

$$L(y) := -[r(t)y']' + p(t)y, \tag{3.4}$$

where $t \in (a,b), -\infty < a < b < \infty, \quad r^{-1}, p \in \mathcal{L}(a,b)$. It is well known that the domain of the Friedrichs extension L_F of the minimal operator defined by L is given by the Dirichlet boundary condition

$$\mathcal{D}(L_F) = \{y \in \mathcal{L}^2(a,b) : L(y) \in \mathcal{L}^2(a,b), y(a) = 0 = y(b)\}. \tag{3.5}$$

If the operator L is *singular* at one or both endpoints a, b, it was discovered by Rellich [33] that functions in $\mathcal{D}(L_F)$ behave near a and b like the *principal solution* of a certain non-oscillatory differential equation associated with (3.4). This fact is a natural extension of (3.5) since the principal solution (at a singular point) of a second order differential equation is a solution which is less, near this singular point (in a certain sense), than any other solution of this equation. We refer to the paper [27], where the concept of the principal solution has been introduced and to books [20, 32] for properties of the principal solution of (3.4) and for the extension of this concept to linear Hamiltonian systems. Note also that the results of Rellich had been later extended in various directions, let us mention here at least the papers [22, 26, 29].

Concerning the Friedrichs extension of *difference* operators, the discrete counterpart of the concept of the principal solution is the so-called *recessive* solution. This concept for the second order Sturm-Liouville difference equation

$$\Delta(r_k \Delta x_k) + p_k x_{k+1} = 0, \quad \Delta x_k := x_{k+1} - x_k, \tag{3.6}$$

appears explicitly for the first time in [18], even if it is implicitly contained in a series of earlier papers. The fact that this solution of (3.6) plays the same role in the theory of second order difference operators and Jacobi matrices as the principal solution for differential operators has been established in [3, 8, 19]. In this chapter, we extend some results of these papers to matrix operators defined by (3.2).

3.3 Sturm-Liouville difference operators and symmetric banded matrices

We start this section with the relationship between banded symmetric matrices and Sturm-Liouville difference operators as established in [25]. Consider the $2n$-order

Sturm-Liouville difference operator

$$L(y)_k := \sum_{\mu=0}^{n} (-\Delta)^{\mu} \left[r_k^{[\mu]} \Delta^{\mu} y_{k-\mu} \right], \quad k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad r_k^{[n]} \neq 0, \tag{3.7}$$

where $\Delta y_k = y_{k+1} - y_k$ and $\Delta^{\nu} y_k = \Delta(\Delta^{\nu-1} y_k)$. Expanding the forward differences in (3.7), with the convention that $t_{\mu,\nu} = 0$ for $\mu, \nu < 0$, we get the recurrence relation (3.2) with $t_{i,j}$ given by the formulas

$$\begin{aligned} t_{k,k+j} &= (-1)^j \sum_{\mu=j}^{n} \sum_{\nu=j}^{\mu} \binom{\mu}{\nu} \binom{\mu}{\nu-j} r_{k+\nu}^{[\mu]}, \\ t_{k,k-j} &= (-1)^j \sum_{\mu=j}^{n} \sum_{\nu=0}^{\mu-j} \binom{\mu}{\nu} \binom{\mu}{\nu+j} r_{k+\nu}^{[\mu]}, \end{aligned} \tag{3.8}$$

for $k \in \mathbb{N}_0$ and $j \in \{0, \dots, n\}$. Consequently, one can associate the difference operator L with the matrix operator $\mathcal{T}$, defined via an infinite matrix T, by the formula

$$(\mathcal{T} y)_k := L(y)_k, \quad k \in \mathbb{N}_0, \tag{3.9}$$

where L is related to $\mathcal{T}$ by (3.2) and (3.8). Conversely, having a symmetric banded matrix $T = (t_{\mu,\nu})$ with the bandwidth $2n+1$, one can associate with this matrix the Sturm-Liouville operator (3.7) with $r^{[\mu]}$, $\mu = 0, \dots, n$, given by the formula

$$r_{k+\mu}^{[\mu]} = (-1)^{\mu} \sum_{s=\mu}^{n} \left[\binom{s}{\mu} t_{k,k+s} + \sum_{l=1}^{s-\mu} \frac{s}{l} \binom{\mu+l-1}{l-1} \binom{s-l-1}{s-\mu-l} t_{k-l,k-l+s} \right],$$

where $k \in \mathbb{N}_0, 0 \leq \mu \leq n$.

Sturm-Liouville difference equations are closely related to linear Hamiltonian difference systems (see, e.g., [5]). Let y be a solution of the equation

$$L(y)_k = 0, \quad k \in \mathbb{N}_0, \tag{3.10}$$

and let

$$x_k = \begin{pmatrix} y_{k-1} \\ \Delta y_{k-2} \\ \vdots \\ \Delta^{n-1} y_{k-n} \end{pmatrix}, \quad u_k = \begin{pmatrix} \sum_{\mu=1}^{n} (-1)^{\mu-1} \Delta^{\mu-1} (r_k^{[\mu]} \Delta^{\mu} y_{k-\mu}) \\ \vdots \\ -\Delta(r_k^{[n]} \Delta^n y_{k-n}) + r_k^{[n-1]} \Delta^{n-1} y_{k-n+1} \\ r_k^{[n]} \Delta^n y_{k-n} \end{pmatrix}, \tag{3.11}$$

where we extend $y = \{y_k\}_{k=0}^{\infty}$ by $y_{-1} = \cdots = y_{-n} = 0$. Then $\binom{x}{u}$ solves the linear Hamiltonian difference system

$$\Delta x_k = A x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A^T u_k, \tag{3.12}$$

with

$$B_k = \operatorname{diag}\left\{0, \dots, 0, \frac{1}{r_k^{[n]}}\right\}, \quad C_k = \operatorname{diag}\left\{r_k^{[0]}, \dots, r_k^{[n-1]}\right\}, \tag{3.13}$$

and

$$A = a_{ij} = \begin{cases} 1 & \text{if } j = i+1,\ i = 1, \dots, n-1, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.14}$$

Next we recall Reid's construction of the recessive solution of (3.12) as it is introduced in [1] for three terms matrix recurrence relations. This construction naturally extends to (3.12) (see, e.g., [12]) and the important role is played there by the following concepts introduced in [5]. A $2n \times n$ matrix solution $\binom{X}{U}$ of (3.12) is said to be a *conjoined basis* if $X^T U$ is symmetric and rank $\binom{X}{U} = n$. System (3.12) is said to be *disconjugate* in a discrete interval $[l, m]$, $l, m \in \mathbb{N}$, if the $2n \times n$ matrix solution $\binom{X}{U}$ given by the initial condition $X_l = 0$, $U_l = I$ satisfies

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \quad \text{and} \quad X_k X_{k+1}^{\dagger} (I - A)^{-1} B_k \geq 0 \tag{3.15}$$

for $k = l, \dots, m$. Here Ker, †, and $\geq$ stand for the kernel, Moore-Penrose generalized inverse, and non-negative definiteness of a matrix indicated, respectively. System (3.12) is said to be *non-oscillatory* if there exists $N \in \mathbb{N}$ such that this system is disconjugate on $[N, \infty)$ and it is said to be *oscillatory* in the opposite case. System (3.12) is said to be *eventually controllable* if there exist $N, \kappa \in \mathbb{N}$ such that for any $m \geq N$ the trivial solution $\binom{x}{u} = \binom{0}{0}$ is the only solution for which $x_m = x_{m+1} = \cdots = x_{m+\kappa} = 0$. Note that Hamiltonian system (3.12) corresponding to Sturm-Liouville equation (3.10) is controllable with the constant $\kappa = n$, see [5, Remark 9].

A conjoined basis $\binom{\tilde{X}}{\tilde{U}}$ of (3.12) is said to be the *recessive solution* if $\tilde{X}_k$ are nonsingular, $\tilde{X}_k \tilde{X}_{k+1}^{-1} (I - A)^{-1} B_k \geq 0$, both for large k, and for any other conjoined basis $\binom{X}{U}$ for which the (constant) matrix $X^T \tilde{U} - U^T \tilde{X}$ is nonsingular (such a solution is usually called *dominant*) we have

$$\lim_{k \to \infty} X_k^{-1} \tilde{X}_k = 0. \tag{3.16}$$

The recessive solution is determined uniquely up to a right multiple by a nonsingular $n \times n$ matrix. The equivalent characterization of the recessive solution $\binom{\tilde{X}}{\tilde{U}}$

of eventually controllable Hamiltonian difference systems (3.12) is

$$\lim_{k\to\infty}\left(\sum^{k}\tilde{X}_j^{-1}(I-A)^{-1}B_j\tilde{X}_j^{T-1}\right)^{-1}=0.$$

Note that the existence of a conjoined basis $\binom{X}{U}$ such that its first component X is nonsingular and the second condition in (3.15) holds for large k implies that the first component of any other conjoined basis has the same property, see [7].

The recessive solution $\binom{\tilde{X}}{\tilde{U}}$ of (3.12) can be constructed as follows. Let $l, m \in \mathbb{N}$, $l > m$, be such that (3.12) is disconjugate on $[m, \infty)$, and consider the solution $\binom{X^{[l]}}{U^{[l]}}$ of (3.12) given by the condition $X_m^{[l]} = I$, $X_l^{[l]} = 0$, where I is the identity matrix. Such a solution exists because of disconjugacy of (3.12) on $[m, l]$ and for every $k \in [m, l]$ we have

$$\lim_{l\to\infty}\begin{pmatrix}X_k^{[l]}\\ U_k^{[l]}\end{pmatrix}=\begin{pmatrix}\tilde{X}_k\\ \tilde{U}_k\end{pmatrix}.$$

If (3.12) is rewritten Sturm-Liouville equation (3.7), i.e., the entries in the first row of the matrix $\tilde{X}$ are solutions $\tilde{y}^{[1]}, \dots, \tilde{y}^{[n]}$ of (3.10), we call these solutions *the recessive system of solutions* of (3.7). Nonoscillation and disconjugacy of (3.10) is defined via nonoscillation and disconjugacy of the associated linear Hamiltonian difference system, hence recessive system of solutions of (3.10) exists whenever this equation is non-oscillatory.

Next we recall the result (see [25, Lemma 2]) which relates the quadratic form associated with the matrix T, the quadratic functional associated with (3.7), and the quadratic functional associated with (3.12). Let $y = \{y_k\} \in \ell^2$ and suppose that there exists $N \in \mathbb{N}$ such that $y_k = 0$ for $k \geq N$. If we extend y by $y_{-1} = \cdots = y_{1-n} = 0$, we have the identity

$$\langle Ty, y\rangle = \mathcal{F}(y) = \mathcal{Q}(x, u), \tag{3.17}$$

where

$$\mathcal{F}(y) := \sum_{k=0}^{\infty}\sum_{\mu=0}^{n} r_k^{[\mu]}(\Delta^\mu y_{k+1-\mu})^2,$$

and

$$\mathcal{Q}(x, u) = \sum_{k=0}^{\infty}[u_k^T B_k u_k + x_{k+1}^T C_k x_{k+1}]$$

with x, u in $\mathcal{Q}$ related to y by (3.11) and the matrices A, B, C are given by (3.13), (3.14). According to [5], the quadratic functional $\mathcal{Q}$ is positive for all (x, u) satisfying

$$\Delta x_k = A x_{k+1} + B_k u_k \tag{3.18}$$

with $x_0 = 0$, $x_k = 0$ for large k, $x \not\equiv 0$, if and only if system (3.12) is disconjugate on $[0, \infty)$. Moreover, for such (x, u) we have

$$\mathcal{Q}(x, u) = \sum_{k=0}^{\infty} x_{k+1}^T \big[-\Delta u_k + C_k x_{k+1} - A^T u_k \big]. \tag{3.19}$$

Recall that a pair (x, u) satisfying (3.18) is said to be *admissible* for $\mathcal{Q}$.

We finish this section with some results of the general theory operators defined by even order (formally) symmetric difference expressions or by symmetric banded matrices. The maximal operator associated with the infinite matrix T is defined by

$$(\mathcal{T}_{\text{max}} y)_k = (Ty)_k, \quad k \in \mathbb{N}_0$$

on the domain

$$\mathcal{D} := \mathcal{D}(\mathcal{T}_{\text{max}}) = \{ y \in \ell^2 : \, Ty \in \ell^2 \}.$$

The minimal operator $\mathcal{T}_{\text{min}}$ is the closure of the so-called preminimal operator which is the restriction of $\mathcal{T}_{\text{max}}$ to the domain

$$\mathcal{D}_0 := \big\{ y = \{y_k\} \in \mathcal{D}, \text{ only finitely many } y_k \neq 0 \big\}.$$

Denote the so-called *deficiency indices* by

$$q_{\pm} := \dim \left(\operatorname{Ker} \mathcal{T}_{\text{max}} \mp iI \right). \tag{3.20}$$

We also denote by L_{max} and L_{min} the corresponding Sturm-Liouville difference operators related to $\mathcal{T}_{\text{max}}$ and $\mathcal{T}_{\text{min}}$ by (3.9). If we suppose that $\mathcal{T}$ is bounded below (and since we suppose that the entries $t_{\mu,\nu}$ of T are real), we have $q := q_+ = q_-$. Moreover, $q \in \{0, \dots, n\}$. This is due to the fact that we extended $y = \{y_k\}_{k=0}^{\infty} \in \ell^2$ to negative integers as $y_k = 0$, so we implicitly suppose the boundary conditions $y_{-1} = 0 = \Delta y_{-2} = \cdots = \Delta^{n-1} y_{-n}$. This corresponds to the situation when a $2n$-order symmetric *differential* operator is considered on an interval (a, b) with the boundary conditions $y(a) = 0 = y'(a) = \cdots = y^{(n-1)}(a)$ at the regular left endpoint.

Let $y^{[1]}, y^{[2]} \in \ell^2$ and let $(x^{[1]}, u^{[1]})$ and $(x^{[2]}, u^{[2]})$ be the associated (via (3.11)) sequences of $2n$-dimensional vectors. We define

$$[y^{[1]}, y^{[2]}]_k := \begin{pmatrix} x_k^{[1]} \\ u_k^{[1]} \end{pmatrix}^T \mathcal{J} \begin{pmatrix} x_k^{[2]} \\ u_k^{[2]} \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Using the fact that

$$\Delta \begin{pmatrix} x_k^{[1]} \\ u_k^{[1]} \end{pmatrix}^T \mathcal{J} \begin{pmatrix} x_k^{[2]} \\ u_k^{[2]} \end{pmatrix} = (x_{k+1}^{[1]})^T \left[\Delta u_k^{[2]} - C_k x_{k+1}^{[2]} + A^T u_k^{[2]} \right] - (x_{k+1}^{[2]})^T \left[\Delta u_k^{[1]} - C_k x_{k+1}^{[1]} + A^T u_k^{[1]} \right] = y_k^{[2]} L(y^{[1]})_k - y_k^{[1]} L(y^{[2]})_k$$

we obtain Green's formula (see also [2])

$$\sum_{k=0}^{N} \left[y_k^{[2]} L(y^{[1]})_k - y_k^{[1]} L(y^{[2]})_k \right] = [y^{[1]}, y^{[2]}]_{N+1}.$$

In particular, if $y, z \in \mathcal{D}$, there exists the limit $[y, z]_\infty = \lim_{k\to\infty} [y, z]_k$, and, since $\mathcal{T}_{\max}^* = \mathcal{T}_{\min}$, the domain of $\mathcal{T}_{\min}$ is given by

$$\mathcal{D}(\mathcal{T}_{\min}) = \{y \in \mathcal{D} : \; [y, z]_\infty = 0, \forall z \in \mathcal{D}\}.$$

Similarly as for even order symmetric differential operators, self-adjoint extensions of the minimal operator $L_{\min}$ are defined by boundary conditions at ∞. More precisely, if the operator $\mathcal{T}_{\min}$ is not self-adjoint, i.e., $q \geq 1$, let $y^{[1]}, \dots, y^{[q]} \in \mathcal{D}$ be such that

$$[y^{[i]}, y^{[j]}]_\infty = 0, \quad i, j = 1, \dots, q,$$

and that $y^{[1]}, \dots, y^{[q]}$ are linearly independent modulo $\mathcal{D}(\mathcal{T}_{\min})$ (i.e., no nontrivial combination of $y^{[1]}, \dots, y^{[q]}$ belongs to $\mathcal{D}(\mathcal{T}_{\min})$). Then a self-adjoint extension of $L_{\min}$ (and hence also of $\mathcal{T}_{\min}$) is defined as the restriction of $\mathcal{T}_{\max}$ to the domain

$$\tilde{\mathcal{D}} := \{y \in \mathcal{D} : \; [y, y^{[j]}]_\infty = 0, \; j = 1, \dots, q\}.$$

3.4 Friedrichs extension of symmetric matrix operators

Throughout this section we suppose that there exists $\varepsilon > 0$ such that the minimal operator associated with the matrix T given in (3.1) satisfies

$$\langle Ty, y\rangle \geq \varepsilon \langle y, y\rangle, \quad \text{for } 0 \not\equiv y \in \mathcal{D}(\mathcal{T}_{\min}). \tag{3.21}$$

This means, in view of the previous section, that the associated Sturm-Liouville operator (3.7) and linear Hamiltonian difference system (3.12) are *disconjugate* on $[0, \infty)$. We also suppose that

$$t_{k+n} \neq 0 \quad \text{for } k \in \mathbb{N}.$$

This assumption (which is the typical assumption for tridiagonal matrices which are the special case $n = 1$) means that T given by (3.1) is a "real" $2n+1$ diagonal matrix, i.e., the bandwidth is really $2n + 1$ in each row of T. This assumption is equivalent to the assumption $r_k^{[n]} \neq 0$ in (3.7).

Note that assumption (3.21) essentially means no loss of generality. Friedrichs extension can be constructed for operators bounded below only, i.e., for T satisfying (instead of (3.21)) the assumption $\langle Ty, y\rangle \geq \gamma\langle y, y\rangle$ for some $\gamma \in \mathbb{R}$. However, under this assumption we can apply our construction to the operator defined by $T - (\gamma - \varepsilon)I$, I being the infinite identity matrix, $\varepsilon > 0$, and the results remain unchanged.

The next statement is the main result of this chapter. It reduces to [8, Theorerm 4] for tridiagonal matrices T in (3.1) and it is a discrete counterpart of the main result of [26].

Theorem 3.1. *Let $y^{[1]}, \dots, y^{[n]}$ be the recessive system of solutions of the equation $L(y) = 0$, where L is associated with $\mathcal{T}$ by* (3.9)*. Then the domain of the Friedrichs extension $\mathcal{T}_F$ of $\mathcal{T}_{\min}$ is*

$$\mathcal{D}(\mathcal{T}_F) = \{y \in \mathcal{D}(\mathcal{T}_{\max}) : [y, y^{[j]}]_\infty = 0,\ j = 1, \dots, n\}. \tag{3.22}$$

Proof. The main part of the proof consists in proving that the sequences $y^{[j]}$, $j = 1, \dots, n$, in the recessive system of solutions are in $\mathcal{D}(\mathcal{T}_F)$. Let $\binom{\tilde X}{\tilde U}$ be the recessive solution of Hamiltonian system (3.12) whose columns are formed by $2n$-dimensional vectors $\binom{\tilde x^{[j]}}{\tilde u^{[j]}}$, $j = 1, \dots, n$, related to $y^{[1]}, \dots, y^{[n]}$ by (3.11). Without loss of generality we may suppose that the matrix $\tilde X$ formed by the vectors $\tilde x^{[1]}, \dots, \tilde x^{[n]}$ is nonsingular because of (3.21). Indeed, (3.21) implies disconjugacy in $[0, \infty)$ of Hamiltonian system (3.12) associated with the equation $L(y) = 0$, which means that $\tilde X_0$ is nonsingular by [7]. Further, let

$$\hat X_k = \tilde X_k \sum_{j=0}^{k-1} \tilde B_j, \quad \hat U_k = \tilde U_k \sum_{j=0}^{k-1} \tilde B_j + \tilde X_k^{T-1},$$

where

$$\tilde B_j := \tilde X_{j+1}^{-1}(I - A)^{-1} B_j \tilde X_j^{T-1},$$

is the so-called *associated dominant solution* of (3.12). The fact that $\binom{\hat X}{\hat U}$ is really a solution of (3.12) is proved, e.g., in [2, p.107]. Further, for a fixed $m \in \mathbb{N}$, we denote

$$X_k^{[m]} = \tilde X_k - \hat X_k \hat X_m^{-1} \tilde X_m, \quad U_k^{[m]} = \tilde U_k - \hat U_k \hat X_m^{-1} \tilde X_m.$$

Then according to (3.16)

$$\lim_{m\to\infty} \begin{pmatrix} X_k^{[m]} \\ U_k^{[m]} \end{pmatrix} = \begin{pmatrix} \tilde X_k \\ \tilde U_k \end{pmatrix} \tag{3.23}$$

for every $k \in \mathbb{N}$. Also, $\binom{\tilde{X}}{\tilde{U}}$, $\binom{\hat{X}}{\hat{U}}$ are conjoined bases of (3.12) for which $\tilde{X}_k^T \hat{U}_k - \tilde{U}_k^T \hat{X}_k \equiv I$ holds, which means that the matrix $\begin{pmatrix} \tilde{X} & \hat{X} \\ \tilde{U} & \hat{U} \end{pmatrix}$ is symplectic, i.e.,

$$\begin{pmatrix} \tilde{X} & \hat{X} \\ \tilde{U} & \hat{U} \end{pmatrix}^T \mathcal{J} \begin{pmatrix} \tilde{X} & \hat{X} \\ \tilde{U} & \hat{U} \end{pmatrix} = \mathcal{J}.$$

This implies the identity

$$\begin{pmatrix} \tilde{X} & \hat{X} \\ \tilde{U} & \hat{U} \end{pmatrix} \mathcal{J} \begin{pmatrix} \tilde{X} & \hat{X} \\ \tilde{U} & \hat{U} \end{pmatrix}^T = \mathcal{J}$$

which, in terms of the matrices $\tilde{X}, \tilde{U}, \hat{X}, \hat{U}$, reads as

$$\hat{U}_k \tilde{X}_k^T - \tilde{U}_k \hat{X}_k^T = I, \quad \tilde{X}_k \hat{X}_k^T = \hat{X}_k \tilde{X}_k^T, \quad \tilde{U}_k \hat{U}_k^T = \hat{U}_k \tilde{U}_k^T. \tag{3.24}$$

Denote, for $j \in \{1, \dots, m\}$,

$$x_k^{[j,m]} = \begin{cases} X_k^{[m]} e_j, & k \le m, \\ 0, & k > m, \end{cases} \qquad u_k^{[j,m]} = \begin{cases} U_k^{[m]} e_j, & k < m, \\ 0, & k \ge m, \end{cases}$$

where $e_j = (0, \dots, 0, 1, 0 \dots, 0)^T$, number 1 being the j-th entry, is the standard canonical basis in $\mathbb{R}^n$. Then by (3.23) the first entry of $x_k^{[j,n]}$, denote it by $y_k^{[j,m]} := e_1^T x_k^{[j,m]}$, satisfies (for every $k \in \mathbb{N}$)

$$\lim_{m \to \infty} y_k^{[j,m]} = y_k^{[j]},$$

where $y^{[j]} = \{y_k^{[j]}\}$ have been defined at the beginning of this proof. Then $\Delta x_k^{[j,m]} = A x_{k+1}^{[j,m]} + B_k u_k^{[j,m]}$, i.e., $\binom{x^{[j,m]}}{u^{[j,m]}}$ is admissible for $\mathcal{Q}$ and for $l > m$ we obtain from (3.19) (with the matrices B, C given by (3.13))

$$\begin{aligned} &\mathcal{Q}(x^{[j,m]} - x^{[j,l]}, u^{[j,m]} - u^{[j,l]}) \\ &\qquad = \sum_{k=0}^{\infty} \Big[(x_{k+1}^{[j,m]} - x_{k+1}^{[j,l]})^T C_k (x_{k+1}^{[j,m]} - x_{k+1}^{[j,l]}) \\ &\qquad\qquad + (u_k^{[j,m]} - u_k^{[j,l]})^T B_k (u_k^{[j,m]} - u_k^{[j,l]}) \Big] \end{aligned}$$

$$
\begin{aligned}
= \sum_{k=0}^{m-1} (x_{k+1}^{[j,m]})^T \left[-\Delta u_k^{[j,m]} + C_k x_{k+1}^{[j,m]} - A^T u_k^{[j,m]}\right] \\
- \sum_{k=0}^{m-1} (x_{k+1}^{[j,m]})^T \left[-\Delta u_k^{[j,l]} + C_k x_{k+1}^{[j,l]} - A^T u_k^{[j,l]}\right] \\
+ \sum_{k=0}^{l-1} (x_{k+1}^{[j,l]})^T \left[-\Delta u_k^{[j,l]} + C_k x_{k+1}^{[j,l]} - A^T u_k^{[l]}\right] \\
- \sum_{k=0}^{m-1} (x_{k+1}^{[j,l]})^T \left[-\Delta u_k^{[j,m]} + C_k x_{k+1}^{[j,m]} - A^T u_k^{[j,m]}\right].
\end{aligned}
$$

According to the definition of $\binom{x_k^{[\cdot]}}{u_k^{[\cdot]}}$, only the last summand in the previous expression is nonzero, denote it by $(*)$. For this expression we have (taking into account that $-\Delta u_k^{[j,m]} + C_k x_{k+1}^{[j,m]} - A^T u_k^{[j,m]} = 0$ for $k = 0, \ldots, m-2$)

$$
\begin{aligned}
(*) &= \sum_{k=0}^{m-1} (x_{k+1}^{[j,l]})^T \left[-\Delta u_k^{[j,m]} + C_k x_{k+1}^{[j,m]} - A^T u_k^{[j,m]}\right] \\
&= e_j^T (X_m^{[l]})^T (I - A^T) U_{m-1}^{[m]} e_j.
\end{aligned}
$$

Hamiltonian system (3.12) can be written in the recurrence form

$$x_{k+1} = \tilde{A} x_k + \tilde{A} B_k u_k, \quad u_{k+1} = C_k \tilde{A} x_k + (C_k \tilde{A} B_k + I - A^T) u_k,$$

where $\tilde{A} = (I - A)^{-1}$, whose matrix is symplectic as can be verified by a direct computation, see also [5] or [6], and hence

$$
\begin{pmatrix} \tilde{A} & \tilde{A} B_k \\ C_k \tilde{A} & C_k \tilde{A} B_k + I - A^T \end{pmatrix}^{-1} = \begin{pmatrix} B_k \tilde{A}^T C_k + I - A & -B_k \tilde{A}^T \\ -\tilde{A}^T C_k & \tilde{A}^T \end{pmatrix}.
$$

This means that (3.12) can be written as the the so-called reversed symplectic system (in the matrix form)

$$X_k = (B_k \tilde{A}^T C_k + I - A) X_{k+1} - \tilde{B}_k A^T U_{k+1}, \quad U_k = -\tilde{A}^T C_k X_{k+1} + \tilde{A}^T U_{k+1}.$$

Using the second equation in this reversed system with $k = m-1$, $\binom{X}{U} = \binom{X^{[m]}}{U^{[m]}}$, and taking into account that $X_m^{[m]} = 0$, we have

$$
\begin{aligned}
(X_m^{[l]})^T (I - A^T) U_{m-1}^{[m]} &= (X_m^{[l]})^T U_m^{[m]} \\
&= \left(\tilde{X}_m - \hat{X}_m \hat{X}_l^{-1} \tilde{X}_l\right)^T \left(\tilde{U}_m - \hat{U}_m \hat{X}_m^{-1} \tilde{X}_m\right) \\
&= \left(\hat{X}_m^{-1} \tilde{X}_m - \hat{X}_l^{-1} \tilde{X}_l\right)^T \hat{X}_m^T \left(\tilde{U}_m \hat{X}_m^T - \hat{U}_m \tilde{X}_m^T\right) \hat{X}_m^{T-1} \\
&= \hat{X}_l^{-1} \tilde{X}_l - \hat{X}_m^{-1} \tilde{X}_m,
\end{aligned}
$$

where we have used the identities (3.24).

Consequently, by (3.16),

$$\mathcal{Q}(x^{[j,m]} - x^{[j,l]}, u^{[j,m]} - u^{[j,l]}) = e_j^T \big[\hat{X}_l^{-1}\tilde{X}_l - \hat{X}_m^{-1}\tilde{X}_m\big] e_j \to 0$$

as $m, l \to \infty$, i.e., by (3.17),

$$\langle T(y^{[j,m]} - y^{[j,l]}), y^{[j,m]} - y^{[j,l]}\rangle \to 0.$$

By the same computation we find that (for $i = 1, \dots, n$)

$$Q(\tilde{x}^{[j]} - x^{[j,m]}, \tilde{u}^{[j]} - u^{[i,m]}) = \langle T(y^{[i]} - y^{[i,m]}), y^{[i]} - y^{[i,m]}\rangle \to 0$$

as $m \to \infty$, which means, by (3.21), that $y^{[j,m]} \to y^{[j]}$ in ℓ^2 as $m \to \infty$. Consequently, in view of (3.3), $y^{[j]} \in \mathcal{D}(\mathcal{T}_F)$, $j = 1, \dots, n$.

Now, we prove that (3.22) really characterizes the domain of the Friedrichs extension of $\mathcal{T}_{\min}$. Here we essentially follow the idea introduced in [26]. We have

$$[y^{[i]}, y^{[j]}] = e_i^T \begin{pmatrix} \tilde{X} \\ \tilde{U} \end{pmatrix}^T \mathcal{J} \begin{pmatrix} \tilde{X} \\ \tilde{U} \end{pmatrix} e_j = e_i^T (\tilde{X}^T \tilde{U} - \tilde{U}^T \tilde{X}) e_j = 0,$$

since the recessive system of solutions of $L(y) = 0$ determines (via (3.11)) a conjoined bases of (3.12). Hence, the domain given by the right-hand side of (3.22), we denote it by $\tilde{\mathcal{D}}$, is the domain of a self-adjoint realization of $\mathcal{T}_{\min}$. Note that boundary conditions in (3.22) need not to be linearly independent relative $\mathcal{D}(L_{\min})$, see Remark 1 (ii) below. Now, let $y \in \mathcal{D}(\mathcal{T}_F)$, then also (by self-adjointness) $y \in \mathcal{D}(\mathcal{T}_F^*)$. Since $y^{[j]} \in \mathcal{D}(\mathcal{T}_F)$, we have $[y, y^{[j]}] = 0$, $j = 1, \dots, n$, i.e., $\mathcal{D}(\mathcal{T}_F) \subseteq \tilde{\mathcal{D}}$. This, together with the fact that $\tilde{\mathcal{D}}$ is a domain of a self-adjoint extension of $\mathcal{T}_{\min}$ shows that $\mathcal{D}(\mathcal{T}_F) = \tilde{\mathcal{D}}$. □

Concluding remarks.

(i) In the previous theorem we have proved that sequences which are in the domain of the Friedrichs extension of the operator $\mathcal{T}$ behave near ∞ like sequences from the recessive system of solutions of the associated Sturm-Liouville operator (3.7). Consider now again this Sturm-Liouville operator and let $\binom{X}{U}$ be a *dominant* solution of (3.12) associated with (3.7), i.e., (3.16) holds. Theorem 3.1, coupled with (3.16), suggests the conjecture that the domain $\mathcal{D}(\mathcal{T}_F)$ can be also described as follows

$$\mathcal{D}(\mathcal{T}_F) = \big\{y = \{y_k\}_{k=0}^{\infty} \in \ell^2,\ \lim_{k\to\infty} X_k^{-1} x_k = 0,\ x_k \text{ given by (3.11)}\big\}.$$

This conjecture is a subject of the present investigation.

(ii) Observe that similarly to higher order symmetric *differential* expressions, n boundary conditions hidden in (3.22) need not to be linearly independent, see

[26]. The number of linearly independent conditions among them depends (again similarly to differential expressions) on the number $q = q_{\pm}$ defined in (3.20). In particular, if $q = 0$, i.e., the operator $\mathcal{T}_{\min}$ is self-adjoint and $\mathcal{T}_F = \mathcal{T}_{\min}$, then boundary conditions (3.22) are implied by the assumption $\mathcal{T} y \in \ell^2$ involved in the definition of $\mathcal{D}$. A typical example of this situation is the operator $L(y)_k = \Delta^2 y_{k-1}$ where the recessive solution is $\tilde{y}_k = 1$, i.e., $0 = [y, \tilde{y}]_\infty = \lim_{k\to\infty} \Delta y_k$ and this condition is implied by $y \in \ell^2$, $\Delta^2 y \in \ell^2$, which define $\mathcal{D}$ in this case.

Chapter 4

Critical higher order Sturm-Liouville difference operators

4.1 Introduction

In this chapter, we deal with the $2n$-order Sturm-Liouville difference operators and equations

$$L(y)_k := \sum_{\nu=0}^{n} (-\Delta)^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k-\nu} \right) = 0, \quad k \in \mathbb{Z}, \tag{4.1}$$

and the associated matrix operators

$$(\mathcal{T}y)_k = \sum_{j=k-n}^{k+n} t_{k,j} y_j, \quad k \in \mathbb{Z}. \tag{4.2}$$

defined by infinite symmetric banded matrices

$$T = (t_{\mu,\nu}), \quad t_{\mu,\nu} = t_{\nu,\mu},\ \mu, \nu \in \mathbb{Z}, \quad t_{\mu,\nu} = 0 \quad \text{for } |\mu - \nu| > n. \tag{4.3}$$

This chapter is motivated by some results presented in [19], where perturbations of second order Sturm-Liouville operators and of associated tridiagonal matrices are investigated. The concept of a *critical* operator is introduced in [19] and it is shown there that a "small negative perturbation" of a non-negative critical operator leads to the so-called supercritical operator. We recall these concepts in more details in the next section.

The results of this chapter are based again on the relationship of (4.1) to linear Hamiltonian difference systems

$$\Delta x_k = A x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A^T u_k. \tag{4.4}$$

We use the concept of the recessive solution of (4.4) to introduce the so-called p-critical Sturm-Liouville operator and we investigate perturbations of such operators.

The chapter is organized as follows. In the next section we recall necessary preliminaries, including the relationship between banded symmetric matrices, Sturm-Liouville difference operators, and linear Hamiltonian difference systems. Section 3 is devoted to the study of perturbations of critical Sturm-Liouville operators of the form (4.1). In the last section we study perturbations of one-term even order difference operators.

4.2 Preliminaries

We start with the relationship between Sturm-Liouville operators (4.1) and matrix operators (4.2) defined by banded symmetric matrices (4.3) as established, e.g., in [25]. Expanding the forward differences in (4.1), we get the recurrence relation (4.2) with $t_{i,j}$ given by the formulas

$$\begin{aligned} t_{k,k+j} &= (-1)^j \sum_{\mu=j}^{n} \sum_{\nu=j}^{\mu} \binom{\mu}{\nu} \binom{\mu}{\nu-j} r_{k+\nu}^{[\mu]}, \\ t_{k,k-j} &= (-1)^j \sum_{\mu=j}^{n} \sum_{\nu=0}^{\mu-j} \binom{\mu}{\nu} \binom{\mu}{\nu+j} r_{k+\nu}^{[\mu]}, \end{aligned} \tag{4.5}$$

for $k \in \mathbb{Z}$ and $j \in \{0, \dots, n\}$. These formulas are essentially the same as (3.8). We repeat them here for the reader's convenience, similarly as some other formulas in this section. Consequently, one can associate the difference operator L given by (4.1) with the matrix operator $\mathcal{T}$, defined via an infinite matrix T, by the formula

$$(\mathcal{T}y)_k = L(y)_k, \quad k \in \mathbb{Z}.$$

The substitution

$$x_k^{[y]} = \begin{pmatrix} y_{k-1} \\ \Delta y_{k-2} \\ \vdots \\ \Delta^{n-1} y_{k-n} \end{pmatrix}, \quad u_k^{[y]} = \begin{pmatrix} \sum_{\nu=1}^{n} (-\Delta)^{\nu-1} \left(r_k^{[\nu]} \Delta^\nu y_{k-\nu} \right) \\ \vdots \\ -\Delta(r_k^{[n]} \Delta^n y_{k-n}) + r_k^{[n-1]} \Delta^{n-1} y_{k-n+1} \\ r_k^{[n]} \Delta^n y_{k-n} \end{pmatrix}, \tag{4.6}$$

converts (4.1) to linear Hamiltonian system (4.4) with the matrices A, B, C given by the formulas

$$A = a_{ij} = \begin{cases} 1 & \text{if } j = i+1,\ i = 1, \dots, n-1, \\ 0 & \text{elsewhere}, \end{cases} \tag{4.7}$$

$$B_k = \operatorname{diag}\left\{0,\dots,0,\frac{1}{r_k^{[n]}}\right\}, \quad C_k = \operatorname{diag}\left\{r_k^{[0]},\dots,r_k^{[n-1]}\right\}. \tag{4.8}$$

A $2n \times n$ matrix solution $\binom{X}{U}$ of (4.4) is said to be a *conjoined basis* if $X^T U$ is symmetric and rank $\binom{X}{U} = n$. Following [5], system (4.4) is said to be *right disconjugate* in a discrete interval $[l,m]$, $l,m \in \mathbb{Z}$, if the $2n \times n$ matrix solution $\binom{X}{U}$ given by the initial condition $X_l = 0$, $U_l = I$ satisfies

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \quad \text{and} \quad X_k X_{k+1}^{\dagger}(I-A)^{-1}B_k \geq 0 \tag{4.9}$$

for $k = l,\dots,m-1$. Here Ker, †, and $\geq$ stand for the kernel, Moore-Penrose generalized inverse, and non-negative definiteness of a matrix indicated, respectively. Similarly, (4.4) is said to be *left disconjugate* on $[l,m]$ if the solution given by the initial condition $X_m = 0$, $U_m = -I$ satisfies

$$\operatorname{Ker} X_k \subseteq \operatorname{Ker} X_{k+1} \quad \text{and} \quad X_{k+1} X_k^{\dagger} B_k (I-A)^{T-1} \geq 0, \quad k \in [l,m), \tag{4.10}$$

see [6]. System (4.4) is disconjugate on $\mathbb{Z}$ if it is right disconjugate (which is the same as left disconjugate, see, e.g., [6, Theorem 1]) on $[l,m]$ for every $l,m \in \mathbb{Z}$, $l < m$. System (4.4) is said to be *non-oscillatory at* ∞ (*non-oscillatory at* $-\infty$) if there exists $l \in \mathbb{Z}$ ($m \in \mathbb{Z}$) such that it is right disconjugate on $[l,m]$ for every $m > l$ (left disconjugate on $[l,m]$ for every $l < m$). Nonoscillation and disconjugacy of (4.1) is defined via nonoscillation and disconjugacy of the associated linear Hamiltonian difference system. An equivalent approach to disconjugacy of (4.1), based on the nonexistence of a pair of generalized zero points of multiplicity n of a solution of (4.1) is presented in [5].

If $\binom{X}{U}$ is a conjoined basis of (4.4) such that X_k are nonsingular for $k \in \{l, l+1,\dots,m\}$, then

$$\begin{aligned}\tilde{X}_k &= X_k \sum_{j=l}^{k-1} X_{j+1}^{-1}(I-A)^{-1}B_j X_j^{T-1},\\ \tilde{U}_k &= U_k \sum_{j=l}^{k-1} X_{j+1}^{-1}(I-A)^{-1}B_j X_j^{T-1} + X_k^{T-1},\end{aligned}$$

$k \in \{l, l+1,\dots,m\}$, is also a conjoined basis of (4.4), which together with $\binom{X}{U}$ generates the solutions space of (4.4), see [10].

Nonnegativity of the difference operator L given by (4.1) and of the associated matrix operator (4.2) is defined as follows. Denote by

$$\ell^2(\mathbb{Z}) := \left\{ y = \{y_k\}_{k\in\mathbb{Z}}, \sum_{k\in\mathbb{Z}} |y_k|^2 < \infty \right\},$$
$$\ell_0^2(\mathbb{Z}) := \big\{ y = \{y_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}), \text{ only finitely many } y_k \neq 0 \big\}$$

and by $\langle \cdot, \cdot \rangle$ the usual scalar product in $\ell^2(\mathbb{Z})$. Then by summation by parts we have

$$\langle L(y), y\rangle = \mathcal{F}(y) := \sum_{k=-\infty}^{\infty} \sum_{\nu=0}^{n} r_k^{[\nu]} \left(\Delta^\nu y_{k-\nu}\right)^2, \quad \forall y \in \ell_0^2(\mathbb{Z})$$

and according to [25, Lemma 2]

$$\langle \mathcal{T} y, y\rangle = \mathcal{F}(y) = \sum_{k\in\mathbb{Z}} \left[(x_{k+1}^{[y]})^T C_k x_{k+1}^{[y]} + (u_k^{[y]})^T B_k u_k^{[y]} \right], \quad \forall y \in \ell_0^2(\mathbb{Z}),$$

where the matrices B and C are given by (4.8) and $x^{[y]}, u^{[y]}$ are related to y by (4.6). Now we say that the operator L is *non-negative* and we write $L \geq 0$ if $\mathcal{F}(y) \geq 0$ for every $y \in \ell_0^2(\mathbb{Z})$.

A conjoined basis $\binom{\tilde{X}}{\tilde{U}}$ of (4.4) is said to be the *recessive solution* at ∞ if $\tilde{X}_k$ are nonsingular, $\tilde{X}_k \tilde{X}_{k+1}^{-1} (I-A)^{-1} B_k \geq 0$, both for large k, and for any other conjoined basis $\binom{X}{U}$, for which the (constant) matrix $X^T \tilde{U} - U^T \tilde{X}$ is nonsingular (such a solution is usually called *dominant* at ∞), we have

$$\lim_{k\to\infty} X_k^{-1} \tilde{X}_k = 0. \tag{4.11}$$

The recessive solution at ∞ is determined uniquely up to a right multiple by a nonsingular $n \times n$ matrix and exists whenever (4.4) is non-oscillatory at ∞. The equivalent characterization of the recessive solution $\binom{\tilde{X}}{\tilde{U}}$ at ∞ of Hamiltonian difference systems (4.4) is

$$\lim_{k\to\infty} \left(\sum_{j=M}^{k} \tilde{X}_{j+1}^{-1} (I-A)^{-1} B_j \tilde{X}_j^{T-1} \right)^{-1} = 0 \quad \text{for some large } M, \tag{4.12}$$

see [2]. The recessive solution at $-\infty$ is defined analogously, $\lim_{k\to\infty}$ in (4.11) and (4.12) is replaced by $\lim_{k\to-\infty}$ and the summation limit (4.12) is $\sum_k^{j=M}$. According to [5], $\mathcal{F}(y) > 0$ for every $0 \not\equiv y \in \ell_0^2(\mathbb{Z})$ if and only if (4.1) is disconjugate on $\mathbb{Z}$ and disconjugacy of (4.1) on $\mathbb{Z}$ is equivalent (by [7]) to the fact that the recessive solution at ∞ (at $-\infty$) of the associated Hamiltonian system (4.4) has no left (right) focal point in $\mathbb{Z}$, i.e., (4.9) (resp. (4.10)) holds for $k \in \mathbb{Z}$.

If we denote by $\tilde{x}^{[j]}$, $\tilde{u}^{[j]}$, $j = 1, \dots, n$, the column vectors, which form the recessive solution $\binom{\tilde{X}}{\tilde{U}}$ of (4.4) with A, B, C given by (4.7) and (4.8), and by $\tilde{y}^{[j]}$ the first entry of $\tilde{x}^{[j]}$, then we call $\tilde{y}^{[j]}$ the *recessive system of solutions of* (4.1) *at* ∞. The recessive system of solutions of (4.1) at $-\infty$ is defined analogously.

We finish this section by recalling some results of [19], which deal with the second order Sturm-Liouville difference equation

$$\tau(y) := -\Delta(a_k \Delta y_{k-1}) + c_k y_k = 0, \quad a_k > 0, \tag{4.13}$$

and the corresponding three-term symmetric recurrence relation

$$\tau(y) = -a_{k+1} y_{k+1} + b_k y_k - a_k y_{k-1} = 0, \quad b_k = a_{k+1} + a_k + c_k.$$

Suppose that the equation (4.13) is disconjugate on $\mathbb{Z}$, i.e., $\tau \geq 0$. Operator τ is said to be *critical* if the recessive solutions at ∞ and $-\infty$ are linearly dependent, in the opposite case τ is said to be *subcritical*. If $\tau \not\geq 0$, i.e., (4.13) is not disconjugate, τ is said to be *supercritical*.

Proposition 4.1 ([19])**.** *The following statements are equivalent.*

(i) The operator τ is critical on $\mathbb{Z}$.

(ii) For any $\varepsilon > 0$ and $m \in \mathbb{Z}$, the operator $\tilde{\tau}$, which we get from τ by replacing a_m by $a_m + \varepsilon$, is supercritical on $\mathbb{Z}$, i.e., $\tilde{\tau} \not\geq 0$.

(iii) For any $\varepsilon > 0$ and $m \in \mathbb{Z}$, the operator $\hat{\tau}$, which we get from τ by replacing b_m by $b_m - \varepsilon$, is supercritical on $\mathbb{Z}$, i.e., $\hat{\tau} \not\geq 0$.

4.3 Critical and supercritical operators

Suppose that (4.1) is disconjugate on $\mathbb{Z}$ and let $\hat{y}^{[i]}$ and $\tilde{y}^{[i]}$, $i = 1, \dots, n$, be the recessive systems of solutions of $L(y) = 0$ at $-\infty$ and ∞, respectively. Introduce the linear space

$$\mathcal{H} = \operatorname{Lin}\{\hat{y}^{[1]}, \dots, \hat{y}^{[n]}\} \cap \operatorname{Lin}\{\tilde{y}^{[1]}, \dots, \tilde{y}^{[n]}\}. \tag{4.14}$$

In this section, $|J|$ stands for a number of elements of a set J.

Definition 4.1. Let (4.1) be disconjugate on $\mathbb{Z}$ and let $\dim \mathcal{H} = p \in \{1, \dots, n\}$. Then we say that the operator L (or equation (4.1)) is *p-critical* on $\mathbb{Z}$. If $\dim \mathcal{H} = 0$, we say that L is *subcritical* on $\mathbb{Z}$. If (4.1) is not disconjugate on $\mathbb{Z}$, i.e., $L \not\geq 0$, we say that L is *supercritical* on $\mathbb{Z}$.

The following statement and the construction in its proof will be useful in the proof of the main result of this section, Theorem 4.1.

Lemma 4.1. *Suppose that* (4.1) *is p-critical for some $p \in \{1,\dots,n\}$. Then for every $\delta > 0$ there exists a sequence $y_k \in \ell_0^2(\mathbb{Z})$ such that*

$$\mathcal{F}(y) := \sum_{k=-\infty}^{\infty} \sum_{\nu=0}^{n} r_k^{[\nu]} \left(\Delta^\nu y_{k-\nu}\right)^2 < \frac{\delta}{2}.$$

Proof. Denote by (X^-,U^-) and (X^+,U^+) the recessive solutions of (4.4) (with A, B, C given by (4.7), (4.8)) at $-\infty$ and ∞, respectively. Further, let h be a solution of (4.1), which is contained both in recessive systems of solutions of this equation at ∞ and $-\infty$, and let $(x^{[h]},u^{[h]})$ be the associated solution of (4.4). Let K,L,M,N be arbitrary integers such that $N-M>n, M-L>n$, and $L-K>n$ and let $(x^{[f]},u^{[f]})$ and $(x^{[g]},u^{[g]})$ be the solutions of (4.4) (see [10]) given by the formulas

$$\begin{aligned}
x_k^{[f]} &= X_k^- \left(\sum_{j=K}^{k-1} \mathcal{B}_j^-\right) \left(\sum_{j=K}^{L-1} \mathcal{B}_j^-\right)^{-1} \left(X_L^-\right)^{-1} x_L^{[h]},\\
u_k^{[f]} &= U_k^- \left(\sum_{j=K}^{k-1} \mathcal{B}_j^-\right) \left(\sum_{j=K}^{L-1} \mathcal{B}_j^-\right)^{-1} \left(X_L^-\right)^{-1} x_L^{[h]}\\
&\qquad + \left(X_k^-\right)^{T-1} \left(\sum_{j=K}^{L-1} \mathcal{B}_j^-\right)^{-1} \left(X_L^-\right)^{-1} x_L^{[h]},\\
x_k^{[g]} &= X_k^+ \left(\sum_{j=k}^{N-1} \mathcal{B}_j^+\right) \left(\sum_{j=M}^{N-1} \mathcal{B}_j^+\right)^{-1} \left(X_M^+\right)^{-1} x_M^{[h]},\\
u_k^{[g]} &= U_k^+ \left(\sum_{j=k}^{N-1} \mathcal{B}_j^+\right) \left(\sum_{j=M}^{N-1} \mathcal{B}_j^+\right)^{-1} \left(X_M^+\right)^{-1} x_M^{[h]}\\
&\qquad - \left(X_k^+\right)^{T-1} \left(\sum_{j=M}^{N-1} \mathcal{B}_j^+\right)^{-1} \left(X_M^+\right)^{-1} x_M^{[h]},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{B}_k^- &= (X_{k+1}^-)^{-1}(I-A)^{-1}B_k(X_k^-)^{T-1},\\
\mathcal{B}_k^+ &= (X_{k+1}^+)^{-1}(I-A)^{-1}B_k(X_k^+)^{T-1},
\end{aligned}$$

and denote by f and g the solutions of (4.1), which generate $(x^{[f]}, u^{[f]})$ and $(x^{[g]}, u^{[g]})$, respectively, i.e., f and g are the first entries of $x^{[f]}$ and $x^{[g]}$, respectively. Using the convention that $\sum_K^{K-1} = 0 = \sum_N^{N-1}$ we have

$$x_K^{[f]} = 0, \quad x_L^{[f]} = x_L^{[h]}, \quad x_M^{[h]} = x_M^{[g]}, \quad x_N^{[g]} = 0.$$

Let us introduce the sequence

$$y_k := \begin{cases} 0, & k \in (-\infty, K-1], \\ f_k, & k \in [K, L-1], \\ h_k, & k \in [L, M-1], \\ g_k, & k \in [M, N-1], \\ 0, & k \in [N, \infty). \end{cases} \tag{4.15}$$

Now, we compute

$$\begin{aligned} \mathcal{F}(y) &= \sum_{k=-\infty}^{\infty} \sum_{\nu=0}^{n} r_k^{[\nu]} \left(\Delta^\nu y_{k-\nu}\right)^2 \\ &= \sum_{k=K}^{L-1} \left[u_k^{[f]T} B_k u_k^{[f]} + x_{k+1}^{[f]T} C_k x_{k+1}^{[f]}\right] + \sum_{k=L}^{M-1} \left[u_k^{[h]T} B_k u_k^{[h]} + x_{k+1}^{[h]T} C_k x_{k+1}^{[h]}\right] \\ &\qquad + \sum_{k=M}^{N-1} \left[u_k^{[g]T} B_k u_k^{[g]} + x_{k+1}^{[g]T} C_k x_{k+1}^{[g]}\right]. \end{aligned}$$

Using summation by parts we obtain (see [11, Lemma 3.1])

$$\begin{aligned} \mathcal{F}(y) &= x_k^{[f]T} u_k^{[f]} \Big|_K^L + x_k^{[h]T} u_k^{[h]} \Big|_L^M + x_k^{[g]T} u_k^{[g]} \Big|_M^N \\ &= x_L^{[h]T} \left(u_L^{[f]} - u_L^{[h]}\right) + x_M^{[h]T} \left(u_M^{[h]} - u_M^{[g]}\right). \end{aligned}$$

Because

$$u_L^{[f]} = U_L^- \left(X_L^-\right)^{-1} x_L^{[h]} + \left(X_L^-\right)^{T-1} \left(\sum_{j=K}^{L-1} \mathcal{B}_j^-\right)^{-1} \left(X_L^-\right)^{-1} x_L^{[h]},$$

and

$$u_M^{[g]} = U_M^+ \left(X_M^+\right)^{-1} x_M^{[h]} - \left(X_M^+\right)^{T-1} \left(\sum_{j=M}^{N-1} \mathcal{B}_j^+\right)^{-1} \left(X_M^+\right)^{-1} x_M^{[h]},$$

we have

$$\mathcal{F}(y) = x_L^{[h]T} \left(X_L^-\right)^{T-1} \left(\sum_{j=K}^{L-1} \mathcal{B}_j^- \right)^{-1} \left(X_L^-\right)^{-1} x_L^{[h]} \\ + x_M^{[h]T} \left(X_M^+\right)^{T-1} \left(\sum_{j=M}^{N-1} \mathcal{B}_j^+ \right)^{-1} \left(X_M^+\right)^{-1} x_M^{[h]}.$$

The definition of $\mathcal{B}^-$, $\mathcal{B}^+$ implies that (compare (4.12))

$$\left(\sum_{j=K}^{L-1} \mathcal{B}_j^- \right)^{-1} \to 0, \quad \text{for } K \to -\infty,$$
$$\left(\sum_{j=M}^{N-1} \mathcal{B}_j^+ \right)^{-1} \to 0, \quad \text{for } N \to \infty.$$

Hence $\mathcal{F}(y) < \frac{\delta}{2}$ for K sufficiently small and N sufficiently large. □

The main result of this section read as follows.

Theorem 4.1. *Let the operator L be p-critical on $\mathbb{Z}$, and let $m \in \mathbb{Z}$ and $\varepsilon > 0$ be arbitrary. Further, let $J \subseteq \{0, \dots, n-1\}$ with $|J| = n-p+1$ and let us consider the sequences*

$$\hat{r}_m^{[\mu]} = \begin{cases} r_m^{[\mu]} - \varepsilon, & \textit{for } \mu \in J, \\ r_m^{[\mu]}, & \textit{otherwise}, \end{cases}$$
$$\hat{r}_k^{[\mu]} = r_k^{[\mu]}, \quad \textit{for } k \neq m, \quad (\mu = 0, \dots, n).$$

Then the operator

$$\widehat{L}(y) := \sum_{\nu=0}^{n} (-\Delta)^\nu \left(\hat{r}_k^{[\nu]} \Delta^\nu y_{k-\nu} \right)$$

is supercritical on $\mathbb{Z}$, i.e., for the associated quadratic form we have $\widehat{\mathcal{F}} \ngeq 0$.

Proof. Let $\binom{X^-}{U^-}$ and $\binom{X^+}{U^+}$ be the recessive solutions at $-\infty$ and ∞ of (4.4) associated to the equation $L(y) = 0$.

Let us introduce the linear space

$$\widetilde{\mathcal{H}} := \text{Lin} \left\{ \begin{pmatrix} x^{[1]-} \\ u^{[1]-} \end{pmatrix}, \dots, \begin{pmatrix} x^{[n]-} \\ u^{[n]-} \end{pmatrix} \right\} \cap \text{Lin} \left\{ \begin{pmatrix} x^{[1]+} \\ u^{[1]+} \end{pmatrix}, \dots, \begin{pmatrix} x^{[n]+} \\ u^{[n]+} \end{pmatrix} \right\},$$

where $\binom{x^{[j]-}}{u^{[j]-}}$, $\binom{x^{[j]+}}{u^{[j]+}}$, $j = 1, \dots, n$, are the columns of $\binom{X^-}{U^-}$ and $\binom{X^+}{U^+}$, respectively. According to the relationship between (4.1) and (4.4) the dimension of this space is the same as dimension of the space from Definition 4.1, i.e., $\dim \widetilde{\mathcal{H}} = p$, and let $z^{[i]} = \binom{x^{[i]}}{u^{[i]}}$, $i = 1, \dots, p$, be a basis of $\widetilde{\mathcal{H}}$.

At first, we show that x-part of any base vector $z^{[i]}$ is nonzero at any index $k \in \mathbb{Z}$, i.e.,

$$x_k^{[i]} \neq 0, \quad k \in \mathbb{Z}, \quad i \in \{1, \dots, p\}.$$

By contradiction, let $x_k^{[j]} = 0$ for some $j \in \{1, \dots, p\}$ and some $k \in \mathbb{Z}$. Because there exist nonzero constant vectors $c^-, c^+ \in \mathbb{R}^n$ such that

$$\begin{pmatrix} x^{[j]} \\ u^{[j]} \end{pmatrix} = \begin{pmatrix} X^- \\ U^- \end{pmatrix} c^-, \quad \begin{pmatrix} x^{[j]} \\ u^{[j]} \end{pmatrix} = \begin{pmatrix} X^+ \\ U^+ \end{pmatrix} c^+,$$

we have $x^{[j]} = X^\pm c^\pm$, which implies that the matrices $X_k^\pm$ are both singular. This contradicts the disconjugacy of equation (4.1) (see [7, Theorem 1]). This also means that $x_k^{[j]}$, $j = 1, \dots, p$, are linearly independent, because none of their nontrivial linear combinations can be a zero vector.

We have obtained that

$$\operatorname{rank}(x_k^{[1]}, \dots, x_k^{[p]}) = p \quad \text{for any } k \in \mathbb{Z}.$$

Let us denote

$$P = (x_{m+1}^{[1]}, \dots, x_{m+1}^{[p]}), \quad P \in \mathbb{R}^{n \times p},$$

where m is the integer in which $r^{[\mu]}$ are changed to $r^{[\mu]} - \varepsilon$, $\mu \in J$, and consider the square matrix $\widetilde{P} \in \mathbb{R}^{p \times p}$ that consists of the linearly independent lines of the matrix P.

Hence, the system

$$\widetilde{P} \begin{pmatrix} d_1 \\ \vdots \\ d_p \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

has a unique solution $d \in \mathbb{R}^p, d \neq 0$.

Now, let us introduce the solution $z^{[h]} = \binom{x^{[h]}}{u^{[h]}}$ of (4.4) given by

$$z^{[h]} := \sum_{j=1}^{p} d_j z^{[j]}$$

and let h be the solution of (4.1) which generates $z^{[h]}$, i.e.,

$$x_k^{[h]} = \begin{pmatrix} h_{k-1} \\ \Delta h_{k-2} \\ \vdots \\ \Delta^{n-1} h_{k-n} \end{pmatrix}. \tag{4.16}$$

Because $x^{[j]}$ are linearly independent and $d \neq 0$, we have $0 \neq x^{[h]} \in \mathbb{R}^n$. The construction of $z^{[h]}$ implies that at least p entries of $x^{[h]}_{m+1}$ are nonzero, i.e., $(x^{[h]}_{m+1})_{i+1} = \Delta^i h_{m-i} = 1$ (at least) for $i \in \tilde{J}$, $\tilde{J} \subseteq \{0, \dots, n-1\}$ with $|\tilde{J}| = p$.

Now, let $y = \{y_k\}_{k\in\mathbb{Z}}$ be the sequence given by (4.15) with h from (4.16). We have

$$\widehat{\mathcal{F}}(y) = \sum_{k=-\infty}^{\infty} \left[\sum_{\nu=0}^{n} \hat{r}_k^{[\nu]} \left(\Delta^\nu y_{k-\nu}\right)^2 \right] = \sum_{k=-\infty}^{\infty} \left[u_k^{[y]T} B_k u_k^{[y]} + x_{k+1}^{[y]T} \widehat{C}_k x_{k+1}^{[y]} \right],$$

where $\widehat{C}_k = \operatorname{diag} \left\{ \hat{r}_k^{[0]}, \dots, \hat{r}_k^{[n-1]} \right\}$ and $x^{[y]}$ and $u^{[y]}$ are given by formulas (4.6). Denote $\widetilde{C}_k := C_k - \widehat{C}_k$. Then $\widetilde{C}_k = 0$ for $k \neq m$ and $\widetilde{C}_m$ is the diagonal matrix with the diagonal entries $(\widetilde{C}_m)_{i+1,i+1} = \varepsilon$, $i \in J$, and $(\widetilde{C}_m)_{i+1,i+1} = 0$, $i \notin J$. Hence

$$\begin{aligned} \widehat{\mathcal{F}}(y) &= \sum_{k=-\infty}^{\infty} \left[u_k^{[y]T} B_k u_k^{[y]} + x_{k+1}^{[y]T} (C_k - \widetilde{C}_k) x_{k+1}^{[y]} \right] \\ &= \sum_{k=-\infty}^{\infty} \left[u_k^{[y]T} B_k u_k^{[y]} + x_{k+1}^{[y]T} C_k x_{k+1}^{[y]} \right] - \sum_{k=-\infty}^{\infty} \left[x_{k+1}^{[y]T} \widetilde{C}_k x_{k+1}^{[y]} \right]. \end{aligned}$$

and

$$\sum_{k=-\infty}^{\infty} \left[x_{k+1}^{[y]T} \widetilde{C}_k x_{k+1}^{[y]} \right] = \varepsilon \sum_{i\in J} (\Delta^i h_{m-i})^2 \geq \varepsilon \sum_{i\in J\cap\tilde{J}} (\Delta^i h_{m-i})^2 = \varepsilon |J \cap \tilde{J}| =: \delta > 0$$

since $|J| = n - p + 1$ and $|\tilde{J}| = p$, i.e., $J \cap \tilde{J} \neq \emptyset$. Using Lemma 4.1 and the construction from its proof we obtain that

$$\sum_{k=-\infty}^{\infty} \left[u_k^{[y]T} B_k u_k^{[y]} + x_{k+1}^{[y]T} C_k x_{k+1}^{[y]} \right] < \frac{\delta}{2}$$

if K and N in (4.15) are sufficiently close to $-\infty$ and ∞, respectively.

Together we have

$$\widehat{\mathcal{F}}(y) < \frac{\delta}{2} - \delta = -\frac{\delta}{2} < 0,$$

i.e., $\widehat{L}$ is supercritical. □

The following statement is a direct consequence of Theorem 4.1.

Corollary 4.1. *Let the operator L be p-critical on $\mathbb{Z}$, $p \in \{1, \dots, n\}$, and let $m \in \mathbb{Z}$ and $\varepsilon > 0$ be arbitrary. Further, let $\widetilde{J} \subseteq \{0, \dots, n-1\}$ with $|\widetilde{J}| = n-p+1$ and consider the sequence*

$$\tilde{r}_m^{[\mu]} = \begin{cases} r_m^{[\mu]} + \varepsilon, & \text{for } \mu \in \widetilde{J}, \\ r_m^{[\mu]}, & \text{otherwise}, \end{cases}$$
$$\tilde{r}_k^{[\mu]} = r_k^{[\mu]}, \quad \text{for } k \neq m, \quad (\mu = 0, \dots, n).$$

Then

$$\widetilde{L}(y) := \sum_{\nu=0}^{n} (-\Delta)^\nu \left(\tilde{r}_k^{[\nu]} \Delta^\nu y_{k-\nu} \right)$$

is at most q-critical for some $q \in \{0, \dots, p-1\}$.

Proof. Suppose that $\widetilde{L}$ is q-critical with $q \geq p$. We get L from $\widetilde{L}$ by changing $\tilde{r}_m^{[\mu]}$ to $\tilde{r}_m^{[\mu]} - \varepsilon$. Theorem 4.1 then implies that L is supercritical, a contradiction. □

Remark 4.1. In Theorem 4.1 we have formulated the statement in terms of perturbations of coefficients $r^{[\mu]}$ in L. Using formulas (4.5), this statement can be reformulated in terms of perturbation of the entries $t_{i,j}$ of T, similarly as it is done in Proposition 4.1 in case $n = 1$.

4.4 One term operators

In this section, we deal with perturbations of the one-term difference operator (equation)

$$l(y) := (-\Delta)^n (r_k \Delta^n y_k) = 0, \quad r_k > 0, \; k \in \mathbb{Z}. \tag{4.17}$$

Since this equation can be solved explicitly, we can find an explicit condition for its criticality. To avoid technical difficulties, in the main part of this section we deal with the case $n = 2$, but at the end of the section we suggest how the results can be extended to general n.

Equation (4.17) has solutions $y^{[1]} = 1, \dots, y^{[n]} = k^{(n-1)} = k(k-1)\cdots(k-n+1)$ which satisfy $\Delta^n y = 0$, such solutions we will call *polynomial*. In addition to polynomial solutions, (4.17) is also solved by sequences for which $\Delta^n y_k = k^{(j-1)} r_k^{-1}$, $j = 1, \dots, n$, such solutions we will call *nonpolynomial*. A similar analysis as in the continuous case (see [15]) shows that only polynomial solutions can be simultaneously contained both in the recessive systems of solutions at ∞ and $-\infty$.

We start with a statement which describes the structure of the solution space of (4.17).

Lemma 4.2 ([11, Sec. 2]). *Equation* (4.17) *is disconjugate on* $\mathbb{Z}$ *and possesses a system of solutions* $y^{[j]}, \tilde{y}^{[j]}$, $j = 1, \dots, n$, *such that*

$$y^{[1]} \prec \cdots \prec y^{[n]} \prec \tilde{y}^{[1]} \prec \cdots \prec \tilde{y}^{[n]} \tag{4.18}$$

as $k \to \infty$, *where* $f \prec g$ *as* $k \to \infty$ *for a pair of sequences* f, g *means that* $\lim_{k\to\infty}(f_k/g_k) = 0$. *If* (4.18) *holds, the solutions* $y^{[j]}$ *form the recessive system of solutions at* ∞, *while* $\tilde{y}^{[j]}$ *form the dominant system,* $j = 1, \dots, n$. *The analogous statement holds for the ordered system of solutions as* $k \to -\infty$.

To compute nonpolynomial solutions, we need the following formula (which can be verified by a direct computation or by induction). Let z_k be any sequence and

$$y_k = \frac{1}{(n-1)!} \sum_{j=0}^{k-1} (k-j-1)^{(n-1)} z_j,$$

then $\Delta^n y_k = z_k$. Moreover, the generalized power function

$$(k-j)^{(n)} = (k-j)(k-j-1)\cdots(k-j-n+1)$$

has the "binomial" expansion

$$(k-j)^{(n)} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} k^{(n-i)} (j+i-1)^{(i)}. \tag{4.19}$$

In the remaining part of this chapter we denote by $\mathcal{V}^+$ and $\mathcal{V}^-$ the solutions subspaces of (4.17) generated by the recessive system of solutions at ∞ and $-\infty$, respectively.

Theorem 4.2. *Let* $n = 2$.
(i) If

$$\sum_{k=0}^{\infty} r_k^{-1} = \infty = \sum_{k=-\infty}^{0} r_k^{-1}, \tag{4.20}$$

then (4.17) *is 2-critical on* $\mathbb{Z}$ *and* $\mathcal{V}^+ \cap \mathcal{V}^- = \operatorname{Lin}\{1, k\}$.
(ii) If at least one of the infinite series in (4.20) *is convergent, but*

$$\sum_{k=0}^{\infty} k^2 r_k^{-1} = \infty = \sum_{k=-\infty}^{0} k^2 r_k^{-1}, \tag{4.21}$$

then (4.17) *is 1-critical on* $\mathbb{Z}$ *and* $\mathcal{V}^+ \cap \mathcal{V}^- = \operatorname{Lin}\{1\}$.
(iii) If at least one of the infinite series in (4.21) *is convergent,* (4.17) *is subcritical on* $\mathbb{Z}$, *i.e.,* $\mathcal{V}^+ \cap \mathcal{V}^- = \emptyset$.

Proof. In the proof we will use Lemma 4.2 (with $n = 2$) to show that a given solution y of (4.17) is contained in the recessive system of solutions. To show this, it is sufficient to find linearly independent solutions $y^{[1]}$, $y^{[2]}$ such that $y \prec y^{[i]}$, $i = 1, 2$ (for $k \to \infty$ or $k \to -\infty$).
(i) Let

$$y_k^{[1]} = \sum_{j=0}^{k-1}(k-j-1)r_j^{-1}, \quad y_k^{[2]} = \sum_{j=0}^{k-1}(k-j-1)jr_j^{-1}.$$

Then by the L'Hospital rule for $i = 1, 2$

$$\lim_{k\to\infty} \frac{y_k^{[i]}}{k} = \lim_{k\to\infty} \sum_{j=0}^{k-1} j^{i-1} r_j^{-1} = \infty,$$

i.e., $y_k = k \prec y_k^{[i]}$, $i = 1, 2$. Since $\bar{y}_k = 1 \prec y_k = k$, we have $\mathcal{V}^+ = \operatorname{Lin}\{1, k\}$. For $k < 0$ we set

$$y_k^{[1]} = \sum_{j=k-1}^{0}(k-j-1)r_j^{-1}, \quad y_k^{[2]} = \sum_{j=k-1}^{0}(k-j-1)jr_j^{-1}$$

and similarly as above we show that $\mathcal{V}^- = \operatorname{Lin}\{1, k\}$.
(ii) Suppose that (4.21) holds and, e.g., $\sum^{\infty} r_k^{-1} < \infty$ (the case $\sum_{-\infty} r_k^{-1} < \infty$ can be treated analogically). We show that $y_k = k \notin \mathcal{V}^+$ by constructing a nonpolynomial solution $\bar{y}$ for which $\bar{y}_k \prec k$ as $k \to \infty$. Such a solution is

$$\bar{y}_k = (k-1)\sum_{j=0}^{\infty} r_j^{-1} - \sum_{j=0}^{k-1}(k-j-1)r_j^{-1}. \tag{4.22}$$

We have $\lim_{k\to\infty}(\bar{y}_k/k) = \lim_{k\to\infty} \Delta\bar{y}_k = \lim_{k\to\infty} \sum_{j=k}^{\infty} r_j^{-1} = 0$, i.e., $\bar{y}_k \prec k$. On the other hand, if (4.21) holds, there are two linearly independent solutions greater than $y = 1$. One of them is obviously $y_k = k$ and the second one is

$$y_k = (k-1)\sum_{j=0}^{\infty} jr_j^{-1} - \sum_{j=0}^{k-1}(k-j-1)jr_j^{-1}, \tag{4.23}$$

if $\sum^{\infty} jr_j^{-1} < \infty$ or the solution (4.22) if $\sum^{\infty} jr_j^{-1} = \infty$. Consider, e.g., the case of solution (4.23). We have $\Delta y_k = \sum_{j=k}^{\infty} jr_j^{-1}$ and using summation by parts

$$\begin{aligned} y_k =& \sum_{j=0}^{k-1} \Delta y_j = [j\Delta y_j]_0^k - \sum_{j=0}^{k-1} j\Delta^2 y_j \\ =& k\sum_{j=k}^{\infty} jr_j^{-1} + \sum_{j=0}^{k-1} j^2 r_j^{-1} \to \infty \quad \text{as } k \to \infty, \end{aligned}$$

i.e., $y_k \succ y = 1$.
(iii) Suppose, e.g., that $\sum^\infty j^2 r_j^{-1} < \infty$ (the complementary case is again analogical). Then both solutions

$$y_k^{[1]} = \sum_{j=0}^{k-1} (k-j-1) r_j^{-1} - (k-1) \sum_{j=0}^{\infty} r_j^{-1} + \sum_{j=0}^{\infty} j r_j^{-1},$$
$$y_k^{[2]} = \sum_{j=0}^{k-1} (k-j-1) j r_j^{-1} - (k-1) \sum_{j=0}^{\infty} j r_j^{-1} + \sum_{j=0}^{\infty} j^2 r_j^{-1}$$

satisfy $y_k^{[i]} \to 0$ as $k \to \infty$, i.e., $y^{[i]} \prec y = 1$. To show this, consider, e.g., the solution $y^{[1]}$, we have

$$|y_k^{[1]}| = \left| -(k-1) \sum_{j=k}^{\infty} r_j^{-1} + \sum_{j=k}^{\infty} j r_j^{-1} \right| \le (k-1) \sum_{j=k}^{\infty} r_j^{-1} + \sum_{j=k}^{\infty} j r_j^{-1}$$
$$\le \sum_{j=k}^{\infty} j r_j^{-1} + \sum_{j=k}^{\infty} j r_j^{-1} \to 0 \quad \text{as } k \to \infty.$$

Hence $\mathcal{V}^+ \cap \mathcal{V}^- = \emptyset$. □

Remark 4.2. The previous result may be used, e.g., for improving the main result of [9], where a conjugacy of the second order Sturm-Liouville difference equation via a phase theory is studied. Combining these two concepts, some interesting new results may be obtained.

Finally, we briefly mention the case of general $n \in \mathbb{N}$. The continuous case, i.e., the equation $(r(t)y^{(n)})^{(n)} = 0$ is treated in detail in [15] and computations given there show that the situation is technically rather complicated in the general case. Nevertheless, computations for $n = 2$ together with the continuous case and formula (4.19) suggest the following conjecture, which we hope to prove in a subsequent paper.

Conjecture 4.1. *Let* $m \in \{0, \dots, n-1\}$ *and suppose that*

$$\sum_{j=-\infty}^{0} j^{2(n-m-1)} r_j^{-1} = \infty = \sum_{j=0}^{\infty} j^{2(n-m-1)} r_j^{-1}.$$

Then $\operatorname{Lin}\{1, \dots, k^{(m)}\} \subseteq \mathcal{V}^+ \cap \mathcal{V}^-$.

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Appendix A

Curriculum vitae

Petr Hasil

Personal Details

Date of Birth	23/03/1982
Nationality	Czech

Education

2006 –	Ph.D. student	Mathematical Analysis Masaryk University, Faculty of Science
2006	Master's degree	Mathematical Analysis Masaryk University, Faculty of Science
2004	Bachelor's degree	(General) Mathematics Masaryk University, Faculty of Science

Professional experience

2008 –	Assistant	Mendel University in Brno Department of Mathematics

Professional stays

March – July 2009	Ulm University (Ulm, Germany)

Teaching activities

Linear algebra	Ulm University, Faculty of Mathematics and Economics
Mathematical Analysis	Masaryk University, Faculty of Science
Basic courses of Mathematics	Masaryk University, Faculty of Informatics
Basic courses of Mathematics	Mendel University in Brno

Appendix B

Publications

- O. DOŠLÝ, P. HASIL, *Friedrichs extension of operators defined by symmetric banded matrices*, Linear Algebra Appl. **430** (2009), 1966–1975.
- O. DOŠLÝ, P. HASIL, *Critical higher order Sturm-Liouville difference operators*, J. Differ. Equ. Appl., to appear.
- O. DOŠLÝ, P. HASIL, *Critical oscillation constant for half-linear differential equations with periodic coefficients*, submitted.
- P. HASIL, *Conditional oscillation of half-linear differential equations with periodic coefficients*, Arch. Math. **44** (2008), 119–131.
- P. HASIL, *On positivity of the three term $2n$-order difference operators*, Stud. Univ. Žilina (Math. Ser.), **23** (2009), 51–58.
- P. HASIL, *Criterion of p-criticality for one term $2n$-order difference operators*, in preparation.