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On a Nonlocal Boundary Value Problem for Linear Systems of Functional Differential Equations

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Abstrakt

V předložené práci je studována otázka existence a jednoznačnosti řešení systému funkcionálních differenciálních rovnic

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p(x)(t) + q(t) \tag{R}$$

a jejich zvláštních případů (zejména rovnice pantografu) vyhovujících okrajové podmínce

$$\int_a^b [\mathrm{d}\Phi(t)]x(t) = c_0, \tag{P}$$

a jejim význačným zvláštním případům (mnohabodová, integrální podmínka).

V obecné části předpokládáme, že $p : C(I, \mathbb{R}^n) \to L(I, \mathbb{R}^n)$ je lineární silně ohraničený operátor, $\Phi : I \to \mathbb{R}^{n\times n}$ je maticová funkce s ohraničenou variací, $q \in L(I, \mathbb{R}^n)$ a $c_0 \in \mathbb{R}^n$. V dalších částech je práce zaměřena zejména na systémy obyčejných diferenciálních rovnic s více odkloněnými, resp. zpožděnými, argumenty a na zvláštní případy okrajových podmínek – mnohabodovou a integrální.

Práce obsahuje šest kapitol, které jsou dále členěny do patnácti podkapitol. V první kapitole jsou definovány základní pojmy a prezentovány jejich vlastnosti ve formě převzatých tvrzení. Ve druhé kapitole jsou uvedeny věty o jednoznačné řešitelnosti úlohy (R), (P) a úlohy s malým parametrem. Kapitoly 3 a 4 jsou věnovány lineárním systémům s konečně mnoha odkloněnými argumenty. Pátá kapitola se zabývá zobecněnou rovnicí pantografu s mnohabodovou okrajovou podmínkou. Zde jsou uvedeny detailnější výsledky pro případy konstantního a proporcionálního zpoždění. Poslední kapitola obsahuje konstrukci řešení zobecněné rovnice pantografu s mnohabodovou okrajovou podmínkou a konkrétní příklady.

Práce vychází z publikací [1, 2, 6 – 9, 15, 18 – 21] a tam zmíněné literatury. Nové a v práci uvedené výsledky byly publikovány v [10 – 14].

Abstract

In the present thesis, the question on the existence and uniqueness of a solution of the system of functional differential equations

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p(x)(t) + q(t) \tag{E}$$

and its special cases (particularly pantograph equation) with boundary condition

$$\int_a^b [\mathrm{d}\Phi(t)]x(t) = c_0, \tag{C}$$

and its special cases (multi-point, integral condition) is studied.

We assume in general that $p : C(I, \mathbb{R}^n) \to L(I, \mathbb{R}^n)$ is a linear strongly bounded operator, $\Phi : I \to \mathbb{R}^{n\times n}$ is a matrix function of bounded variation, $q \in L(I, \mathbb{R}^n)$ and $c_0 \in \mathbb{R}^n$. We study particularly the systems of ordinary differential equations with more deviating (especially delayed) arguments and special boundary conditions – multi-point and integral.

The thesis consists of six chapters, which are divided into fifteen sections. In Chapter 1, we present definitions and quoted statements. In Chapter 2, the results dealing with the unique solvability of the problem (E), (C) and the problem with a small parameter are established. Chapter 3 and 4 are devoted to the linear systems with deviating arguments. A generalized pantograph equation with multi-point boundary condition is studied in Chapter 5. We preset there also corollaries for systems with constant and proportional delays. In Chapter 6, we construct the solution of the generalized equation of the pantograph with multi-point boundary condition and give some examples.

The results which we present here are contained in the papers [10 – 14] and can be regarded as a continuation of the papers [1, 2, 6 – 9, 15, 18 – 21] and the references given therein.

Contents

Notation

$\mathbb{R}$ is the set of all real numbers, i.e., $\mathbb{R} =]-\infty, +\infty[$;

$\mathbb{R}_+ = [0, +\infty[,\ I = [a, b]$;

χ_I is the characteristic function of the interval I, i.e.,

$$\chi_I(t) = \begin{cases} 1 & \text{for } t \in I \\ 0 & \text{for } t \notin I \end{cases};$$

$\mathbb{R}^n$ is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in \mathbb{R}$ $(i = 1, \dots, n)$ and the norm

$$||x|| = \sum_{i=1}^{n} |x_i|;$$

$\mathbb{R}_+^n = \{(x_i)_{i=1}^n \in \mathbb{R}^n : x_i \geq 0\ (i = 1, \dots, n)\}$;

$\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrices $X = (x_{ij})_{i,j=1}^n$ with elements $x_{ij} \in \mathbb{R}$ $(i, j = 1, \dots, n)$ and the norm

$$||X|| = \sum_{i,j=1}^{n} |x_{ij}|;$$

$\mathbb{R}_+^{n\times n} = \{(x_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n} : x_{ij} \geq 0\ (i, j = 1, \dots, n)\}$;

If $x, y \in \mathbb{R}^n$ and $X, Y \in \mathbb{R}^{n\times n}$, then

$$x \leq y \Leftrightarrow y - x \in \mathbb{R}_+^n,\ X \leq Y \Leftrightarrow Y - X \in \mathbb{R}_+^{n\times n};$$

$$|x| = (|x_i|)_{i=1}^n,\ |X| = (|x_{ij}|)_{i,j=1}^n;$$

$\det(X)$ is the determinant of the matrix $X \in \mathbb{R}^{n\times n}$;

X^{-1} is the inverse matrix to the matrix $X \in \mathbb{R}^{n\times n}$;

$r(X)$ is the spectral radius of the matrix $X \in \mathbb{R}^{n\times n}$;

E is the unit matrix;

Θ is the zero matrix;

Remark.

1. Let $X, Y \in \mathbb{R}_+^{n\times n}$. If $X \leq Y$ then $r(X) \leq r(Y)$.

2. Let $X = (x_{ij})_{i,j=1}^n \in \mathbb{R}_+^{n\times n}$ and $\sum_{i=1}^n x_{ij} < 1$ for $j = 1, \dots, n$. Then $r(X) < 1$.

A vector (matrix) function is said to be continuous, integrable, etc., if such are its elements.

$C(I, \mathbb{R}^n)$ is the space of continuous vector functions $x : I \to \mathbb{R}^n$ with the norm

$$||x||_C = \max\{||x(t)|| : t \in I\};$$

If $x = (x_i)_{i=1}^n \in C(I, \mathbb{R}^n)$, then

$$|x|_C = (||x_i||_C)_{i=1}^n;$$

$C(I, \mathbb{R}^{n\times n})$ is the set of continuous matrix functions $X : I \to \mathbb{R}^{n\times n}$;

If $X = (x_{ij})_{i,j=1}^n \in C(I, \mathbb{R}^{n\times n})$, then

$$|X|_C = (||x_{ij}||_C)_{i,j=1}^n;$$

$\widetilde{C}(I, \mathbb{R}^n)$ is the space of absolutely continuous vector functions $x : I \to \mathbb{R}^n$ with the norm

$$||x||_C = ||x(a)|| + \int_a^b ||x'(t)||\,\mathrm{d}t;$$

$\widetilde{C}(I, \mathbb{R}^{n\times n})$ is the set of absolutely continuous matrix functions $X : I \to \mathbb{R}^{n\times n}$;

$L^p(I, \mathbb{R}^n)$, where $1 \le p < +\infty$ is the space of vector functions $x : I \to \mathbb{R}^n$ with elements integrable in the p-th power with the norm

$$||x||_{L^p} = \left(\int_a^b ||x(t)||^p\,\mathrm{d}t\right)^{\frac{1}{p}};$$

$L^{+\infty}(I, \mathbb{R}^n)$ is the space of measurable and bounded vector functions $x : I \to \mathbb{R}^n$ with the norm

$$||x||_{L^{+\infty}} = \text{ess sup}\{||x(t)|| : t \in I\};$$

If $x = (x_i)_{i=1}^n \in L^p(I, \mathbb{R}^n)$, then

$$|x|_{L^p} = (||x_i||_{L^p})_{i=1}^n;$$

$L(I, \mathbb{R}^n) = L^1(I, \mathbb{R}^n)$,

$L^p(I, \mathbb{R}^{n\times n})$, where $1 \le p < +\infty$ is the set of matrix functions $X : I \to \mathbb{R}^{n\times n}$ with elements integrable in the p-th power;

If $X = (x_{ij})_{i,j=1}^n \in L^p(I, \mathbb{R}^{n\times n})$, then

$$|X|_{L^p} = (||x_{ij}||_{L^p})_{i,j=1}^n;$$

$L(I, \mathbb{R}^{n\times n}) = L^1(I, \mathbb{R}^{n\times n})$
If $Z \in C(I, \mathbb{R}^{n\times n})$ is a matrix function with columns $z_1, \dots, z_n$ and $g : C(I, \mathbb{R}^n) \to L(I, \mathbb{R}^n)$ is a linear operator, then $g(Z)$ stands for the matrix function with columns $g(z_1), \dots, g(z_n)$.

We use also the following terminology: absolutely continuous function, function with bounded variation, measure of the subset of the interval I (Lebesgue measure - denoted by mes), essential minimum (denoted by $\operatorname{vrai\,min}$). This terminology is commonly used in theory of ordinary and functional differential equations in the meaning of Carathéodory. By a solution of the considered system we understand a vector function absolutely continuous on the considered interval. Further, we use Lebesgue integral, Lebesgue measure and other terms of the modern mathematical and functional analysis (for definitions see [17, 24]). There are Carathéodory conditions for ordinary differential equation in mentioned publications. Definition of Carathéodory conditions for functional differential equations is published in [3].

Introduction

The publications [8] and [9] deal with boundary value problems for linear functional differential equations. General results are frequently illustrated on the cases of linear systems of differential equations with one deviating argument with Cauchy and periodic boundary conditions. Systems with more deviating arguments with multi-point or integral boundary condition are studied rarely. More detailed results for multi-point boundary condition are known only for linear systems of ordinary differential equations (see [6, 7]). Also the problems with integral boundary condition are studied in literature insufficiently.

In the present thesis, we supplement and improve the theory of boundary value problem of linear functional differential equations in that parts, which are not studied in detail yet.

The thesis consists of six chapters, which are divided into fifteen sections. In Chapter 1, we present definitions and quoted statements. We recall the theorem about unique solvability of the linear system of functional differential equations

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p(x)(t) + q(t)$$

with the linear boundary condition

$$l(x) = c_0,$$

where $p : C(I, \mathbb{R}^n) \to L(I, \mathbb{R}^n)$ is a linear strongly bounded operator, $l : C(I, \mathbb{R}^n) \to \mathbb{R}^n$ is linear bounded operator, $q \in L(I, \mathbb{R}^n)$ and $c_0 \in \mathbb{R}^n$. We understand the solution of this problem in a sense of Carathéodory, i.e., the solution is a vector function absolutely continuous on the interval I, it satisfies this system of functional differential equations almost everywhere on I and fulfils the boundary condition. The theorem is grounded on the Fredholm property of this problem. The proof of this theorem was published in [8] and [9].

Chapter 2 is devoted to the investigation of the question of the existence and uniqueness of a solution of the above mentioned system satisfying boundary condition

$$\int_a^b [\mathrm{d}\Phi(t)]x(t) = c_0, \tag{C}$$

where $\Phi : I \to \mathbb{R}^{n\times n}$ is a matrix function of bounded variation and $c_0 \in \mathbb{R}^n$. The use is made here of a method developed in [8] and [9]. In Section 2.2, we present a theorem about unique solvability and corollaries for problems with multi-point, Cauchy, periodic and integral boundary conditions. The system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \varepsilon p(x)(t) + q(t)$$

with small parameter $\varepsilon > 0$ is studied in Section 2.3. We introduce a theorem and corollaries for unique solvability of this system with boundary conditions (C) and its special cases (multi-point, Cauchy, periodic and integral boundary conditions). The similar results for such systems with boundary condition $\int_a^b x(t)\,\mathrm{d}\sigma(t) = c_0$ (where $\sigma : I \to \mathbb{R}$ is a function of bounded variation) were published in paper [10] and [11].

Linear systems with deviating argument are studied in Chapter 3. In Section 3.1, we present theorems for the unique solvability of the systems

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = P(t)x(\tau(t)) + q_0(t), \quad x(t) = u(t) \text{ for } t \notin I,$$

and

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \varepsilon P(t)x(\tau(t)) + q_0(t), \quad x(t) = u(t) \text{ for } t \notin I,$$

with boundary condition (C), where $P \in L(I, \mathbb{R}^{n\times n})$, $\tau : I \to \mathbb{R}$ is a measurable function, $u : \mathbb{R} \setminus I \to \mathbb{R}^n$ is a continuous and bounded function, $q_0 \in L(I, \mathbb{R}^n)$ and $\varepsilon > 0$ is a small parameter. There are again corollaries for special cases of the condition (C). The systems with more deviating arguments

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{\mu} P_i(t)x(\tau_i(t)) + q_0(t), \quad x(t) = u(t) \text{ for } t \notin I$$

with multi-point, Cauchy, periodic and integral boundary conditions are studied in Section 3.2. Here, $P_i \in L(I, \mathbb{R}^{n\times n})$, $\tau_i : I \to \mathbb{R}$ are measurable functions, $u : \mathbb{R} \setminus I \to \mathbb{R}^n$ is a continuous and bounded function, and $q_0 \in L(I, \mathbb{R}^n)$. The similar results for such systems with boundary condition $\int_a^b x(t)\,\mathrm{d}\sigma(t) = c_0$ (where $\sigma : I \to \mathbb{R}$ is a function with bounded variation) were published in paper [11].

Chapter 4 is devoted to more detailed study of the general multi-point and general integral boundary conditions for the system with more deviating arguments. Results from Chapter 3 are supplemented with other criteria which are essential for solving of larger class of boundary value problems with use of some other assumptions on the right side of the studied system. The results from Section 4.1 and Section 4.2 were published in paper [13]. Section 4.3 and Section 4.4 include consequences which have not been published yet. In Section 4.3, we specify coefficients of the matrices P_i and deviations of the arguments τ_i to get some other special criteria of the unique solvability of studied problems. Section 4.4 is devoted to the corollaries for problems with Cauchy and periodic boundary conditions.

Generalized pantograph equation with multi-point boundary condition is studied in Chapter 5. This equation is a special type of systems of differential equations with more deviating arguments which was studied in previous chapters. Independent variable t (representing time) is considered on the interval $[0, T]$ and there are delayed arguments in a special form. It allows to get other deeper results. We introduce also corollaries for such systems with Cauchy and periodic boundary conditions. In Section 5.3, we present also corollaries for systems with constant and proportional delays. New results of this chapter were published in paper [12].

In Chapter 6, we construct the solution of the generalized equation of the pantograph with multi-point boundary condition using the method of successive approximation and give some examples. The method of successive approximation can be used for all mentioned problems. Results of this chapter are published in paper [14].

1. Definitions and Quoted Statements

The following notation on the interval $I = [a, b]$ is used in accordance with publications [7, 8, 9] and references mentioned therein.

Definition 1.1. *A linear operator $p : C(I, \mathbb{R}^n) \to L(I, \mathbb{R}^n)$ is said to be* ***strongly bounded*** *if there exists a summable function $\eta \in L(I, \mathbb{R}_+)$ such that*

$$||p(x)(t)|| \leq \eta(t)||x||_C \quad for \quad t \in I, x \in C(I, \mathbb{R}^n).$$

Definition 1.2. *A linear operator $l : C(I, \mathbb{R}^n) \to R^n$ is said to be* ***bounded*** *if there exists a number $\alpha \in \mathbb{R}_+$ such that*

$$||l(x)|| \leq \alpha||x||_C \quad for \quad x \in C(I, \mathbb{R}^n).$$

On the bounded interval $I = [a, b]$ consider the **linear system of functional differential equations**

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p(x)(t) + q(t) \tag{1.1}$$

with the **linear boundary condition**

$$l(x) = c_0, \tag{1.2}$$

where $p : C(I, \mathbb{R}^n) \to L(I, \mathbb{R}^n)$ is a linear strongly bounded operator, $l : C(I, \mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded operator, $q \in L(I, \mathbb{R}^n)$ and $c_0 \in \mathbb{R}^n$.

Definition 1.3. *An absolutely continuous vector function $x : I \to \mathbb{R}^n$ is said to be a* ***solution of system*** (1.1) *if it satisfies this system almost everywhere on I. A solution x of system* (1.1) *is said to be a* ***solution of the boundary value problem*** (1.1), (1.2) *if it satisfies condition* (1.2).

Along with the boundary value problem (1.1), (1.2) consider the corresponding homogeneous problem

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p(x)(t), \tag{1.1$_0$}$$

$$l(x) = 0. \tag{1.2$_0$}$$

The following theorem is proved in the basic publication dealing with linear boundary value problems for functional differential equations [9].

Theorem 1.1. *The problem* (1.1), (1.2) *is uniquely solvable if and only if the corresponding homogeneous problem* (1.1_0), (1.2_0) *has only the trivial solution.*

Proof of this theorem is published in [9], page 345 and [8], page 13.

We use the Hölder, Minkowski (see [24]) and Levin (see [19]) inequalities for finding effective conditions of solvability of the mentioned problems.

Lemma 1.1 (Hölder)**.** *Let* $p, q > 1$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(I, \mathbb{R})$ *and* $g \in L^q(I, \mathbb{R})$. *Then*

$$\int_a^b |f(t)g(t)|\,\mathrm{d}t \le \left(\int_a^b |f(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}.$$

Lemma 1.2 (Minkowski)**.** *Let* $p \ge 1$ *and* $f, g \in L^p(I, \mathbb{R})$. *Then*

$$\left(\int_a^b |f(t) + g(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}} \le \left(\int_a^b |f(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}.$$

Lemma 1.3 (Levin)**.** *Let* $u \in \widetilde{C}(I, \mathbb{R})$, $u' \in L^p(I, \mathbb{R})$ *with* $1 < p < +\infty$ *and let* $t_0 \in I$ *be such that* $u(t_0) = 0$. *Then*

$$\int_a^b |u(t)|^p\,\mathrm{d}t \le \left(\frac{b-a}{l_p}\right)^p \int_a^b |u'(t)|^p\,\mathrm{d}t,$$

where

$$l_p = (p-1)^{\frac{1}{p}} \left(\frac{p}{\pi} \sin\frac{\pi}{p}\right)^{-1}.$$

Remark. For $p = 2$ in Lemma 1.3 we get Wirtinger inequality (see [7]).

2. General Linear Boundary Value Problem

2.1. Statement of the Problem

On the bounded interval $I = [a,b]$, consider the system of functional differential equations

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p(x)(t) + q(t) \tag{2.1}$$

with the boundary condition

$$\int_a^b [\mathrm{d}\Phi(t)]x(t) = c_0, \tag{2.2}$$

where $p : C(I,\mathbb{R}^n) \to L(I,\mathbb{R}^n)$ is a linear strongly bounded operator, $\Phi : I \to \mathbb{R}^{n\times n}$ is a matrix function of bounded variation, $q \in L(I,\mathbb{R}^n)$, $c_0 \in \mathbb{R}^n$. We understand the integral in condition (2.2) as Lebesgue-Stieltjes integral.

Condition (2.2) fulfils assumptions of the first chapter, i.e., $l(x) = \int_a^b [\mathrm{d}\Phi(t)]x(t)$ and from characteristic properties of Lebesgue-Stieltjes integral

$$||l(x)|| = ||\int_a^b [\mathrm{d}\Phi(t)]x(t)|| \leq \sum_{i=1}^n \int_a^b ||\mathrm{d}\Phi(t)||\,|x_i(t)| \leq \alpha \sum_{i=1}^n ||x_i||_C = \alpha ||x||_C,$$

where $\alpha = \int_a^b ||\mathrm{d}\Phi(t)|| \in \mathbb{R}_+$ (see also Riesz's theorem, [5]).

Consider now the matrix function $\Phi : I \to \mathbb{R}^{n\times n}$ piecewise absolutely continuous on the interval I, i.e., suppose that there exists a finite number of points $t_j \in I\,(j = 1,\dots,\nu)$ such that $a = t_1 < \dots < t_\nu = b$ and the function Φ is absolutely continuous on every interval $]t_j, t_{j+1}[\, \subset [a,b]\,(j = 2,\dots,\nu-2)$ and on the intervals $[t_1,t_2[$ and $]t_{\nu-1},t_\nu]$. Moreover, suppose that there exist finite limits $\lim_{t\to t_j^-} \Phi(t)\,(j = 2,\dots,\nu)$ and $\lim_{t\to t_j^+} \Phi(t)\,(j = 1,\dots,\nu-1)$.

Therefore, the function Φ may have a discontinuity of the first kind in some point $t_j\,(j = 1,\dots,\nu)$. Put

$$S_{\Phi_j} = \lim_{t\to t_j^+} \Phi(t) - \lim_{t\to t_j^-} \Phi(t) \ \ (j = 2,\dots,\nu-1),$$

$$S_{\Phi_1} = \lim_{t\to t_1^+} \Phi(t) - \Phi(a), \quad S_{\Phi_\nu} = \Phi(b) - \lim_{t\to t_\nu^-} \Phi(t).$$

Note that, in this situation, the condition (2.2) can be rewritten as

$$\int_a^b \Phi'(t)x(t)\mathrm{d}t + \sum_{j=1}^{\nu} S_{\Phi_j} x(t_j) = c_0. \tag{2.3}$$

By a special choice of the function Φ we get from (2.2) the following boundary conditions:

– multi-point condition

$$\sum_{j=1}^{\nu} A_j x(t_j) = c_0, \tag{2.4}$$

where $a = t_1 < \ldots < t_\nu = b$, $A_j \in \mathbb{R}^{n\times n}$ $(j = 1, \ldots, \nu)$, if $\Phi(t) = \sum_{j=1}^{\nu-1} A_j \chi_{]t_j,b]}(t) + A_\nu \chi_{[b,b]}(t)$;

– integral condition

$$\int_a^b A(t)x(t)\,\mathrm{d}t = c_0, \tag{2.5}$$

where $A \in L(I, \mathbb{R}^{n\times n})$, if $\Phi(t) = \int_a^t A(s)\,\mathrm{d}s$ for $t \in I$.

Special types of condition (2.4) are

– multi-point condition

$$\sum_{j=1}^{\nu} \lambda_j x(t_j) = c_0, \tag{2.6}$$

where $a = t_1 < \ldots < t_\nu = b$ and $\lambda_j \in \mathbb{R}$ $(j = 1, \ldots, \nu)$;

– Cauchy condition

$$x(t_0) = c_0, \tag{2.7}$$

where $t_0 \in I$;

– periodic condition

$$x(b) - x(a) = c_0. \tag{2.8}$$

Special type of condition (2.5) is

– integral condition

$$\int_a^b x(t)\,\mathrm{d}t = c_0. \tag{2.9}$$

General problem (2.1), (2.2) and problems (2.1), (2.7) and (2.1), (2.8) are studied in detail in publications [8, 9]. We study effective criteria for the unique solvability of the problem (2.1) (and its special types) with boundary conditions (2.6) - (2.9) in this and in the following chapter. In Chapter 4 we study the special types of the system (2.1) with boundary conditions (2.4) and (2.5).

The problem (2.1), (2.2) is special type of the problem (1.1), (1.2). Therefore, the following theorem about Fredholm property of the problem results from Theorem 1.1.

Theorem 2.1. *The problem* (2.1), (2.2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p(x)(t), \tag{2.1$_0$}$$

$$\int_a^b [\mathrm{d}\Phi(t)]x(t) = 0 \tag{2.2$_0$}$$

has only the trivial solution.

2.2. Existence and Uniqueness Theorems

Let $t_0 \in I$ be an arbitrary but fixed point. According to [8, 9] we define the following sequences of operators $p^k : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$ and matrices $\Lambda_k \in \mathbb{R}^{n\times n}$, $k \in \mathbb{N}$:

$$p^0(x)(t) = x(t), \qquad p^k(x)(t) = \int_{t_0}^t p\,(p^{k-1}(x))(s)\,\mathrm{d}s, \tag{2.10}$$

$$\Lambda_k = \sum_{j=0}^{k-1} \int_a^b [\mathrm{d}\Phi(t)]p^j(E)(t). \tag{2.11}$$

If the matrix Λ_k is nonsingular for some $k \in \mathbb{N}$, then we set

$$p^{k,0}(x)(t) = x(t),$$

$$p^{k,m}(x)(t) = p^m(x)(t) - \sum_{i=0}^{m-1} p^i(E)(t)\ \Lambda_k^{-1} \int_a^b [\mathrm{d}\Phi(s)]p^k(x)(s), m \in \mathbb{N}. \tag{2.12}$$

Theorem 2.2. *Let the matrix function* $\Phi : I \to \mathbb{R}^{n\times n}$ *be piecewise absolutely continuous and let either*

$$\Lambda_1 = \int_a^b \Phi'(t)dt + \sum_{j=1}^{\nu} S_{\Phi_j} \tag{2.13}$$

be nonsingular or, for some $k \in \{2, 3, \dots\}$, $\Lambda_1 = \dots = \Lambda_{k-1} = \Theta$ *and*

$$\Lambda_k = \int_a^b \Phi'(t) p^{k-1}(E)(t) dt + \sum_{j=1}^{\nu} S_{\Phi_j} p^{k-1}(E)(t_j) \tag{2.14}$$

be nonsingular. Let, moreover, there exist matrices $B, B_j \in \mathbb{R}_+^{n\times n}$ $(j = 1, \dots, \nu)$ *such that*

$$\int_a^b |p(x)(t)|\, \mathrm{d}t \le B\ |x|_C \quad \text{for } x \in C(I, \mathbb{R}^n), \tag{2.15}$$

$$\left| \int_{t_0}^{t_j} |p(x)(t)|\, \mathrm{d}t \right| \le B_j\ |x|_C \quad \text{for } x \in C(I, \mathbb{R}^n) \tag{2.16}$$

and, for some $m \in \mathbb{N}$,

$$r(A) < 1, \tag{2.17}$$

where

$$A = B^m + \sum_{i=0}^{m-1} B^i |\Lambda_k^{-1}| \left(\int_a^b |\Phi'(t)|\, \mathrm{d}t\, B^k + \sum_{j=1}^{\nu} |S_{\Phi_j}| B_j B^{k-1} \right). \tag{2.18}$$

Then the problem (2.1), (2.2) *has a unique solution.*

Proof. According to Theorem 2.1, it is sufficient to show that if x is a solution of the problem (2.1_0), (2.2_0), then $x(t) \equiv 0$. Let x be such a solution and let $t_0 \in I$ be an arbitrary fixed point.

It is clear that functional differential equation (2.1_0) is equivalent to equation

$$x(t) = c + \int_{t_0}^t p(x)(s)\, \mathrm{d}s,$$

where $c = x(t_0)$ and

$$\begin{aligned} x(t) &= c + \int_{t_0}^t p(x)(s)\, \mathrm{d}s = c + p^1(x)(t) = [E + p^1(E)(t)]c + p^2(x)(t) = \\ &= \ldots = [E + p^1(E)(t) + \ldots + p^{j-1}(E)(t)]c + p^j(x)(t) \end{aligned} \tag{2.19}$$

for every $j = 1, 2, \ldots$.

According to the assumption of Λ_k let $j = k$ and from (2.2_0), (2.19) we get

$$0 = \int_a^b [\mathrm{d}\Phi(t)] x(t) = \Lambda_k c + \int_a^b [\mathrm{d}\Phi(t)] p^k(x)(t)$$

and therefore

$$c = -\Lambda_k{}^{-1} \int_a^b [\mathrm{d}\Phi(t)] p^k(x)(t).$$

Thus for every solution x of the system (2.1_0), (2.2_0) and for every $j = 1, 2, \dots$ we have

$$x(t) = p^{k,j}(x)(t), \ t \in I. \tag{2.20}$$

Then from (2.10) and (2.15) we get

$$|p^1(x)|_C = \max\left\{\left|\int_{t_0}^{t} |p(x)(s)|\,\mathrm{d}s\right| : t \in I\right\} \le$$
$$\le \int_a^b |p(x)(t)|\,\mathrm{d}t \le B|x|_C \quad \text{for } x \in C(I, \mathbb{R}^n)$$

and analogously, for an arbitrary $j \in \mathbb{N}$, we have

$$|p^j(x)|_C = \max\left\{\left|\int_{t_0}^{t} |p(p^{j-1}(x))(s)|\,\mathrm{d}s\right| : t \in I\right\} \le$$
$$\le \int_a^b |p(p^{j-1}(x))(t)|\,\mathrm{d}t \le B|p^{j-1}(x)|_C \quad \text{for } x \in C(I, \mathbb{R}^n).$$

By induction, in view of the last inequality, it can be shown that

$$|p^j(x)|_C \le B^j |x|_C \quad \text{for } x \in C(I, \mathbb{R}^n), j = 1, 2, \dots. \tag{2.21}$$

Now (2.21) implies

$$|p^j(E)|_C \le B^j, \ j = 1, 2, \dots. \tag{2.22}$$

Therefore from (2.12), on account of (2.21) and (2.22), we get

$$|p^{k,m}(x)|_C = \left|p^m(x)(\cdot) - \sum_{i=0}^{m-1} p^i(E)(\cdot)\Lambda_k^{-1}\int_a^b [\mathrm{d}\Phi(t)]p^k(x)(t)\right|_C \le$$
$$\le |p^m(x)|_C + \sum_{i=0}^{m-1} |p^i(E)|_C \left|\Lambda_k^{-1}\int_a^b [\mathrm{d}\Phi(t)]p^k(x)(t)\right| \le$$
$$\le B^m|x|_C + \sum_{i=0}^{m-1} B^i|\Lambda_k^{-1}|\left|\int_a^b [\mathrm{d}\Phi(t)]p^k(x)(t)\right|.$$

Further from (2.10), (2.15) and (2.16) it follows that

$$\left|\int_a^b [\mathrm{d}\Phi(t)]p^k(x)(t)\right| =$$
$$= \left|\int_a^b \Phi'(t)\int_{t_0}^{t} p(p^{k-1}(x))(s)\,\mathrm{d}s\ \mathrm{d}t + \sum_{j=1}^{\nu} S_{\Phi_j}\int_{t_0}^{t_j} p(p^{k-1}(x))(s)\,\mathrm{d}s\right| \le$$

$$\leq \left| \int_a^b \Phi'(t) \int_{t_0}^t p(p^{k-1}(x))(s) \, \mathrm{d}s \; \mathrm{d}t \right| + \sum_{j=1}^{\nu} |S_{\Phi_j}| \left| \int_{t_0}^{t_j} |p(p^{k-1}(x))(s)| \, \mathrm{d}s \right| \leq$$

$$\leq \int_a^b |\Phi'(t)| \int_a^b |p(p^{k-1}(x))(s)| \, \mathrm{d}s \; \mathrm{d}t + \sum_{j=1}^{\nu} |S_{\Phi_j}| B_j \, |p^{k-1}(x)|_C \leq$$

$$\leq \left(\int_a^b |\Phi'(t)| \, \mathrm{d}t \right) B^k |x|_C + \sum_{j=1}^{\nu} |S_{\Phi_j}| B_j \, B^{k-1} \, |x|_C =$$

$$= \left(\int_a^b |\Phi'(t)| \, \mathrm{d}t B^k + \sum_{j=1}^{\nu} |S_{\Phi_j}| B_j B^{k-1} \right) |x|_C \quad \text{for } x \in C(I, \mathbb{R}^n).$$

Thus for every solution x of the problem (2.1_0), (2.2_0) and for suitable $k, m \in \mathbb{N}$ we have

$$|p^{k,m}(x)|_C \leq A |x|_C, \tag{2.23}$$

where A is defined by (2.18).

From (2.20) it follows that

$$|x|_C = |p^{k,m}(x)|_C \leq A |x|_C.$$

Since $r(A) < 1$, there exists $(A - E)^{-1}$ and

$$|x|_C \leq (A - E)^{-1} 0 = 0.$$

Therefore $x(t) \equiv 0$ on I. □

Theorem 2.2 yields the following corollaries for the boundary conditions (2.6) – (2.9).

Corollary 2.1. *Let either*

$$\sum_{j=1}^{\nu} \lambda_j \neq 0 \tag{2.24}$$

or

$$\sum_{j=1}^{\nu} \lambda_j = 0 \tag{2.25}$$

and the matrix

$$\Lambda = \sum_{j=1}^{\nu} \lambda_j \int_a^{t_j} p(E)(t) \, \mathrm{d}t$$

be nonsingular. Let, moreover, there exist matrices $B, B_j \in \mathbb{R}_+^{n\times n}$ $(j = 1, \dots, \nu)$ *and* $m \in \mathbb{N}$ *such that the conditions* (2.15), (2.16) *with* $t_0 = a$ *hold and*

$$r(A) < 1,$$

where

$$A = B^m + \frac{1}{\sum_{j=1}^{\nu} \lambda_j} \sum_{i=0}^{m-1} B^i \sum_{j=1}^{\nu} |\lambda_j| B_j \quad \text{if} \quad \sum_{j=1}^{\nu} \lambda_j \neq 0$$

and

$$A = B^m + \sum_{i=0}^{m-1} B^i |\Lambda^{-1}| \sum_{j=1}^{\nu} |\lambda_j| B_j B \quad \text{if} \quad \sum_{j=1}^{\nu} \lambda_j = 0.$$

Then the problem (2.1), (2.6) *has a unique solution.*

Proof. In the case of the multi-point condition (2.6), the function Φ in (2.2) can be defined by

$$\Phi(t) = \left[\sum_{j=1}^{\nu-1} \lambda_j \chi_{]t_j, b]}(t) + \lambda_\nu \chi_{[b,b]}(t) \right] E \quad \text{for} \quad t \in I.$$

Choose $t_0 = a$. Then, from (2.14) we get

$$\Lambda_1 = E \sum_{j=1}^{\nu} \lambda_j$$

and if $\sum_{j=1}^{\nu} \lambda_j = 0$, then $\Lambda_1 = \Theta$ and

$$\Lambda_2 = \sum_{j=1}^{\nu} p^1(E)(t_j)\lambda_j = \sum_{j=1}^{\nu} \lambda_j \int_a^{t_j} p(E)(t)\, \mathrm{d}t = \Lambda.$$

If (2.24) holds, then the matrix Λ_1 is nonsingular and from (2.18) we obtain

$$A = B^m + \frac{1}{\sum_{j=1}^{\nu} \lambda_j} \sum_{i=0}^{m-1} B^i \sum_{j=1}^{\nu} |\lambda_j| B_j.$$

If (2.25) holds and $\det(\Lambda) \neq 0$, then $\Lambda_1 = \Theta$, the matrix Λ_2 is nonsingular and from (2.18) we obtain

$$A = B^m + \sum_{i=0}^{m-1} B^i |\Lambda^{-1}| \sum_{j=1}^{\nu} |\lambda_j| B_j B.$$

Consequently, the assumptions of Theorem 2.2 are fulfilled. □

Corollary 2.2. *Let there exist a matrix* $B \in \mathbb{R}_+^{n\times n}$ *and* $m \in \mathbb{N}$ *such that*

$$r(B^m) < 1$$

and the inequality (2.15) *be fulfilled. Then the problem* (2.1), (2.7) *has a unique solution.*

Proof. In the case of the Cauchy condition, the function Φ in (2.2) can be defined as

$$\Phi(t) = (1 - \chi_{[a,t_0]}(t))E \quad \text{for} \quad t \in I.$$

It follows from Corollary 2.1 with $t_1 = a, t_2 = t_0, t_3 = b$, $A_1 = \Theta = A_2$. □

Corollary 2.3. *Let the matrix*

$$\Lambda = \int_a^b p(E)(t)\,\mathrm{d}t$$

be nonsingular and let there exist matrix $B \in \mathbb{R}_+^{n\times n}$ *and* $m \in \mathbb{N}$ *such that the inequality* (2.15) *holds and*

$$r(B^m + \sum_{i=0}^{m-1} B^i|\Lambda^{-1}|B^2) < 1.$$

Then the problem (2.1), (2.8) *has a unique solution.*

Proof. In the case of the periodic condition (2.8), the function Φ in (2.2) can be defined as

$$\Phi(t) = (1 - \chi_{]a,b[}(t))E \quad \text{for} \quad t \in I.$$

Then $\nu = 2$, $t_1 = a$, $t_2 = b$, $S_{\Phi_1} = -E, S_{\Phi_2} = E$. Choose $t_0 = a$. Then, from (2.14) we get

$$\Lambda_1 = \int_a^b \Phi'(t)\,\mathrm{d}t + (S_{\Phi_1} + S_{\Phi_2}) = \Theta$$

and

$$\Lambda_2 = \int_a^b \Phi'(t)p^1(E)(t)\,\mathrm{d}t + S_{\Phi_1}p^1(E)(a) + S_{\Phi_2}p^1(E)(b) =$$
$$= \Theta + \left(\int_a^{t_1} p(E)(t)\,\mathrm{d}t + \int_{t_1}^b p(E)(t)\,\mathrm{d}t\right) = \int_a^b p(E)(t)\,\mathrm{d}t = \Lambda,$$

i.e., the matrix Λ_2 is nonsingular. Moreover,

$$\left|\int_{t_0}^{t_1} |p(x)(t)|\,\mathrm{d}t\right| = \int_a^a |p(x)(t)|\,\mathrm{d}t = 0 \quad \text{for } x \in C(I, \mathbb{R}^n)$$

and

$$\left|\int_{t_0}^{t_2} |p(x)(t)|\,\mathrm{d}t\right| = \int_a^b |p(x)(t)|\,\mathrm{d}t \le B|x|_C \quad \text{for } x \in C(I,\mathbb{R}^n),$$

and thus, the condition (2.16) holds with $B_1 = \Theta$ and $B_2 = B$.

Furthermore, from (2.18) we obtain

$$A = B^m + \sum_{i=1}^{m-1} B^i |\Lambda^{-1}| B^2,$$

i.e., the condition (2.17) is fulfilled.

Therefore, the assumptions of Theorem 2.2 are satisfied. □

Corollary 2.4. *Let there exist a matrix $B \in \mathbb{R}_+^{n\times n}$ and $m \in \mathbb{N}$ such that*

$$r(B^m + \sum_{i=0}^{m-1} B^{i+1}) < 1$$

and the inequality (2.15) *holds. Then the problem* (2.1), (2.9) *has a unique solution.*

Proof. In the case of the integral condition (2.9), the function Φ in (2.2) can be defined as

$$\Phi(t) = tE \quad \text{for} \quad t \in I.$$

Choose $t_0 = a$. Then from (2.14) we get

$$\Lambda_1 = \int_a^b \Phi'(t)\,\mathrm{d}t = \int_a^b E\mathrm{d}t = E(b-a),$$

i.e., the matrix Λ_1 is nonsingular.

Moreover, in a similar manner as in proof of Theorem 2.2 it can be shown that, in view of (2.15), we have

$$|p^{1,m}(x)|_C \le B^m|x|_C + \sum_{i=0}^{m-1} B^i \frac{1}{b-a}\left|\int_a^b \Phi'(t)p^1(x)(t)\,\mathrm{d}t\right| \le$$

$$\le B^m|x|_C + \sum_{i=0}^{m-1} B^i \int_a^b |p(x)(t)|\,\mathrm{d}t \le \left(B^m + \sum_{i=0}^{m-1} B^{i+1}\right)|x|_C \quad \text{for } x \in C(I,\mathbb{R}^n),$$

i.e., the assumptions of Theorem 2.2 hold with $k = 1$ and $A = B^m + \sum_{i=0}^{m-1} B^{i+1}$ (see also Theorem 1.3.1 in [8]).

□

2.3. Linear System with a Small Parameter

On bounded interval $I = [a, b]$ consider the system of functional differential equations with a small parameter $\varepsilon > 0$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \varepsilon p(x)(t) + q(t), \tag{2.26}$$

where $p : C(I, \mathbb{R}^n) \to L(I, \mathbb{R}^n)$ is a linear strongly bounded operator and $q \in L(I, \mathbb{R}^n)$.

For any $\varepsilon > 0$ and $x \in C(I, \mathbb{R}^n)$ set

$$p_\varepsilon(x)(t) = \varepsilon p(x)(t).$$

Then from (2.10) and (2.11) we get

$$p_\varepsilon^k(x)(t) = \varepsilon^k p^k(x)(t), \quad \Lambda_{k,\varepsilon} = \varepsilon^{k-1}\Lambda_k, \; k \in \mathbb{N}.$$

Theorem 2.3. *Let the matrix function $\Phi : I \to \mathbb{R}^{n\times n}$ be piecewise absolutely continuous and let either Λ_1 given by* (2.13) *be nonsingular, or for some $k \in \{2, 3, \ldots\}$, $\Lambda_1 = \cdots = \Lambda_{k-1} = \Theta$ and Λ_k given by* (2.14) *be nonsingular. Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in]0, \varepsilon_0[$, problem* (2.26), (2.2) *has a unique solution.*

Proof. Note in the first place that since operator p is strongly bounded, i.e., there exists $\eta \in L(I, \mathbb{R}_+)$ such that $||p(x)(t)|| \le \eta(t)||x||_C$ for $t \in I$, $x \in C(I, \mathbb{R}^n)$. Therefore, there exist matrices $B, B_j \in \mathbb{R}_+^{n\times n}$ $(j = 1, \ldots, \nu)$ such that the inequalities (2.15), (2.16) hold. Indeed, we can put $B = \int_a^b \eta(t)\,\mathrm{d}t\ (1)_{i,j=1}^n$ and $B_j = \left|\int_{t_0}^{t_j} \eta(t)\,\mathrm{d}t\right| (1)_{i,j=1}^n$.

Using (2.15) and (2.16) from

$$p_\varepsilon^{k,1}(x)(t) = p_\varepsilon^1(x)(t) - \Lambda_{k,\varepsilon}^{-1} \int_a^b [\mathrm{d}\Phi(t)] p_\varepsilon^k(x)(t)$$

we find

$$|p_\varepsilon^{k,1}(x)|_C = \varepsilon |p^{k,1}(x)|_C \le A_\varepsilon |x|_C \quad \text{for } x \in C(I, \mathbb{R}^n), \tag{2.27}$$

where $A_\varepsilon = \varepsilon A$ and

$$A = B + |\Lambda_k^{-1}| \left(\int_a^b |\Phi'(t)|\,\mathrm{d}t B^k + \sum_{j=1}^{\nu} |S_{\Phi_j}| B_j B^{k-1} \right).$$

Clearly, if

$$\varepsilon_0 = 1/r(A),$$

then

$$r(A_\varepsilon) < 1 \text{ for } \varepsilon \in]0, \varepsilon_0[. \tag{2.28}$$

However, in view of Theorem 2.2 (Theorem 1.3.1 with $m = 1, m_0 = 0$ in [8]), conditions (2.27) and (2.28) guarantee the unique solvability of problem (2.26), (2.2) for arbitrary $\varepsilon \in]0, \varepsilon_0[$. □

Remark. Instead of the condition $\Lambda_1 = \cdots = \Lambda_{k-1} = \Theta$, Λ_k is nonsingular we can assume that, for some $k \in \{2, 3, \ldots\}$, $\Lambda_1, \ldots, \Lambda_{k-1}$ are singular and Λ_k is nonsingular. (This assumption can be also used in Theorem 2.2.)

In such case

$$\Lambda_{k,\varepsilon} = \sum_{j=0}^{k-1} \int_a^b [\mathrm{d}\Phi(t)] p_\varepsilon^j(E)(t).$$

Following Corollaries can be easily derived from Theorem 2.3.

Corollary 2.5. *Let either $\lambda = \sum_{j=1}^{\nu} \lambda_j \neq 0$ or $\lambda = 0$ and the matrix*

$$\Lambda = \sum_{j=1}^{\nu} \lambda_j \int_a^{t_j} p(E)(t)\,\mathrm{d}t$$

be nonsingular. Then there exists $\varepsilon_0 > 0$ such, that for any $\varepsilon \in]0, \varepsilon_0[$, problem (2.26), (2.6) *has a unique solution.*

Corollary 2.6. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in]0, \varepsilon_0[$, problem* (2.26), (2.7) *has a unique solution.*

Corollary 2.7. *Let the matrix*

$$\Lambda = \int_a^b p(E)(t)\,\mathrm{d}t$$

be nonsingular. Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in]0, \varepsilon_0[$, problem (2.26), (2.8) *has a unique solution.*

Corollary 2.8. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0[$ problem* (2.26), (2.9) *has a unique solution.*

3. Linear System with Deviating Arguments I

3.1. Linear System with a Deviating Argument and a Small Parameter

On the bounded interval $I = [a, b]$, consider the linear systems of differential equations with a deviating argument

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = P(t)x(\tau(t)) + q_0(t), \quad x(t) = u(t) \text{ for } t \notin I, \tag{3.1}$$

and

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \varepsilon P(t)x(\tau(t)) + q_0(t), \quad x(t) = u(t) \text{ for } t \notin I, \tag{3.2}$$

where $P \in L(I, \mathbb{R}^{n\times n})$, $\tau : I \to \mathbb{R}$ is a measurable function, $u : \mathbb{R} \setminus I \to \mathbb{R}^n$ is a continuous and bounded function, $q_0 \in L(I, \mathbb{R}^n)$ and $\varepsilon > 0$ is a small parameter.

Along with systems (3.1) and (3.2) consider the boundary condition (2.2), i.e.,

$$\int_a^b [\mathrm{d}\Phi(t)]x(t) = c_0,$$

where $\Phi : I \to \mathbb{R}^{n\times n}$ is a matrix function of bounded variation and $c_0 \in \mathbb{R}^n$.

We first study the problem (3.1), (2.2). In view of [8] and [9] we put

$$\tau^0(t) = \begin{cases} a & \text{if } \tau(t) < a \\ \tau(t) & \text{if } a \le \tau(t) \le b \\ b & \text{if } \tau(t) > b \end{cases}, \tag{3.3}$$

$$p(x)(t) = \chi_I(\tau(t))P(t)x(\tau^0(t)), \tag{3.4}$$

$$q(t) = (1 - \chi_I(\tau(t)))P(t)u(\tau(t)) + q_0(t). \tag{3.5}$$

It is obvious that $p : C(I, \mathbb{R}^n) \to L(I, \mathbb{R}^n)$ is a linear operator and $q \in L(I, \mathbb{R}^n)$. Moreover, $||p(x)(t)|| \le \alpha(t)||x||_C$ for any $x \in C(I, \mathbb{R}^n)$ and almost all $t \in I$, where $\alpha(t) = ||\chi_I(\tau(t))P(t)||$. It is also clear that $\alpha \in L(I, \mathbb{R}_+)$. Therefore p is a strongly bounded operator.

The problem (3.1), (2.2) can be rewritten to the form (2.1), (2.2), where the function τ^0, the operator p and the vector function q are given by (3.3)–(3.5), respectively. Therefore, according to Theorem 2.1 the following assertion is valid.

Theorem 3.1. *The problem* (3.1), (2.2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \chi_I(\tau(t))P(t)x(\tau^0(t)) \tag{3.1$_0$}$$

with boundary condition (2.2_0) *has only the trivial solution.*

In what follows we use forms and notations from [9].

For an arbitrary matrix function $P \in L(I, R^{n\times n})$ we set

$$[P(t)]_{\tau,0} = \Theta, \quad [P(t)]_{\tau,1} = \chi_I(\tau(t))P(t),$$

$$[P(t)]_{\tau,i+1} = [P(t)]_{\tau,1}\int_a^{\tau^0(t)} [P(s)]_{\tau,i}\,\mathrm{d}s \quad (i = 1, 2, \ldots),$$

where τ^0 is a function given by (3.3). Let $t_0 = a$. With respect to (2.10) and (2.11) we get

$$\Lambda_k = \int_a^b [\mathrm{d}\Phi(t)] + \sum_{j=0}^{k-1}\int_a^b [\mathrm{d}\Phi(t)]\int_a^t [P(s)]_{\tau,j}\,\mathrm{d}s \quad \text{for } k \in \mathbb{N}. \tag{3.6}$$

Put

$$A_{k,m} = \int_a^b [|P(t)|]_{\tau,m}\,\mathrm{d}t + \\ + \left[E + \sum_{i=1}^{m-1}\int_a^b [|P(t)|]_{\tau,i}\,\mathrm{d}t\right]|\Lambda_k^{-1}|\int_a^b |\mathrm{d}\Phi(t)|\int_a^t [|P(s)|]_{\tau,k}\mathrm{d}s. \tag{3.7}$$

Theorem 3.2. *Let there exist* $k, m \in \mathbb{N}$ *such that* $\det(\Lambda_k) \neq 0$ *and* $r(A_{k,m}) < 1$, *where the matrices* Λ_k *and* $A_{k,m}$ *are given by* (3.6) and (3.7), *respectively. Then the problem* (3.1), (2.2) *has a unique solution.*

Proof. According to above introduced notation it can be easily verified that the assumptions of Theorem 2.2 are valid. In particular the inequality (2.23) is satisfied with $A = A_{k,m}$. Thus the assertion follows from Theorem 2.2 (see also Theorem 1.3.1 with $m_0 = 0$ in [8]). □

For the special cases of (2.2) – multi-point condition, periodic condition and integral condition – Theorem 3.2 yields the following corollaries.

Corollary 3.1. *Let either* $\lambda := \sum_{j=1}^{\nu} \lambda_j \neq 0$ *and, for some* $m \in \mathbb{N}$, *the relation* $r(A_{1,m}) < 1$ *holds, where*

$$A_{1,m} = \int_a^b [|P(t)|]_{\tau,m}\,\mathrm{d}t + \frac{1}{|\lambda|}\left[E + \sum_{i=1}^{m-1}\int_a^b [|P(t)|]_{\tau,i}\,\mathrm{d}t\right]\sum_{j=1}^{\nu}|\lambda_j|\int_a^{t_j}[|P(t)|]_{\tau,1}\,\mathrm{d}t$$

or $\lambda = 0$, *the matrix*

$$\Lambda = \sum_{j=1}^{\nu}\lambda_j\int_a^{t_j}\chi_I(\tau(t))P(t)\,\mathrm{d}t$$

be nonsingular and, for some $m \in \mathbb{N}$, *the relation* $r(A_{2,m}) < 1$ *holds, where*

$$A_{2,m} = \int_a^b [|P(t)|]_{\tau,m}\,\mathrm{d}t + \\ + \left[E + \sum_{i=1}^{m-1}\int_a^b [|P(t)|]_{\tau,i}\,\mathrm{d}t\right]|\Lambda^{-1}|\sum_{j=1}^{\nu}|\lambda_j|\int_a^{t_j}[|P(t)|]_{\tau,2}\,\mathrm{d}t.$$

Then the problem (3.1), (2.6) *has a unique solution.*

Remark. This method is suitable for multi-point boundary value problems with $\nu \geq 2$. The criteria for the problem with Cauchy condition at a point $t_0 \in I$ ($\nu = 1$) are not optimal. We can find optimal criteria by changing lower bound of integration from a to t_0.

Corollary 3.2. *Let*

$$\Lambda_2 = \int_a^b [P(t)]_{\tau,1}\,\mathrm{d}t$$

be a nonsingular matrix and let there exist $m \in \mathbb{N}$ *such that* $r(A_{2,m}) < 1$, *where*

$$A_{2,m} = \int_a^b [|P(t)|]_{\tau,m}\,\mathrm{d}t + \left[E + \sum_{i=1}^{m-1}\int_a^b [|P(t)|]_{\tau,i}\,\mathrm{d}t\right]|\Lambda_2^{-1}|\int_a^b [|P(t)|]_{\tau,2}\,\mathrm{d}t.$$

Then the problem (3.1), (2.8) *has a unique solution.*

Corollary 3.3. *Let there exist* $m \in \mathbb{N}$ *such that* $r(A_{1,m}) < 1$, *where*

$$A_{1,m} = \int_a^b [|P(t)|]_{\tau,m}\,\mathrm{d}t + \left[E + \sum_{i=1}^{m-1}\int_a^b [|P(t)|]_{\tau,i}\,\mathrm{d}t\right]\int_a^b [|P(t)|]_{\tau,1}\,\mathrm{d}t.$$

Then the problem (3.1), (2.9) *has a unique solution.*

Remark. If $\tau(t) \equiv t$ we get results for ordinary differential equations published in [7].

Consider now the special type of functional differential equation with a small parameter (2.26) - system of differential equations with deviating argument and a small parameter (3.2). With the use of (3.3), (3.4) and

$$q(t) = \varepsilon(1 - \chi_I(\tau(t)))P(t)u(\tau(t)) + q_0(t),$$

the problem (3.2), (2.2) can be rewritten into the form (2.26), (2.2).

From Theorem 2.3 we can get directly conditions for the unique solvability of the problem (3.2), (2.2).

Theorem 3.3. *Let the function $\Phi : I \to \mathbb{R}$ be piecewise absolutely continuous and let either*

$$\Lambda_1 = \int_a^b [\mathrm{d}\Phi(t)]$$

be nonsingular or, for some $k \in \{2, 3, \ldots\}$, $\Lambda_1 = \cdots = \Lambda_{k-1} = \Theta$ and

$$\Lambda_k = \int_a^b [\mathrm{d}\Phi(t)] \int_a^t [P(s)]_{\tau,k-1} \, \mathrm{d}s$$

be nonsingular. Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in]0, \varepsilon_0[$, problem (3.2), (2.2) *has a unique solution.*

Remark. Analogically to remark on page 26: Instead of the condition $\Lambda_1 = \cdots = \Lambda_{k-1} = \Theta$, Λ_k is nonsingular we can assume that, for some $k \in \{2, 3, \ldots\}$, $\Lambda_1, \ldots, \Lambda_{k-1}$ are singular and Λ_k is nonsingular.

The following corollaries can be easily derived from Theorem 3.3.

Corollary 3.4. *Let either $\lambda := \sum_{j=1}^{\nu} \lambda_j \neq 0$ or let $\lambda = 0$ and the matrix*

$$\Lambda = \sum_{j=1}^{\nu} \lambda_j \int_a^{t_j} \chi_I(\tau(t))P(t) \, \mathrm{d}t$$

be nonsingular. Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in]0, \varepsilon_0[$, problem (3.2), (2.6) *has a unique solution.*

Corollary 3.5. *There exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in]0, \varepsilon_0[$, *problem* (3.2), (2.7) *has a unique solution.*

Corollary 3.6. *Let*

$$\Lambda = \int_a^b \chi_I(\tau(t))P(t)\,\mathrm{d}t$$

be a nonsingular matrix. Then there exists $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in]0, \varepsilon_0[$, *problem* (3.2), (2.8) *has a unique solution.*

Corollary 3.7. *There exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in]0, \varepsilon_0[$, *problem* (3.2), (2.9) *has a unique solution.*

3.2. Linear System with More Deviating Arguments

On the bounded interval $I = [a, b]$, we consider the linear system of differential equations with deviating arguments

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{\mu} P_i(t)x(\tau_i(t)) + q_0(t), \quad x(t) = u(t) \text{ for } t \notin I, \tag{3.8}$$

where $P_i \in L(I, \mathbb{R}^{n\times n})$, $\tau_i : I \to \mathbb{R}$ are measurable functions, $u : \mathbb{R} \setminus I \to \mathbb{R}^n$ is a continuous and bounded function, and $q_0 \in L(I, \mathbb{R}^n)$.

Put

$$\tau_i^0(t) = \begin{cases} a & \text{if } \tau_i(t) < a \\ \tau_i(t) & \text{if } a \le \tau_i(t) \le b \\ b & \text{if } \tau_i(t) > b \end{cases}. \tag{3.9}$$

For any $x \in C(I, \mathbb{R}^n)$ and $t \in I$, we set

$$p(x)(t) = \sum_{i=1}^{\mu} \chi_I(\tau_i(t))P_i(t)x(\tau_i^0(t)), \tag{3.10}$$

$$q(t) = \sum_{i=1}^{\mu} (1 - \chi_I(\tau_i(t)))P_i(t)u(\tau_i(t)) + q_0(t).$$

It is obvious that $p : C(I, \mathbb{R}^n) \to L(I, \mathbb{R}^n)$ is a linear operator and $q \in L(I, \mathbb{R}^n)$. Moreover, $||p(x)(t)|| \le \alpha(t)||x||_C$ for any $x \in C(I, \mathbb{R}^n)$ and almost all $t \in I$, where $\alpha(t) = \sum_{i=1}^{\mu} ||\chi_I(\tau_i(t))P_i(t)||$. It is also clear that $\alpha \in L(I, \mathbb{R}_+)$. Therefore p is a strongly bounded operator.

Consequently, the system (3.8) can be rewritten to the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p(x)(t) + q(t)$$

and therefore the following theorem is valid (see Theorem 2.1).

Theorem 3.4. *The problem* (3.8), (2.2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{\mu} \chi_I(\tau_i(t)) P_i(t) x(\tau_i^0(t)) \tag{3.8$_0$}$$

with boundary condition (2.2_0) *has only the trivial solution.*

Remark. If $\mu = 1$ we get Theorem 3.1.

We introduce criteria for special types of boundary condition (2.2).

Corollary 3.8. *Let either*

$$\lambda := \sum_{j=1}^{\nu} \lambda_j \neq 0$$

and $r(A_{1,1}) < 1$, *where*

$$A_{1,1} = \sum_{i=1}^{\mu} \int_a^b \chi_I(\tau_i(t)) |P_i(t)| \,\mathrm{d}t + \frac{1}{|\lambda|} \sum_{j=1}^{\nu} |\lambda_j| \sum_{i=1}^{\mu} \int_a^{t_j} \chi_I(\tau_i(t)) |P_i(t)| \,\mathrm{d}t$$

or $\lambda = 0$ *and matrix*

$$\Lambda_2 = \sum_{j=1}^{\nu} \lambda_j \sum_{i=1}^{\mu} \int_a^{t_j} \chi_I(\tau_i(t)) P_i(t) \,\mathrm{d}t$$

be nonsingular and $r(A_{2,1}) < 1$, *where*

$$A_{2,1} = \sum_{i=1}^{\mu} \int_a^b \chi_I(\tau_i(t)) |P_i(t)| \,\mathrm{d}t + \\ + |\Lambda_2^{-1}| \sum_{j=1}^{\nu} |\lambda_j| \sum_{i=1}^{\mu} \int_a^{t_j} \chi_I(\tau_i(t)) |P_i(t)| \sum_{k=1}^{\mu} \int_a^{\tau_i^0(t)} \chi_I(\tau_k(s)) |P_k(s)| \,\mathrm{d}s \,\mathrm{d}t.$$

Then the problem (3.8), (2.6) *has a unique solution.*

Corollary 3.9. *Let*

$$\Lambda_2 = \sum_{i=1}^{\mu} \int_a^b \chi_I(\tau_i(t)) P_i(t) \, \mathrm{d}t$$

be a nonsingular matrix and $r(A_{2,1}) < 1$, *where*

$$A_{2,1} = \sum_{i=1}^{\mu} \int_a^b \chi_I(\tau_i(t)) |P_i(t)| \, \mathrm{d}t + \\ + |\Lambda_2^{-1}| \sum_{i=1}^{\mu} \int_a^b \chi_I(\tau_i(t)) |P_i(t)| \sum_{j=1}^{\mu} \int_a^{\tau_i^0(t)} \chi_I(\tau_j(s)) |P_j(s)| \, \mathrm{d}s \, \mathrm{d}t.$$

Then the problem (3.8), (2.8) *has a unique solution.*

Corollary 3.10. *Let* $r(A_{1,1}) < 1$, *where*

$$A_{1,1} = 2 \sum_{i=1}^{\mu} \int_a^b \chi_I(\tau_i(t)) |P_i(t)| \, \mathrm{d}t$$

. Then the problem (3.8), (2.9) *has a unique solution.*

Remark. Another criteria of the solvability of the problem (3.8), (2.2) will be derived in Chapter 4.

Remark. Criteria of the solvability for the problem with more deviating arguments and a small parameter can be deduced analogously to Chapter 2.3.

4. Linear System with Deviating Arguments II

4.1. Statement of the Problem

On the bounded interval $I = [a, b]$, consider the linear differential equation with deviating arguments (3.8), i.e.,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{\mu} P_i(t)x(\tau_i(t)) + q_0(t), \quad x(t) = u(t) \text{ for } t \notin I$$

with the linear boundary conditions (2.4) and (2.5), i.e.,

$$\sum_{k=1}^{\nu} A_k x(t_k) = c_0,$$

$$\int_a^b A(t)x(t)\,\mathrm{d}t = c_0,$$

where $P_i \in L(I, \mathbb{R}^{n\times n})$, $q_0 \in L(I, \mathbb{R}^n)$, $\tau_i : I \to \mathbb{R}$ are measurable functions ($i = 1, \dots, \mu$), $u : \mathbb{R} \setminus I \to \mathbb{R}^n$ is a continuous and bounded function, $t_k \in I$, $A_k \in \mathbb{R}^{n\times n}$ ($k = 1, \dots, \nu$), $A \in L(I, \mathbb{R}^{n\times n})$ and $c_0 \in \mathbb{R}^n$.

It is clear that (2.4) and (2.5) are special cases of (2.2) and, therefore, problems (3.8), (2.4) and (3.8), (2.5) fulfil Fredholm property (see Theorem 3.4).

This chapter is devoted to more detailed study of the general multi-point and general integral boundary conditions for the system with more deviating arguments. Results from Chapter 3 are supplemented with other criteria which are essential for solving larger class of boundary value problems with use of some other assumptions on the right-hand side of the studied system of differential equations (3.8).

Remark. If $\mu = 1$ and $\tau_1(t) \equiv t$ or $\tau_i(t) \equiv t$ for $i = 1, \dots, \mu$ then we get a special type of system (3.8) - the system of ordinary linear differential equations. This system with multi-point boundary condition is thoroughly studied in [7]. If $\tau_i(t) \leq t$ for $i = 1, \dots, \mu$ then we speak about the system with delayed arguments.

Note also that some criteria of the solvability of the linear system with one delayed argument and linear system of functional differential equations (in general meaning) with mentioned boundary conditions are published in [10].

4.2. Existence and Uniqueness Theorems I

In this section, we establish some effective criteria of the unique solvability of the problems (3.8), (2.4) and (3.8), (2.5) using the results and methods from [8].

For the sake of transparentness, for any $t, \zeta \in I$ and $i, j = 1, \dots, \mu$, we set

$$P_{ij}(\zeta, t) = \chi_I(\tau_i(t))|P_i(t)|\left|\int_\zeta^{\tau_i^0(t)} \chi_I(\tau_j(s))|P_j(s)|\,\mathrm{d}s\right|, \quad P_{ij}(t) = P_{ij}(t,t).$$

Theorem 4.1. *Let $P_i \in L(I, \mathbb{R}^{n\times n}), \tau_i : I \to \mathbb{R}$ be measurable functions ($i = 1, \dots, \mu$) and $t_0 = \min\{t_j : j = 1, 2, \dots, \nu\}$. Let either the matrix*

$$\Lambda_1 = \sum_{k=1}^{\nu} A_k \tag{4.1}$$

be nonsingular and $r(S_1) < 1$, where

$$S_1 = \sum_{i=1}^{\mu} |\chi_I(\tau_i)P_i|_L + |\Lambda_1^{-1}| \sum_{k=1}^{\nu}\sum_{i=1}^{\mu} |A_k| \int_{t_0}^{t_k} \chi_I(\tau_i(t))|P_i(t)|\,\mathrm{d}t, \tag{4.2}$$

or the matrix

$$\Lambda_2 = \Lambda_1 + \sum_{k=1}^{\nu}\sum_{i=1}^{\mu} A_k \int_{t_0}^{t_k} \chi_I(\tau_i(t))P_i(t)\,\mathrm{d}t \tag{4.3}$$

be nonsingular and $r(S_2) < 1$, where

$$S_2 = \sum_{i=1}^{\mu} |\chi_I(\tau_i)P_i|_L + |\Lambda_2^{-1}| \sum_{k=1}^{\nu}\sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |A_k| \int_{t_0}^{t_k} P_{ij}(t_0, t)\,\mathrm{d}t. \tag{4.4}$$

Then the problem (3.8), (2.4) *has a unique solution.*

To prove this theorem we need the following lemma.

Lemma 4.1. *Let $P_i \in L(I, \mathbb{R}^{n\times n}), \tau_i : I \to \mathbb{R}$ be measurable functions ($i = 1, \dots, \mu$), $t_0 \in I$, $x \in C(I, \mathbb{R}^n)$ and $\zeta \in I$. Then for almost every $t \in I$*

$$\left|\int_{t_0}^{t} \chi_I(\tau_i(s))P_i(s)x(\tau_i^0(s))\,\mathrm{d}s\right| \le \left|\int_{t_0}^{t} \chi_I(\tau_i(s))|P_i(s)|\,\mathrm{d}s\right| |x|_C \tag{4.5}$$

and for every $i, j \in \{1, \dots, \mu\}$ and almost every $t \in I$

$$\left|\chi_I(\tau_i(t))P_i(t)\int_\zeta^{\tau_i^0(t)} \chi_I(\tau_j(s))P_j(s)x(\tau_j^0(s))\,\mathrm{d}s\right| \le P_{ij}(\zeta, t)\,|x|_C. \tag{4.6}$$

If x is a solution of the homogeneous equation (3.8_0)*, then*

$$\left|\sum_{i=1}^{\mu}\chi_I(\tau_i(t))P_i(t)x(\tau_i^0(t))\right| \le \left[\left|\sum_{i=1}^{\mu}\chi_I(\tau_i(t))P_i(t)\right| + \sum_{i=1}^{\mu}\sum_{j=1}^{\mu}P_{ij}(t)\right]|x|_C \qquad (4.7)$$

for almost every $t \in I$*.*

Proof. Let $x \in C(I,\mathbb{R}^n)$ and $\zeta \in I$. Then

$$\left|\int_{t_0}^{t}\chi_I(\tau_i(s))P_i(s)x(\tau_i^0(s))\,\mathrm{d}s\right| \le \left|\int_{t_0}^{t}\chi_I(\tau_i(s))|P_i(s)||x(\tau_i^0(s))|\,\mathrm{d}s\right| \le$$
$$\le \left|\int_{t_0}^{t}\chi_I(\tau_i(s))|P_i(s)|\,\mathrm{d}s\right||x|_C \quad \text{for } t \in I$$

and

$$\left|\chi_I(\tau_i(t))P_i(t)\int_{\zeta}^{\tau_i^0(t)}\chi_I(\tau_j(s))P_j(s)x(\tau_j^0(s))\,\mathrm{d}s\right| \le$$
$$\le \chi_I(\tau_i(t))|P_i(t)|\left|\int_{\zeta}^{\tau_i^0(t)}\chi_I(\tau_j(s))|P_j(s)|\,\mathrm{d}s\right||x|_C =$$
$$= P_{ij}(\zeta,t)\,|x|_C \quad \text{for almost every } t \in I.$$

If $x \in \widetilde{C}(I,\mathbb{R}^n)$ is a solution of equation (3.8_0), then

$$\left|\sum_{i=1}^{\mu}\chi_I(\tau_i(t))P_i(t)x(\tau_i^0(t))\right| =$$
$$= \left|\sum_{i=1}^{\mu}\chi_I(\tau_i(t))P_i(t)[x(\tau_i^0(t)) - x(t) + x(t)]\right| \le$$
$$\le \left|\sum_{i=1}^{\mu}\chi_I(\tau_i(t))P_i(t)x(t)\right| + \left|\sum_{i=1}^{\mu}\chi_I(\tau_i(t))P_i(t)[x(\tau_i^0(t)) - x(t)]\right| \le$$
$$\le \left|\sum_{i=1}^{\mu}\chi_I(\tau_i(t))P_i(t)\right||x|_C + \sum_{i=1}^{\mu}\chi_I(\tau_i(t))|P_i(t)|\left|\int_{t}^{\tau_i^0(t)}x'(s)\,\mathrm{d}s\right| \le$$
$$\le \left[\left|\sum_{i=1}^{\mu}\chi_I(\tau_i(t))P_i(t)\right| + \right.$$
$$\left. + \sum_{i=1}^{\mu}\chi_I(\tau_i(t))|P_i(t)|\left|\int_{t}^{\tau_i^0(t)}\sum_{j=1}^{\mu}\chi_I(\tau_j(s))|P_j(s)|\,\mathrm{d}s\right|\right]|x|_C =$$
$$= \left[\left|\sum_{i=1}^{\mu}\chi_I(\tau_i(t))P_i(t)\right| + \sum_{i=1}^{\mu}\sum_{j=1}^{\mu}P_{ij}(t)\right]|x|_C \quad \text{for almost every } t \in I.$$

□

Proof of Theorem 4.1. According to Theorem 3.4, it is sufficient to show that if x is a solution of equation (3.8_0), satisfying boundary condition

$$\sum_{k=1}^{\nu} A_k x(t_k) = 0, \tag{2.4$_0$}$$

then $x(t) \equiv 0$.

Let x be a solution of the boundary value problem (3.8_0), (2.4_0). The integration of (3.8_0) from t_0 to t, in view of (3.10), results in

$$x(t) = c + \int_{t_0}^{t} p(x)(s)\,\mathrm{d}s \tag{4.8}$$

and, by iteration in (4.8), we get

$$x(t) = \left[E + \int_{t_0}^{t} p(E)(s)\,\mathrm{d}s\right] c + \int_{t_0}^{t} p\left(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s, \tag{4.8$_1$}$$

where $c = x(t_0)$ and

$$p(E)(s) = \sum_{i=1}^{\mu} \chi_I(\tau_i(s)) P_i(s),$$

$$p\left(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\right)(s) = \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} \chi_I(\tau_i(s)) P_i(s) \int_{t_0}^{\tau_i^0(s)} \chi_I(\tau_j(\xi)) P_j(\xi) x(\tau_j^0(\xi))\,\mathrm{d}\xi.$$

First, suppose that the matrix Λ_1 given by (4.1) is nonsingular and $r(S_1) < 1$, where the matrix S_1 is defined by (4.2). Then, by virtue of (2.4_0), (4.1) and (4.8), we get

$$0 = \sum_{k=1}^{\nu} A_k \left[c + \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s\right] = \Lambda_1 c + \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s$$

and thus

$$c = -\Lambda_1^{-1} \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s.$$

Therefore, (4.8) implies

$$x(t) = \int_{t_0}^{t} p(x)(s)\,\mathrm{d}s - \Lambda_1^{-1} \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s, \tag{4.9}$$

and in view of (4.5) we get

$$\begin{aligned}|x(t)| &\le \int_a^b |p(x)(t)|\,\mathrm{d}t + |\Lambda_1^{-1}|\sum_{k=1}^{\nu}|A_k|\int_{t_0}^{t_k}|p(x)(s)|\,\mathrm{d}s \le \\ &\le \Bigg[\int_a^b \sum_{i=1}^{\mu}\chi_I(\tau_i(t))|P_i(t)|\,\mathrm{d}t + \\ &\quad + |\Lambda_1^{-1}|\sum_{k=1}^{\nu}|A_k|\int_{t_0}^{t_k}\sum_{i=1}^{\mu}\chi_I(\tau_i(t))|P_i(t)|\,\mathrm{d}t\Bigg]|x|_C = \\ &= \Bigg[\sum_{i=1}^{\mu}|\chi_I(\tau_i)P_i|_L + |\Lambda_1^{-1}|\sum_{k=1}^{\nu}\sum_{i=1}^{\mu}|A_k|\int_{t_0}^{t_k}\chi_I(\tau_i(t))|P_i(t)|\,\mathrm{d}t\Bigg]|x|_C,\end{aligned}$$

i.e.,

$$|x|_C \le S_1|x|_C.$$

Whence, together with the assumption $r(S_1) < 1$, we get

$$(E - S_1)|x|_C \le 0 \Rightarrow |x|_C \le (E - S_1)^{-1}0 = 0.$$

Therefore $x(t) \equiv 0$.

Now suppose that the matrix Λ_2 given by (4.3) is nonsingular, and $r(S_2) < 1$, where the matrix S_2 is given by (4.4). From (2.4_0), (4.8_1) and (4.3) we get

$$\begin{aligned}0 &= \sum_{k=1}^{\nu}A_k\left[E + \int_{t_0}^{t_k}p(E)(s)\,\mathrm{d}s\right]c + \sum_{k=1}^{\nu}A_k\int_{t_0}^{t_k}p\left(\int_{t_0}^{\cdot}p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s = \\ &= \left[\Lambda_1 + \sum_{k=1}^{\nu}A_k\int_{t_0}^{t_k}p(E)(s)\,\mathrm{d}s\right]c + \sum_{k=1}^{\nu}A_k\int_{t_0}^{t_k}p\left(\int_{t_0}^{\cdot}p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s = \\ &= \Lambda_2\,c + \sum_{k=1}^{\nu}A_k\int_{t_0}^{t_k}p\left(\int_{t_0}^{\cdot}p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s\end{aligned}$$

and thus

$$c = -\Lambda_2^{-1}\sum_{k=1}^{\nu}A_k\int_{t_0}^{t_k}p\left(\int_{t_0}^{\cdot}p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s.$$

Therefore (4.8) yields

$$x(t) = \int_{t_0}^{t}p(x)(s)\,\mathrm{d}s - \Lambda_2^{-1}\sum_{k=1}^{\nu}A_k\int_{t_0}^{t_k}p\left(\int_{t_0}^{\cdot}p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s. \qquad (4.9_1)$$

Hence, on account of (4.5) and (4.6), we get

$$
\begin{aligned}
|x(t)| &\le \left|\int_a^b p(x)(t)\,\mathrm{d}t\right| + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} |A_k| \int_{t_0}^{t_k} \left| p\Big(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\right| \mathrm{d}s \le \\
&\le \left[\sum_{i=1}^{\mu} |\chi_I(\tau_i) P_i|_L + |\Lambda_2^{-1}| \sum_{k=1}^{\nu}\sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |A_k| \int_{t_0}^{t_k} P_{ij}(t_0,t)\,\mathrm{d}t\right] |x|_C = \\
&= S_2 |x|_C.
\end{aligned}
$$

Whence, together with the assumption $r(S_2) < 1$, we get $x(t) \equiv 0$. □

Remark. Consider the case where $t_0 = a$. If we put $A_k = \lambda_k E$, where $\lambda_k \in \mathbb{R}$, $k = 1, 2, \ldots, \mu$ in Theorem 4.1 we get Corollary 3.8 where $\Lambda_1 = \lambda E, A_{1,1} = S_1$ and $A_{2,1} = S_2$.

The following example demonstrates the situation when $\Lambda_1 \neq \Theta$ is a singular matrix and Λ_2 is a nonsingular matrix.

Example 4.1. *Let* $I = [0, 1]$, $n = 2, \mu = 1, \nu = 2$, $t_1 = 0, t_2 = 1$, $\tau_1(t) \equiv t$, $c = \left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right)$, $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $P(t) = \left(\begin{smallmatrix} 0 & 1 \\ p(t) & 0 \end{smallmatrix}\right)$, $q(t) = \left(\begin{smallmatrix} 0 \\ q(t) \end{smallmatrix}\right)$. *Consider the system* (3.8) *in the form*

$$
\begin{aligned}
x_1'(t) &= x_2(t) \\
x_2'(t) &= p(t)x_1(t) + q(t)
\end{aligned}
$$

with the boundary condition (2.4) *in the form*

$$x_1(0) = c_1,\ \ x_1(1) = c_2.$$

For this problem

$$\Lambda_1 = A_1 + A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

is singular matrix and

$$\Lambda_2 = \Lambda_1 + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \int_0^1 p(t)\mathrm{d}t & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is nonsingular matrix.

According to Theorem 4.1, the considered problem has a solution if $r(S_2) < 1$, *where*

$$S_2 = \begin{pmatrix} 0 & 1 \\ 2|p|_L - \int_0^1 s|p(s)|\,\mathrm{d}s & 0 \end{pmatrix}.$$

Therefore, the considered problem has a unique solution under the assumption $2|p|_L < 1 + \int_0^1 s|p(s)|\,\mathrm{d}s$.

Other criteria of the solvability of the multi-point boundary value problem can be derived using the properties of deviations $\tau_j (j = 1, \dots, \nu)$.

Theorem 4.2. *Let* $P_i \in L(I, \mathbb{R}^{n\times n}), \tau_i : I \to \mathbb{R}$ *be measurable functions* $(i = 1, \dots, \mu)$ *and* $t_0 = \min\{t_j : j = 1, 2, \dots, \nu\}$. *Let either the matrix* Λ_1 *given by* (4.1) *be nonsingular and* $r(S_1) < 1$, *where*

$$\begin{aligned} S_1 = &\left|\sum_{i=1}^{\mu} \chi_I(\tau_i) P_i\right|_L + \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |P_{ij}|_L + \\ &+ |\Lambda_1^{-1}| \sum_{k=1}^{\nu} |A_k| \int_{t_0}^{t_k} \left[\left|\sum_{i=1}^{\mu} \chi_I(\tau_i(t)) P_i(t)\right| + \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} P_{ij}(t)\right] \mathrm{d}t, \end{aligned} \tag{4.10}$$

or the matrix Λ_2 *given by* (4.3) *be nonsingular and* $r(S_2) < 1$, *where*

$$S_2 = \left|\sum_{i=1}^{\mu} \chi_I(\tau_i) P_i\right|_L + \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |P_{ij}|_L + |\Lambda_2^{-1}| \sum_{k=1}^{\nu}\sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |A_k| \int_{t_0}^{t_k} P_{ij}(t_0, t)\,\mathrm{d}t. \tag{4.11}$$

Then the problem (3.8), (2.4) *has a unique solution.*

Proof. According to Theorem 3.4, it is sufficient to show that if x is a solution of the problem (3.8_0), (2.4_0) then $x(t) \equiv 0$. Let x be a solution of boundary value problem (3.8_0), (2.4_0).

First, suppose that the matrix Λ_1 given by (4.1) is nonsingular and $r(S_1) < 1$, where the matrix S_1 is defined by (4.10). Analogously to the proof of Theorem 4.1 we get equation (4.9). Then the relation (4.9), in view of (4.7), implies

$$\begin{aligned} |x(t)| \le &\int_a^b |p(x)(s)|\,\mathrm{d}s + |\Lambda_1|^{-1} \sum_{k=1}^{\nu} |A_k| \int_{t_0}^{t_k} |p(x)(s)|\,\mathrm{d}s \le \\ \le &\left[\left|\sum_{i=1}^{\mu} \chi_I(\tau_i) P_i\right|_L + \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |P_{ij}|_L + \right. \\ &\left. + |\Lambda_1|^{-1} \sum_{k=1}^{\nu} |A_k| \int_{t_0}^{t_k} \left(\left|\sum_{i=1}^{\mu} \chi_I(\tau_i(s)) P_i(s)\right| + \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} P_{ij}(s)\right) \mathrm{d}s\right] |x|_C, \end{aligned}$$

i.e.,

$$|x|_C \leq S_1 |x|_C.$$

Since we suppose that $r(S_1) < 1$, we get $x(t) \equiv 0$.

Now suppose that the matrix Λ_2 given by (4.3) is nonsingular, and $r(S_2) < 1$, where the matrix S_2 is defined by (4.11). Analogously to the proof of Theorem 4.1 we derive the equality (4.9_1). Then, the relation (4.9_1), in view of (4.6) - (4.7), yields

$$\begin{aligned}|x(t)| &\leq \int_a^b |p(x)(s)|\,\mathrm{d}s + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} |A_k| \int_{t_0}^{t_k} \left| p\Big(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi \Big)(s) \right| \mathrm{d}s \leq \\ &\leq \Bigg[\left| \sum_{i=1}^{\mu} \chi_I(\tau_i) P_i \right|_L + \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} |P_{ij}|_L + \\ &\qquad + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} |A_k| \int_{t_0}^{t_k} P_{ij}(t_0, s)\,\mathrm{d}s \Bigg] |x|_C = \\ &= S_2 |x|_C.\end{aligned}$$

Whence, together with the assumption $r(S_2) < 1$, we get $x(t) \equiv 0$. □

Remark. In the case where signs of the matrices P_i, $i = 1, \dots, \mu$ are not the same, the criterion of the Theorem 4.1 is more suitable than the criterion of the Theorem 4.2.

Example 4.2. *Let $\mu = 2$ and*

$$x'(t) = P_1(t)x(\tau_1(t)) + P_2(t)x(\tau_2(t)) + q_0(t),$$

where $P_2(t) = -\frac{1}{2}P_1(t)$ and τ_1, τ_2 are arbitrary deviating arguments.

Since

$$|\chi_I(\tau_1(t))P_1(t) - \frac{1}{2}\chi_I(\tau_2(t))P_1(t)| \leq [\chi_I(\tau_1(t)) + \frac{1}{2}\chi_I(\tau_2(t))]\ |P_1(t)|$$

we get $S_k^2 \leq S_k^1, k = 1, 2$, where S_k^2 are matrices given in Theorem 4.2 and S_k^1 are matrices given in Theorem 4.1. Therefore, $r(S_k^2) \leq r(S_k^1)$ for $k = 1, 2$.

For integral problem (3.8), (2.5) we get from Theorem 3.4 the following theorems.

Theorem 4.3. *Let $P_i \in L(I, \mathbb{R}^{n\times n}), \tau_i : I \to \mathbb{R}$ be measurable functions $(i = 1, \dots, \mu)$ and $A \in L(I, \mathbb{R}^{n\times n})$. Let either*

$$\Lambda_1 = \int_a^b A(t)\,\mathrm{d}t \tag{4.12}$$

be nonsingular matrix and $r(S_1) < 1$, where

$$S_1 = \sum_{i=1}^{\mu} |\chi_I(\tau_i)P_i|_L + |\Lambda_1^{-1}| \sum_{i=1}^{\mu} \int_a^b |A(t)| \int_a^t \chi_I(\tau_i(s))|P_i(s)|\,\mathrm{d}s\,\mathrm{d}t, \tag{4.13}$$

or

$$\Lambda_2 = \Lambda_1 + \sum_{i=1}^{\mu} \int_a^b A(t) \int_a^t \chi_I(\tau_i(s))P_i(s)\,\mathrm{d}s\,\mathrm{d}t \tag{4.14}$$

be nonsingular matrix and $r(S_2) < 1$, where

$$\begin{aligned} S_2 = {} & \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |P_{ij}(a,\cdot)|_L + |\Lambda_2^{-1}| \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} \int_a^b |A(t)| \int_a^t P_{ij}(a,s)\,\mathrm{d}s\,\mathrm{d}t + \\ & + \left|\sum_{i=1}^{\mu} \chi_I(\tau_i)P_i\right|_L |\Lambda_2^{-1}| \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} \int_a^b |A(t)| \int_a^t P_{ij}(a,s)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{4.15}$$

Then the problem (3.8), (2.5) *has a unique solution.*

Proof. According to Theorem 3.4, it is sufficient to show that if x is a solution of the equation (3.8_0), satisfying boundary condition

$$\int_a^b A(t)x(t)\,\mathrm{d}t = 0, \tag{2.5_0}$$

then $x(t) \equiv 0$.

Let x be a solution of boundary value problem (3.8_0), (2.5_0). Analogously to the proof of Theorem 4.1 it can be shown that relations (4.8) and (4.8_1) with $t_0 = a$ are fulfilled. If the matrix Λ_1 given by (4.12) is nonsingular then from (2.5_0) and (4.8) we get

$$0 = \int_a^b A(t)\left[c + \int_a^t p(x)(s)\,\mathrm{d}s\right]\mathrm{d}t,$$

where $c = x(a)$, i.e.,

$$c = -\Lambda_1^{-1} \int_a^b A(t) \int_a^t p(x)(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Therefore, by virtue of (4.5), it follows from from (4.8) that

$$|x(t)| \le \left|\int_a^t p(x)(s)\,\mathrm{d}s\right| + |\Lambda_1^{-1}| \int_a^b |A(t)| \int_a^t |p(x)(s)|\,\mathrm{d}s\,\mathrm{d}t \le$$
$$\le \left[\sum_{i=1}^{\mu} |\chi_I(\tau_i)P_i|_L + |\Lambda_1^{-1}| \sum_{i=1}^{\mu} \int_a^b |A(t)| \int_a^t \chi_I(\tau_i(s))|P_i(s)|\,\mathrm{d}s\,\mathrm{d}t\right] |x|_C$$
$$= S_1|x|_C.$$

Consequently $|x|_C \le S_1|x|_C$. Since we suppose in this case that $r(S_1) < 1$, we get $x(t) \equiv 0$.

If the matrix Λ_2 given by (4.14) is nonsingular, then from of (2.5_0) and (4.8_1) we get

$$0 = \int_a^b A(t)\left[E + \int_a^t p(E)(s)\,\mathrm{d}s\right]\mathrm{d}t\ c +$$
$$+ \int_a^b A(t) \int_a^t p\Big(\int_a^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\,\mathrm{d}s\,\mathrm{d}t =$$
$$= \Lambda_2\, c + \int_a^b A(t) \int_a^t p\Big(\int_a^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\,\mathrm{d}s\,\mathrm{d}t,$$

i.e.,

$$c = -\Lambda_2^{-1} \int_a^b A(t) \int_a^t p\Big(\int_a^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Therefore, by virtue of (4.5) and (4.6), it follows from (4.8_1) that

$$|x(t)| \le \int_a^t \left|p\Big(\int_a^{\cdot} |p(x)(\xi)|\,\mathrm{d}\xi\Big)(s)\right| \mathrm{d}s +$$
$$+ |\Lambda_2^{-1}| \int_a^b |A(t)| \int_a^t \left|p\Big(\int_a^{\cdot} |p(x)(\xi)|\,\mathrm{d}\xi\Big)(s)\right| \mathrm{d}s\,\mathrm{d}t +$$
$$+ \left|\int_a^t p(E)(s)\,\mathrm{d}s\right| |\Lambda_2^{-1}| \int_a^b |A(t)| \int_a^t \left|p\Big(\int_a^{\cdot} |p(x)(\xi)|\,\mathrm{d}\xi\Big)(s)\right| \mathrm{d}s\,\mathrm{d}t \le$$
$$\le \Bigg[\sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |P_{ij}(a,\cdot)|_L + |\Lambda_2^{-1}| \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} \int_a^b |A(t)| \int_a^t P_{ij}(a,s)\,\mathrm{d}s\,\mathrm{d}t +$$
$$+ \left|\sum_{i=1}^{\mu} \chi_I(\tau_i)P_i\right|_L |\Lambda_2^{-1}| \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} \int_a^b |A(t)| \int_a^t P_{ij}(a,s)\,\mathrm{d}s\,\mathrm{d}t\Bigg] |x|_C$$
$$= S_2|x|_C.$$

Consequently $|x|_C \le S_2|x|_C$. Since we suppose in this case that $r(S_2) < 1$, we get $x(t) \equiv 0$. □

Theorem 4.4. *Let* $P_i \in L(I, \mathbb{R}^{n\times n}), \tau_i : I \to \mathbb{R}$ *be measurable functions* $(i = 1, \dots, \mu)$ *and* $A \in L(I, \mathbb{R}^{n\times n})$. *Let either the matrix* Λ_1 *given by* (4.12) *be nonsingular matrix and* $r(S_1) < 1$, *where*

$$S_1 = \left|\sum_{i=1}^{\mu} \chi_I(\tau_i) P_i\right|_L + \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |P_{ij}|_L + \\ + |\Lambda_1^{-1}| \int_a^b |A(t)| \int_a^t \left[\left|\sum_{i=1}^{\mu} \chi_I(\tau_i(s)) P_i(s)\right| + \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} P_{ij}(s)\right] \mathrm{d}s\, \mathrm{d}t, \tag{4.16}$$

or the matrix Λ_2 *given by* (4.14) *be nonsingular and* $r(S_2) < 1$, *where*

$$S_2 = \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |P_{ij}(a,\cdot)|_L + |\Lambda_2^{-1}| \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} \int_a^b |A(t)| \int_a^t P_{ij}(a,s)\, \mathrm{d}s\, \mathrm{d}t + \\ + \left[\left|\sum_{i=1}^{\mu} \chi_I(\tau_i) P_i\right|_L + \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |P_{ij}|_L\right] |\Lambda_2^{-1}| \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} \int_a^b |A(t)| \int_a^t P_{ij}(a,s)\, \mathrm{d}s\, \mathrm{d}t. \tag{4.17}$$

Then the problem (3.8), (2.5) *has a unique solution.*

Proof. The proof is analogous to the proof of Theorem 4.3, but inequality (4.7) should be used instead of inequality (4.5). □

Remark. Analogous criteria for multi-point boundary value problem for linear systems of ordinary differential equations are published in [7] (see Theorem 4.2 and Corollary 4.2). The assertions are identical for nonsingular Λ_1 while the results for nonsingular Λ_2 are new.

Theorems given here for linear systems of differential equations with deviating argument correspond to Corollary 1.3.3 and Corollary 1.3.11 in [8] in the case when the matrix Λ_1 is nonsingular. The results with nonsingular Λ_2 have been published only for a periodic boundary value problem.

Remark. Note also that the criteria for the solvability of the problems (3.8), (2.5) and (2.4) are derived from the inequalities (4.8_1) and (4.8), respectively. It is clear that other effective criteria for the solvability of the problems indicated can be get from (4.8) for the problem (3.8), (2.5) and from (4.8_1) for the problem (3.8), (2.4).

4.3. Existence and Uniqueness Theorems II

In this part, we specify coefficients of the matrices P_i and deviations τ_i $(i = 1, \dots, \nu)$ to get some other special criteria of the unique solvability of the problems (3.8), (2.4) and (3.8), (2.5).

We introduce now two theorems and two corollaries about solvability of the multi-point boundary value problem (3.8), (2.4).

Theorem 4.1^a. *Let, for every $i \in \{1, \dots, \mu\}$, $P_i \in L^p(I, \mathbb{R}^{n\times n})$, where $1 \le p \le +\infty$, $\tau_i : I \to \mathbb{R}$ be measurable functions and $t_0 = \min\{t_j : j = 1, 2, \dots, \nu\}$. Let either the matrix Λ_1 given by* (4.1) *be nonsingular and $r(S_1) < 1$, where*

$$S_1 = (b-a)^{\frac{1}{q}}\left[E + |\Lambda_1^{-1}|\sum_{k=1}^{\nu}|A_k|\right]\sum_{i=1}^{\mu}|\chi_I(\tau_i)P_i|_{L^p} \tag{4.2a}$$

or the matrix Λ_2 given by (4.3) *be nonsingular and $r(S_2) < 1$, where*

$$\begin{aligned} S_2 =& (b-a)^{\frac{1}{q}}\sum_{i=1}^{\mu}|\chi_I(\tau_i)P_i|_{L^p}+ \\ &+ (b-a)^{\frac{2}{q}}|\Lambda_2^{-1}|\sum_{k=1}^{\nu}|A_k|\sum_{i=1}^{\mu}\sum_{j=1}^{\mu}|\chi_I(\tau_i)P_i|_{L^p}|\chi_I(\tau_j)P_j|_{L^p}, \end{aligned} \tag{4.4a}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then the problem (3.8), (2.4) *has a unique solution.*

Remark. If we use more precise estimates for matrices $S_i, i = 1, 2$, we can find criterion for unique solvability of the problem (3.8), (2.4) with matrices

$$S_1 = \left[(b-a)^{\frac{1}{q}}E + |\Lambda_1^{-1}|\sum_{k=1}^{\nu}|A_k|(t_k - t_0)^{\frac{1}{q}}\right]\sum_{i=1}^{\mu}|\chi_I(\tau_i)P_i|_{L^p}$$

and

$$\begin{aligned} S_2 =& (b-a)^{\frac{1}{q}}\sum_{i=1}^{\mu}|\chi_I(\tau_i)P_i|_{L^p}+ \\ &+ (b-a)^{\frac{1}{q}}|\Lambda_2^{-1}|\sum_{k=1}^{\nu}(t_k - t_0)^{\frac{1}{q}}|A_k|\sum_{i=1}^{\mu}\sum_{j=1}^{\mu}|\chi_I(\tau_i)P_i|_{L^p}|\chi_I(\tau_j)P_j|_{L^p}. \end{aligned}$$

This process we use for derivation of following theorems.

Theorem 4.1^b. *Let* $1 \le q_0 \le +\infty$*, for every* $i \in \{1,\dots,\mu\}$*,* $P_i \in L(I,\mathbb{R}^{n\times n})$*,* $\tau_i \in \widetilde{C}(I,\mathbb{R})$ *be monotone functions,* $t_0 = \min\{t_j : j = 1,2,\dots,\nu\}$ *and let there exist* $B_i \in \mathbb{R}_+^{n\times n}$ *such that*

$$\chi_I(\tau_i(t))|P_i(t)| \le B_i|\tau_i'(t)|^{\frac{1}{q_0}} \text{ almost everywhere on } I.$$

Let either the matrix Λ_1 *given by* (4.1) *be nonsingular and* $r(S_1) < 1$*, where*

$$S_1 = (b-a)^{\frac{1}{p_0}} \sum_{i=1}^{\mu} m_i^{\frac{1}{q_0}} B_i + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{\mu} (t_k - t_0)^{\frac{1}{p_0}} m_i^{\frac{1}{q_0}} |A_k| B_i \tag{4.2b}$$

or the matrix Λ_2 *given by* (4.3) *be nonsingular and* $r(S_2) < 1$*, where*

$$\begin{aligned} S_2 =&(b-a)^{\frac{1}{p_0}} \sum_{i=1}^{\mu} m_i^{\frac{1}{q_0}} B_i + \\ &+ |\Lambda_2^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} (b-a)^{\frac{1}{p_0}} (t_k - t_0)^{\frac{1}{p_0}} m_i^{\frac{1}{q_0}} m_j^{\frac{1}{q_0}} |A_k| B_i B_j, \end{aligned} \tag{4.4b}$$

where $p_0 \ge 1, \frac{1}{p_0} + \frac{1}{q_0} = 1, m_i = \operatorname{mes}(\tau_i(I) \cap I), i = 1,\dots,\mu$*. Then the problem* (3.8)*,* (2.4) *has a unique solution.*

For $q_0 = +\infty$ resp. $q_0 = 1$ we get directly from Theorem 4.1^b the following corollaries.

Corollary 4.1. *Let, for every* $i \in \{1,\dots,\mu\}$*,* $P_i \in L(I,\mathbb{R}^{n\times n})$*,* $\tau_i \in \widetilde{C}(I,\mathbb{R})$ *be monotone functions,* $t_0 = \min\{t_j : j = 1,2,\dots,\nu\}$ *and let there exist* $B_i \in \mathbb{R}_+^{n\times n}$ *such that*

$$\chi_I(\tau_i(t))|P_i(t)| \le B_i \text{ almost everywhere on } I.$$

Let either the matrix Λ_1 *given by* (4.1) *be nonsingular and* $r(S_1) < 1$*, where*

$$S_1 = (b-a) \sum_{i=1}^{\mu} B_i + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{\mu} (t_k - t_0)|A_k| B_i$$

or the matrix Λ_2 *given by* (4.3) *be nonsingular and* $r(S_2) < 1$*, where*

$$S_2 = (b-a) \sum_{i=1}^{\mu} B_i + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} (b-a)(t_k - t_0)|A_k| B_i B_j.$$

Then the problem (3.8)*,* (2.4) *has a unique solution.*

Corollary 4.2. *Let, for every* $i \in \{1,\dots,\mu\}$, $P_i \in L(I,\mathbb{R}^{n\times n})$, $\tau_i \in \widetilde{C}(I,\mathbb{R})$ *be monotone functions,* $t_0 = \min\{t_j : j = 1,2,\dots,\nu\}$ *and let there exist* $B_i \in \mathbb{R}_+^{n\times n}$ *such that*

$$\chi_I(\tau_i(t))|P_i(t)| \le B_i|\tau_i'(t)| \text{ almost everywhere on } I.$$

Let either the matrix Λ_1 *given by* (4.1) *be nonsingular and* $r(S_1) < 1$, *where*

$$S_1 = \sum_{i=1}^{\mu} m_i B_i + |\Lambda_1^{-1}| \sum_{k=1}^{\nu}\sum_{i=1}^{\mu} m_i |A_k| B_i,$$

or the matrix Λ_2 *given by* (4.3) *be nonsingular and* $r(S_2) < 1$, *where*

$$S_2 = \sum_{i=1}^{\mu} m_i B_i + |\Lambda_2^{-1}| \sum_{k=1}^{\nu}\sum_{i=1}^{\mu}\sum_{j=1}^{\mu} m_i m_j |A_k| B_i B_j,$$

where $m_i = \mathrm{mes}(\tau_i(I)\cap I), i = 1,\dots,\mu$. *Then the problem* (3.8), (2.4) *has a unique solution.*

To prove the previous theorems we need the following lemma.

Lemma 4.2. *Let, for every* $i \in \{1,\dots,\mu\}$, P_i, τ_i, B_i *fulfil successively the assumptions of Theorem 4.1*^{a} *and Theorem 4.1*^{b}. *Then, for every* $i = 1,\dots,\mu$, $x \in \widetilde{C}(I,\mathbb{R}^n)$ *and an arbitrary* $t \in I$, *the following conditions hold*

a)

$$\left|\int_{t_0}^{t} \chi_I(\tau_i(s))P_i(s)x(\tau_i^0(s))\,\mathrm{d}s\right| \le |t-t_0|^{\frac{1}{q}} |\chi_I(\tau_i)P_i|_{L^p}|x|_C. \tag{4.18}$$

b)

$$\left|\int_{t_0}^{t} \chi_I(\tau_i(s))P_i(s)x(\tau_i^0(s))\,\mathrm{d}s\right| \le |t-t_0|^{\frac{1}{p_0}} m_i^{\frac{1}{q_0}} B_i|x|_C, \tag{4.19}$$

where $m_i = \mathrm{mes}(\tau_i(I)\cap I)$.

Proof. For every $i \in \{1,\dots,\mu\}$ put $P_i = P, \tau_i = \tau, B_i = B$. Then, with the use of Hölder inequality, we get

a)

$$\left|\int_{t_0}^{t} \chi_I(\tau(s))P(s)x(\tau^0(s))\,\mathrm{d}s\right| \le \left|\int_{t_0}^{t} \chi_I(\tau(s))|P(s)|^p\,\mathrm{d}s\right|^{\frac{1}{p}} \left|\int_{t_0}^{t} |x(\tau^0(s))|^q\,\mathrm{d}s\right|^{\frac{1}{q}} \le$$

$$\le |t-t_0|^{\frac{1}{q}}|\chi_I(\tau)P|_{L^p}|x|_C.$$

b)

$$\left|\int_{t_0}^{t} \chi_I(\tau(s))P(s)x(\tau^0(s))\,\mathrm{d}s\right| \le B\left|\int_{t_0}^{t} \chi_I(\tau(s))|x(\tau^0(s))||\tau'(s)|^{\frac{1}{q_0}}\,\mathrm{d}s\right| \le$$

$$\le B|t-t_0|^{\frac{1}{p_0}}\left|\int_{t_0}^{t} \chi_I(\tau(s))|x(\tau^0(s))|^{q_0}|\tau'(s)|\,\mathrm{d}s\right|^{\frac{1}{q_0}} \le$$

$$\le B|t-t_0|^{\frac{1}{p_0}}\left|\int_{\tau^0(t_0)}^{\tau^0(t)} |x(s)|^{q_0}\,\mathrm{d}s\right|^{\frac{1}{q_0}} \le |t-t_0|^{\frac{1}{p_0}}[\mathrm{mes}(\tau(I)\cap I)]^{\frac{1}{q_0}}B|x|_C.$$

□

Proof of Theorem 4.1^a. According to Theorem 3.4, it is sufficient to show that if x is a solution of the problem (3.8_0), (2.4_0) then $x(t) \equiv 0$.

Let x be a solution of the boundary value problem (3.8_0), (2.4_0).

First, suppose that the matrix Λ_1 given by (4.1) is nonsingular and $r(S_1) < 1$, where the matrix S_1 is defined by (4.2a). Analogously to the proof of Theorem 4.1 we get equation (4.9). Then the relation (4.9), in view of (4.18), implies

$$|x(t)| \le \left|\int_{t_0}^{t} \sum_{i=1}^{\mu} \chi_I(\tau_i(s))P_i(s)x(\tau_i^0(s))\,\mathrm{d}s\right| +$$

$$+\left|\Lambda_1^{-1}\sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} \sum_{i=1}^{\mu} \chi_I(\tau_i(s))P_i(s)x(\tau_i^0(s))\,\mathrm{d}s\right| \le$$

$$\le (b-a)^{\frac{1}{q}}\sum_{i=1}^{\mu} |\chi_I(\tau_i)P_i|_{L^p}|x|_C + |\Lambda_1^{-1}|\sum_{k=1}^{\nu}|A_k|\sum_{i=1}^{\mu}(b-a)^{\frac{1}{q}}|\chi_I(\tau_i)P_i|_{L^p}|x|_C =$$

$$= (b-a)^{\frac{1}{q}}\left[E + |\Lambda_1^{-1}|\sum_{k=1}^{\nu}|A_k|\right]\sum_{i=1}^{\mu}|\chi_I(\tau_i)P_i|_{L^p}|x|_C,$$

i.e.,

$$|x|_C \le S_1|x|_C.$$

Since we suppose that $r(S_1) < 1$, we get $x(t) \equiv 0$.

Suppose now that the matrix Λ_2 given by (4.3) is nonsingular and $r(S_2) < 1$, where the matrix S_2 is defined by (4.4a). Analogously to the proof of Theorem 4.1

we derive the equality (4.9_1). Then the relation (4.9_1), in view of (4.18), yields

$$
\begin{aligned}
|x(t)| \le &\left|\int_{t_0}^{t}\sum_{i=1}^{\mu}\chi_I(\tau_i(s))P_i(s)x(\tau_i^0(s))\,\mathrm{d}s\right| + \\
&+\left|\Lambda_2^{-1}\sum_{k=1}^{\nu}A_k\int_{t_0}^{t_k}\sum_{i=1}^{\mu}\sum_{j=1}^{\mu}\chi_I(\tau_i(t))P_i(t)\int_{t_0}^{\tau_i^0(t)}\chi_I(\tau_j(s))P_j(s)x(\tau_j^0(s))\,\mathrm{d}s\,\mathrm{d}t\right| \le \\
&\le (b-a)^{\frac{1}{q}}\sum_{i=1}^{\mu}|\chi_I(\tau_i)P_i|_{L^p}|x|_C + \\
&\quad + (b-a)^{\frac{2}{q}}|\Lambda_2^{-1}|\sum_{k=1}^{\nu}|A_k|\sum_{i=1}^{\mu}\sum_{j=1}^{\mu}|\chi_I(\tau_i)P_i|_{L^p}|\chi_I(\tau_j)P_j|_{L^p}|x|_C,
\end{aligned}
$$

whence, together with the assumption $r(S_2) < 1$, we get $x(t) \equiv 0$. □

Proof of Theorem 4.1^b. The proof is analogous to the proof of Theorem 4.1^a, but inequality (4.19) should be used instead of inequality (4.18). □

We get another type of criterion for the solvability of problem (3.8), (2.4) using the Levin inequality.

Theorem 4.1^c. *Let, for every $i \in \{1,\dots,\mu\}$, $P_i \in L^p(I,\mathbb{R}^{n\times n})$, where $1 < p < +\infty$, $\tau_i \in \widetilde{C}(I,\mathbb{R})$ be monotone functions, $\tau_i^* = \operatorname{vrai\,min}\{|\tau_i'(t)| : t \in I\} > 0$ and $t_0 = \min\{t_j : j = 1,2,\dots,\nu\}$. Let either the matrix Λ_1 given by* (4.1) *be nonsingular and $r(S_1) < 1$, where*

$$
S_1 = \sum_{i=1}^{\mu}\frac{(b-a)^{\frac{1}{q}}}{\sqrt[q_0]{\tau_i^*}\,l(\frac{q_0}{q})}|\chi_I(\tau_i)P_i|_{L^p} + (b-a)^{\frac{1}{q_0}}|\Lambda_1^{-1}|\sum_{k=1}^{\nu}\sum_{i=1}^{\mu}|A_k||\chi_I(\tau_i)P_i|_{L^{p_0}} \tag{4.2e}
$$

or the matrix Λ_2 given by (4.3) *be nonsingular and $r(S_2) < 1$, where*

$$
\begin{aligned}
S_2 = &\sum_{i=1}^{\mu}\frac{(b-a)^{\frac{1}{q}}}{\sqrt[q_0]{\tau_i^*}\,l(\frac{q_0}{q})}|\chi_I(\tau_i)P_i|_{L^p} + \\
&+ (b-a)^{\frac{2}{q_0}}|\Lambda_2^{-1}|\sum_{k=1}^{\nu}\sum_{i=1}^{\mu}\sum_{j=1}^{\mu}|A_k||\chi_I(\tau_i)P_i|_{L^{p_0}}|\chi_I(\tau_j)P_j|_{L^{p_0}},
\end{aligned} \tag{4.4e}
$$

where

$$
l\Big(\frac{q_0}{q}\Big) = \begin{cases} (\frac{q_0}{q}-1)^{\frac{1}{q_0}}(\frac{q_0}{q\pi}\sin\frac{q\pi}{q_0})^{-\frac{1}{q}} & 1 \le q < q_0 < \infty \\ 1 & 1 \le q = q_0 \text{ or } q_0 = \infty, \end{cases}
$$

$q \le q_0$, $\frac{1}{p}+\frac{1}{q} = 1$, $\frac{1}{p_0}+\frac{1}{q_0} = 1$. *Then the problem* (3.8), (2.4) *has a unique solution.*

To prove Theorem 4.1^c we need the following lemma.

Lemma 4.3. *Let, for every* $i \in \{1,\dots,\mu\}$, $P_i, \tau_i, p, q, p_0, q_0$ *fulfil the assumptions of Theorem 4.1*^{c} *and* $\tau_i^* = \operatorname{vrai\,min}\{|\tau_i'(t)| : t \in I\} > 0$. *Then*

$$\left| \int_{t_0}^{\cdot} \chi_I(\tau_i(s)) P_i(s) x(\tau_i^0(s))\,\mathrm{d}s \right|_{L^{q_0}} \le \frac{(b-a)^{\frac{1}{q}}}{\sqrt[q_0]{\tau_i^*}\, l(\frac{q_0}{q})} |\chi_I(\tau_i) P_i|_{L^p} |x|_{L^{q_0}}, \tag{4.20}$$

where

$$l\Big(\frac{q_0}{q}\Big) = \begin{cases} (\frac{q_0}{q} - 1)^{\frac{1}{q_0}} (\frac{q_0}{q\pi} \sin\frac{q\pi}{q_0})^{-\frac{1}{q}} & 1 \le q < q_0 < \infty \\ 1 & 1 \le q = q_0 \text{ or } q_0 = \infty. \end{cases}$$

Proof. With the use of Hölder and Levin inequalities we get for every $i \in \{1,\dots,\mu\}$

$$\begin{aligned}
&\left| \int_{t_0}^{\cdot} \chi_I(\tau_i(s)) P_i(s) x(\tau_i^0(s))\,\mathrm{d}s \right|_{L^{q_0}} \le \\
&\le \left[\int_a^b \left| \int_{t_0}^t \chi_I(\tau_i(s)) |P_i(s)| |x(\tau_i^0(s))|\,\mathrm{d}s \right|^{q_0} \mathrm{d}t \right]^{\frac{1}{q_0}} \le \\
&\le \left[\int_a^b \left(\left| \int_{t_0}^t \chi_I(\tau_i(s)) |P_i(s)|^p\,\mathrm{d}s \right|^{\frac{1}{p}} \left| \int_{t_0}^t \chi_I(\tau_i(s)) |x(\tau_i^0(s))|^q\,\mathrm{d}s \right|^{\frac{1}{q}} \right)^{q_0} \mathrm{d}t \right]^{\frac{1}{q_0}} \le \\
&\le |\chi_I(\tau_i) P_i|_{L^p} \left[\left(\int_a^b \left| \int_{t_0}^t \chi_I(\tau_i(s)) |x(\tau_i^0(s))|^q\,\mathrm{d}s \right|^{\frac{q_0}{q}} \mathrm{d}t \right)^{\frac{q}{q_0}} \right]^{\frac{1}{q}} \le \\
&\le |\chi_I(\tau_i) P_i|_{L^p} (b-a)^{\frac{1}{q}} l^{-1}\Big(\frac{q_0}{q}\Big) \left(\int_a^b \chi_I(\tau_i(s)) |x(\tau_i^0(t))|^{q_0}\,\mathrm{d}t \right)^{\frac{1}{q_0}} \le \\
&\le |\chi_I(\tau_i) P_i|_{L^p} (b-a)^{\frac{1}{q}} l^{-1}\Big(\frac{q_0}{q}\Big) \left(\int_a^b \chi_I(\tau_i(s)) |x(\tau_i^0(t))|^{q_0} \frac{|\tau_i'(t)|}{\tau_i^*}\,\mathrm{d}t \right)^{\frac{1}{q_0}} \le \\
&\le \frac{(b-a)^{\frac{1}{q}}}{\sqrt[q_0]{\tau_i^*}\, l(\frac{q_0}{q})} |\chi_I(\tau_i) P_i|_{L^p} |x|_{L^{q_0}}.
\end{aligned}$$

□

Proof of Theorem 4.1^c. The proof is analogous to the proof of Theorem 4.1^a, but inequality (4.20) should be used instead of inequality (4.18). □

The following theorem is deduced with the use of similar inequalities as in Corollary 4.1. This theorem is more suitable for problems with different signs of the matrix functions P_i (see Example 4.2).

Theorem 4.2[a]. *Let, for every $i \in \{1, \dots, \mu\}$, $P_i \in L(I, \mathbb{R}^{n\times n}), \tau_i : I \to \mathbb{R}$ be measurable function, $t_0 = \min\{t_k : k = 1, 2, \dots, \nu\}$ and let there exist $B, B_i \in \mathbb{R}^{n\times n}$ such that*

$$\left|\sum_{i=1}^{\nu} \chi_I(\tau_i(t))P_i(t)\right| \le B,\ \chi_I(\tau_i(t))|P_i(t)| \le B_i \text{ almost everywhere on } I.$$

Let either the matrix Λ_1 given by (4.1) *be nonsingular and $r(S_1) < 1$, where*

$$S_1 = (b-a)B + |\Lambda_1^{-1}| \sum_{k=1}^{\nu}\sum_{i=1}^{\mu}(t_k - t_0)|A_k|B_i \tag{4.10a}$$

or the matrix Λ_2 given by (4.3) *be nonsingular and $r(S_2) < 1$, where*

$$S_2 = (b-a)B + |\Lambda_2^{-1}| \sum_{k=1}^{\nu}\sum_{i=1}^{\mu}\sum_{j=1}^{\mu}(b-a)(t_k - t_0)|A_k|B_iB_j. \tag{4.11a}$$

Then the problem (3.8), (2.4) *has a unique solution.*

Remark. We can get, from above introduced theorems with $A_k = \lambda_k E, k = 1, 2, \dots, \nu$, criteria for solvability of the problem (3.8) with the boundary condition (2.6).

The following theorems deal with the problem of the unique solvability of system (3.8) with general integral boundary condition (2.5). Proofs are similar to the proof of Theorem 4.3. The inequalities from Lemma 4.2 and Lemma 4.3 are used suitably according to assumptions of each theorem.

Theorem 4.3[a]. *Let, for every $i \in \{1, \dots, \mu\}$, $P_i \in L^p(I, \mathbb{R}^{n\times n})$, where $1 \le p \le +\infty$, $\tau_i : I \to \mathbb{R}$ be measurable functions and $A \in L(I, \mathbb{R}^{n\times n})$. Let either the matrix Λ_1 given by* (4.12) *be nonsingular and $r(S_1) < 1$, where*

$$S_1 = (b-a)^{\frac{1}{q}}\left[E + |\Lambda_1^{-1}||A|_L\right]\sum_{i=1}^{\mu}|\chi_I(\tau_i)P_i|_{L^p}, \tag{4.13a}$$

or the matrix Λ_2 given by (4.14) *be nonsingular and $r(S_2) < 1$, where*

$$\begin{aligned} S_2 =& (b-a)^{\frac{1}{q}}\sum_{i=1}^{\mu}|\chi_I(\tau_i)P_i|_{L^p} + \\ &+ (b-a)^{\frac{2}{q}}|\Lambda_2^{-1}||A|_L\sum_{i=1}^{\mu}\sum_{j=1}^{\mu}|\chi_I(\tau_i)P_i|_{L^p}|\chi_I(\tau_j)P_j|_{L^p}, \end{aligned} \tag{4.15a}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then the problem (3.8), (2.5) *has a unique solution.*

Remark. If we use more precise estimates for matrices $S_i, i = 1, 2$, we can find criterion for unique solvability of the problem (3.8), (2.5) with matrices

$$S_1 = \left[(b-a)^{\frac{1}{q}} E + |\Lambda_1^{-1}| \int_a^b |A(t)|(t-a)^{\frac{1}{q}}\, \mathrm{d}t\right] \sum_{i=1}^{\mu} |\chi_I(\tau_i) P_i|_{L^p}$$

and

$$S_2 = (b-a)^{\frac{1}{q}} \left[E + |\Lambda_2^{-1}| \int_a^b |A(t)|(t-a)^{\frac{1}{q}}\, \mathrm{d}t \sum_{j=1}^{\mu} |\chi_I(\tau_j) P_j|_{L^p}\right] \sum_{i=1}^{\mu} |\chi_I(\tau_i) P_i|_{L^p}.$$

Theorem 4.3^b. *Let $1 \le q_0 \le +\infty$, for every $i \in \{1, \dots, \mu\}$, $P_i \in L(I, \mathbb{R}^{n\times n})$, $\tau_i \in \widetilde{C}(I, \mathbb{R})$ be monotone functions, $A \in L(I, \mathbb{R}^{n\times n})$ and let there exist $B_i \in \mathbb{R}_+^{n\times n}$ such that*

$$\chi_I(\tau_i(t))|P_i(t)| \le B_i |\tau_i'(t)|^{\frac{1}{q_0}} \text{ almost everywhere on } I.$$

Let either the matrix Λ_1 given by (4.12) be nonsingular and $r(S_1) < 1$, where

$$S_1 = (b-a)^{\frac{1}{p_0}} \left[E + |\Lambda_1^{-1}||A|_L\right] \sum_{i=1}^{\mu} m_i^{\frac{1}{q_0}} B_i, \tag{4.13b}$$

or the matrix Λ_2 given by (4.14) be nonsingular and $r(S_2) < 1$, where

$$\begin{aligned} S_2 =& (b-a)^{\frac{1}{p_0}} \sum_{i=1}^{\mu} m_i^{\frac{1}{q_0}} B_i + \\ &+ (b-a)^{\frac{2}{p_0}} |\Lambda_2^{-1}| \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} m_i^{\frac{1}{q_0}} m_j^{\frac{1}{q_0}} |A|_L B_i B_j, \end{aligned} \tag{4.15b}$$

where $\frac{1}{p_0} + \frac{1}{q_0} = 1, m_i = \mathrm{mes}(\tau_i(I) \cap I), i = 1, \dots, \mu$. Then the problem (3.8), (2.5) has a unique solution.

Remark. If we use more precise estimates for matrices $S_i, i = 1, 2$, we can find criterion for unique solvability of the problem (3.8), (2.5) with matrices

$$S_1 = \left[(b-a)^{\frac{1}{p_0}} E + |\Lambda_1^{-1}| \int_a^b |A(t)|(t-a)^{\frac{1}{p_0}}\, \mathrm{d}t\right] \sum_{i=1}^{\mu} m_i^{\frac{1}{q_0}} B_i$$

and

$$S_2 = (b-a)^{\frac{1}{p_0}} \left[E + |\Lambda_2^{-1}| \int_a^b |A(t)|(t-a)^{\frac{1}{p_0}}\, \mathrm{d}t \sum_{j=1}^{\mu} m_j^{\frac{1}{q_0}} B_j\right] \sum_{i=1}^{\mu} m_i^{\frac{1}{q_0}} B_i.$$

For $q_0 = +\infty$ resp. $q_0 = 1$ we get directly from Theorem 4.3^b the following corollaries.

Corollary 4.3. *Let, for every $i \in \{1,\dots,\mu\}$, $P_i \in L(I,\mathbb{R}^{n\times n})$, $\tau_i \in \widetilde{C}(I,\mathbb{R})$ be monotone functions, $A \in L(I,\mathbb{R}^{n\times n})$and let there exist $B_i \in \mathbb{R}_+^{n\times n}$ such that*

$$\chi_I(\tau_i(t))|P_i(t)| \le B_i \text{ almost everywhere on } I.$$

Let either the matrix Λ_1 given by (4.12) *be nonsingular and $r(S_1) < 1$, where*

$$S_1 = (b-a)\Big[E + |\Lambda_1^{-1}||A|_L\Big]\sum_{i=1}^{\mu} B_i,$$

or the matrix Λ_2 given by (4.14) *be nonsingular and $r(S_2) < 1$, where*

$$S_2 = (b-a)\sum_{i=1}^{\mu} B_i + (b-a)^2|\Lambda_2^{-1}|\sum_{i=1}^{\mu}\sum_{j=1}^{\mu}|A|_L B_i B_j.$$

Then the problem (3.8), (2.5) *has a unique solution.*

Corollary 4.4. *Let, for every $i \in \{1,\dots,\mu\}$, $P_i \in L(I,\mathbb{R}^{n\times n})$, $\tau_i \in \widetilde{C}(I,\mathbb{R})$ be monotone functions, $A \in L(I,\mathbb{R}^{n\times n})$ and let there exist $B_i \in \mathbb{R}_+^{n\times n}$ such that*

$$\chi_I(\tau_i(t))|P_i(t)| \le B_i|\tau_i'(t)| \text{ almost everywhere on } I.$$

Let either the matrix Λ_1 given by (4.12) *be nonsingular and $r(S_1) < 1$, where*

$$S_1 = \Big[E + |\Lambda_1^{-1}||A|_L\Big]\sum_{i=1}^{\mu} m_i B_i,$$

or the matrix Λ_2 given by (4.14) *be nonsingular and $r(S_2) < 1$, where*

$$S_2 = \sum_{i=1}^{\mu} m_i B_i + |\Lambda_2^{-1}|\sum_{i=1}^{\mu}\sum_{j=1}^{\mu} m_i m_j |A|_L B_i B_j,$$

where $m_i = \operatorname{mes}(\tau_i(I)\cap I), i = 1,\dots,\mu$. Then the problem (3.8), (2.5) *has a unique solution.*

Theorem 4.3^c. *Let, for every $i \in \{1,\dots,\mu\}$, $P_i \in L^p(I,\mathbb{R}^{n\times n})$, where $1 \le p \le +\infty$, $\tau_i \in \widetilde{C}(I,\mathbb{R})$ be monotone functions, $\tau_i^* = \operatorname{vrai\,min}\{|\tau_i'(t)| : t \in I\} > 0$ and $A \in$*

$L^{q_0}(I, \mathbb{R}^{n\times n})$. *Let either the matrix* Λ_1 *given by* (4.12) *be nonsingular and* $r(S_1) < 1$, *where*

$$S_1 = \sum_{i=1}^{\mu} \frac{(b-a)^{\frac{1}{q}}}{\sqrt[q_0]{\tau_i^* l(\frac{q_0}{q})}} |\chi_I(\tau_i)P_i|_{L^p} + (b-a)^{\frac{1}{q_0}} |\Lambda_1^{-1}||A|_{L^{q_0}} \sum_{i=1}^{\mu} |\chi_I(\tau_i)P_i|_{L^{p_0}} \qquad (4.13c)$$

or the matrix Λ_2 *given by* (4.14) *be nonsingular and* $r(S_2) < 1$, *where*

$$\begin{aligned} S_2 = & \sum_{i=1}^{\mu} \frac{(b-a)^{\frac{1}{q}}}{\sqrt[q_0]{\tau_i^* l(\frac{q_0}{q})}} |\chi_I(\tau_i)P_i|_{L^p} + \\ & + (b-a)^{\frac{2}{q_0}} |\Lambda_2^{-1}||A|_{L^{q_0}} \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |\chi_I(\tau_i)P_i|_{L^{p_0}} |\chi_I(\tau_j)P_j|_{L^{p_0}}, \end{aligned} \qquad (4.15c)$$

where $q \le q_0, \frac{1}{p} + \frac{1}{q} = 1, \frac{1}{p_0} + \frac{1}{q_0} = 1$. *Then the problem* (3.8), (2.5) *has a unique solution.*

The following theorem is more suitable for problems with different signs of the matrix functions P_i.

Theorem 4.4^a. *Let, for every* $i \in \{1, \dots, \mu\}$, $P_i \in L(I, \mathbb{R}^{n\times n}), \tau_i : I \to \mathbb{R}$ *be measurable function,* $A \in L(I, \mathbb{R}^{n\times n})$ *and let there exist* $B, B_i \in \mathbb{R}^{n\times n}$ *such that*

$$\left| \sum_{i=1}^{\nu} \chi_I(\tau_i(t))P_i(t) \right| \le B, \ \ \chi_I(\tau_i(t))|P_i(t)| \le B_i \text{ almost everywhere on } I.$$

Let either the matrix Λ_1 *given by* (4.12) *be nonsingular and* $r(S_1) < 1$, *where*

$$S_1 = (b-a)\left[B + |\Lambda_1^{-1}||A|_L \sum_{i=1}^{\mu} B_i \right] \qquad (4.16a)$$

or the matrix Λ_2 *given by* (4.14) *be nonsingular and* $r(S_2) < 1$, *where*

$$S_2 = (b-a)\left[B + (b-a)|\Lambda_2^{-1}||A|_L \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} B_iB_j \right]. \qquad (4.17a)$$

Then the problem (3.8), (2.5) *has a unique solution.*

Remark. We can get from above introduced theorems criteria for solvability of the problem (3.8) with boundary condition (2.9).

4.4. Cauchy and Periodic Boundary Value Problems

The following corollaries are deduced from Theorems 4.1 and 4.2. We can analogously get other criteria for the unique solvability of the Cauchy and periodic boundary value problems from Theorems 4.1^a - 4.1^c and 4.2^a.

Corollary 4.5. *Let* $r(S) < 1$, *where either*

$$S = \sum_{i=1}^{\mu} |\chi_I(\tau_i)P_i|_L \quad \text{or} \quad S = \left|\sum_{i=1}^{\mu} \chi_I(\tau_i)P_i\right|_L + \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |P_{ij}|_L.$$

Then the problem (3.8), (2.7) *has a unique solution.*

Proof. The validity of the corollary follows immediately from Theorems 4.1 and 4.2 with $\nu = 1, A_1 = E$ and $t_1 = t_0$ because, in this case, the matrix Λ_1 given by (4.1) is the unit matrix. □

Corollary 4.6. *Let*

$$\Lambda = \sum_{i=1}^{\mu} \int_a^b \chi_I(\tau_i(t))P_i(t)\,\mathrm{d}t$$

be a nonsingular matrix and let $r(S) < 1$, *where either*

$$S = \sum_{i=1}^{\mu} |\chi_I(\tau_i)P_i|_L + |\Lambda^{-1}| \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |P_{ij}(a,\cdot)|_L$$

or

$$S = \left|\sum_{i=1}^{\mu} \chi_I(\tau_i)P_i\right|_L + \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |P_{ij}|_L + |\Lambda^{-1}| \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |P_{ij}(a,\cdot)|_L.$$

Then the problem (3.8), (2.8) *has a unique solution.*

Proof. The validity of the corollary follows immediately from Theorems 4.1 and 4.2 with $\nu = 2, A_1 = -E, A_2 = E, t_0 = t_1 = a$ and $t_2 = b$, because, in this case, the matrices Λ_1 and Λ_2 given by (4.1) and (4.3), respectively, satisfy $\Lambda_1 = \Theta$ and $\Lambda_2 = \Lambda$. □

5. Multi-point Boundary Value Problem for Pantograph Equation

5.1. Statement of the Problem

For the sake of transparentness of the results, the interval $[0, T]$ was chosen as interval I in this chapter.

On the bounded interval $I = [0, T]$, consider the system of linear differential equations with deviating arguments

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{\mu} P_i(t)x(\tau_i(t)) + q_0(t), \quad x(t) = u(t) \text{ for } t < 0 \tag{5.1}$$

with the multi-point boundary condition (2.4), i.e.,

$$\sum_{k=1}^{\nu} A_k x(t_k) = c_0,$$

where $T > 0$, $P_i \in L(I, \mathbb{R}^{n \times n})$ for $i = 1, \ldots, \mu$, $q_0 \in L(I, \mathbb{R}^n)$, $c_0 \in \mathbb{R}^n$, $t_k \in I$, $A_k \in \mathbb{R}^{n \times n}$ $(k = 1, \ldots, \nu)$, $\tau_i : I \to \mathbb{R}$ $(i = 1, \ldots, \mu)$ are absolutely continuous nondecreasing delays (i.e., $\tau_i(t) \leq t$ for every $t \in I$), and $u :]-\infty, 0[\to \mathbb{R}^n$ is a continuous and bounded vector function.

Remark. If $\mu = 2$ and $\tau_1(t) \equiv t$ then system (5.1) represents the equation of the pantograph that is frequently studied in the literature (usually for $n = 1$). The paper [23], about simulation of an electricity transmission between wiring and locomotive, increased interest in the equation of the pantograph. Authors studied an asymptotic behaviour of solutions for $t \to +\infty$ and their numerical approximations (see [1, 2, 15, 18, 20] and references therein). They used criteria of the existence and uniqueness of a solution of the Cauchy problem.

In this part, we give effective criteria for the solvability of the generalized equation of the pantograph with a multi-point condition. We put other assumptions on delays of the arguments τ_i.

For any $i = 1, \dots, \mu$ and $t \in I$, we put

$$\tau_i^0(t) = \begin{cases} 0 & \text{if } \tau_i(t) < 0 \\ \tau_i(t) & \text{if } 0 \le \tau_i(t) \end{cases} .$$

This problem with delayed arguments is a special type of the problem (3.8), (2.2). That is why this problem fulfils Fredholm property, i.e., the following theorem follows immediately from Theorem 3.4.

Theorem 5.1. *The problem* (5.1), (2.4) *is uniquely solvable if and only if the corresponding homogeneous system*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{\mu} \chi_I(\tau_i(t)) P_i(t) x(\tau_i^0(t)) \tag{5.1$_0$}$$

with boundary condition (2.4$_0$) *has only the trivial solution.*

5.2. Existence and Uniqueness Theorems

In this section, we establish some efficient criteria of the unique solvability of the problem (5.1), (2.4) using the results and methods from [8].

Theorem 5.2. *Let, for every* $i \in \{1, \dots, \mu\}$, $P_i \in L(I, \mathbb{R}^{n\times n})$, $\tau_i \in \widetilde{C}(I, \mathbb{R})$ *be nondecreasing delays,* $t_0 = \min\{t_j : j = 1, 2, \dots, \nu\}$ *and let there exist matrices* $B_i \in \mathbb{R}_+^{n\times n}$ $(i = 1, \dots, \mu)$ *such that*

$$\chi_I(\tau_i(t))|P_i(t)| \le B_i \tau_i'(t) \quad \textit{almost everywhere on } I. \tag{5.2}$$

Let, moreover, either the matrix Λ_1 *given by* (4.1) *be nonsingular and* $r(S_1) < 1$, *where*

$$S_1 = \sum_{i=1}^{\mu} B_i \tau_i^0(T) + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{\mu} |A_k| B_i (\tau_i^0(t_k) - \tau_i^0(t_0)), \tag{5.3}$$

or the matrix Λ_2 *given by* (4.3) *be nonsingular and* $r(S_2) < 1$, *where*

$$S_2 = \sum_{i=1}^{\mu} B_i \tau_i^0(T) + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} |A_k| B_i B_j (\tau_i^0(t_k) - \tau_i^0(t_0)) \tau_j^0(t_k). \tag{5.4}$$

Then the problem (5.1), (2.4) *has a unique solution.*

Remark. This theorem does not follow from Theorem 4.1^b. The assumption of delayed arguments allows a finer formulation. That is why the proof is introduced independently, as in [12].

Proof. According to Theorem 5.1, it is sufficient to show that if x is a solution of the problem (5.1_0), (2.4_0) then $x(t) \equiv 0$. Let x be such a solution. The integration of (5.1_0) from t_0 to t, in view of (3.10), results in

$$x(t) = c + \int_{t_0}^{t} p(x)(s)\,\mathrm{d}s \tag{5.5}$$

and, by iteration in (5.5), we get

$$x(t) = \left[E + \int_{t_0}^{t} p(E)(s)\,\mathrm{d}s\right] c + \int_{t_0}^{t} p\left(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s, \tag{5.5_1}$$

where $c = x(t_0)$ and

$$p(E)(s) = \sum_{i=1}^{\mu} \chi_I(\tau_i(s)) P_i(s),$$

$$p\left(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\right)(s) = \sum_{i=1}^{\mu}\sum_{j=1}^{\mu} \chi_I(\tau_i(s)) P_i(s) \int_{t_0}^{\tau_i^0(s)} \chi_I(\tau_j(\xi)) P_j(\xi) x(\tau_j^0(\xi))\,\mathrm{d}\xi.$$

First suppose that the matrix Λ_1 given by (4.1) is nonsingular and $r(S_1) < 1$, where the matrix S_1 is defined by (5.3). Then from (2.4_0), by virtue of (4.1) and (5.5), we get

$$0 = \sum_{k=1}^{\nu} A_k \left[c + \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s\right] = \Lambda_1 c + \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s$$

and thus

$$c = -\Lambda_1^{-1} \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s.$$

Therefore, (5.5) implies

$$x(t) = \int_{t_0}^{t} p(x)(s)\,\mathrm{d}s - \Lambda_1^{-1} \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s$$

and, in view of (5.2), we get

$$
\begin{aligned}
|x(t)| &\le \left|\int_{t_0}^{t} p(x)(s)\,\mathrm{d}s\right| + |\Lambda_1^{-1}|\sum_{k=1}^{\nu}|A_k|\left|\int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s\right| \le \\
&\le \Bigg[\sum_{i=1}^{\mu}\left|\int_{t_0}^{t}\chi_I(\tau_i(s))|P_i(s)|\,\mathrm{d}s\right| + \\
&\qquad + |\Lambda_1^{-1}|\sum_{k=1}^{\nu}|A_k|\sum_{i=1}^{\mu}\int_{t_0}^{t_k}\chi_I(\tau_i(s))|P_i(s)|\,\mathrm{d}s\Bigg]|x|_C \le \\
&\le \Bigg[\sum_{i=1}^{\mu}\int_{0}^{T} B_i\chi_I(\tau_i(s))\tau_i'(s)\,\mathrm{d}s + \\
&\qquad + |\Lambda_1^{-1}|\sum_{k=1}^{\nu}|A_k|\sum_{i=1}^{\mu}\int_{t_0}^{t_k} B_i\chi_I(\tau_i(s))\tau_i'(s)\,\mathrm{d}s\Bigg]|x|_C \le \\
&\le \Bigg[\sum_{i=1}^{\mu} B_i\tau_i^0(T) + |\Lambda_1^{-1}|\sum_{k=1}^{\nu}\sum_{i=1}^{\mu}|A_k|B_i(\tau_i^0(t_k) - \tau_i^0(t_0))\Bigg]|x|_C,
\end{aligned}
$$

i.e.,

$$|x|_C \le S_1|x|_C.$$

Whence, together with the assumption $r(S_1) < 1$, we get

$$(E - S_1)|x|_C \le 0 \Rightarrow |x|_C \le (E - S_1)^{-1}0 = 0.$$

Therefore $x(t) \equiv 0$.

Now suppose that the matrix Λ_2 given by (4.3) is nonsingular, and $r(S_2) < 1$, where the matrix S_2 is defined by (5.4). From (2.4_0), (5.5_1) and (4.3) we get

$$
\begin{aligned}
0 &= \sum_{k=1}^{\nu} A_k\left[E + \int_{t_0}^{t_k} p(E)(s)\,\mathrm{d}s\right]c + \sum_{k=1}^{\nu} A_k\int_{t_0}^{t_k} p\left(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s = \\
&= \left[\Lambda_1 + \sum_{k=1}^{\nu} A_k\int_{t_0}^{t_k} p(E)(s)\,\mathrm{d}s\right]c + \sum_{k=1}^{\nu} A_k\int_{t_0}^{t_k} p\left(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s = \\
&= \Lambda_2\, c + \sum_{k=1}^{\nu} A_k\int_{t_0}^{t_k} p\left(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s
\end{aligned}
$$

and thus

$$c = -\Lambda_2^{-1}\sum_{k=1}^{\nu} A_k\int_{t_0}^{t_k} p\left(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s.$$

Therefore (5.5) yields

$$x(t) = \int_{t_0}^{t} p(x)(s)\,\mathrm{d}s - \Lambda_2^{-1}\sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p\Big(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\,\mathrm{d}s.$$

Hence, on account of (5.2) and the notation $\widetilde{P}_i(t) = \chi_I(\tau_i(t))P_i(t)$, $\widetilde{\tau}_i'(t) = \chi_I(\tau_i(t))\tau_i'(t)$, we get

$$\begin{aligned}
|x(t)| \le{}& \Big|\int_{t_0}^{t} p(x)(s)\,\mathrm{d}s\Big| + |\Lambda_2^{-1}|\sum_{k=1}^{\nu}|A_k|\Big|\int_{t_0}^{t_k} p\Big(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\,\mathrm{d}s\Big| \le\\
&\le \sum_{i=1}^{\mu}\Big|\int_{t_0}^{t}\widetilde{P}_i(s)x(\tau_i^0(s))\,\mathrm{d}s\Big| +\\
&\qquad + |\Lambda_2^{-1}|\sum_{k=1}^{\nu}|A_k|\sum_{i=1}^{\mu}\sum_{j=1}^{\mu}\Big|\int_{t_0}^{t_k}\widetilde{P}_i(s)\int_{t_0}^{\tau_i^0(s)}\widetilde{P}_j(\xi)x(\tau_j^0(\xi))\,\mathrm{d}\xi\,\mathrm{d}s\Big| \le\\
&\le \sum_{i=1}^{\mu} B_i\tau_i^0(T)|x|_C +\\
&\qquad + |\Lambda_2^{-1}|\sum_{k=1}^{\nu}|A_k|\sum_{i=1}^{\mu}\sum_{j=1}^{\mu}\int_{t_0}^{t_k}|\widetilde{P}_i(s)|\Big|\int_{t_0}^{\tau_i^0(s)}|\widetilde{P}_j(\xi)||x(\tau_j^0(\xi))|\,\mathrm{d}\xi\Big|\,\mathrm{d}s \le\\
&\le \sum_{i=1}^{\mu} B_i\tau_i^0(T)|x|_C +\\
&\qquad + |\Lambda_2^{-1}|\sum_{k=1}^{\nu}|A_k|\sum_{i=1}^{\mu}\sum_{j=1}^{\mu} B_iB_j\int_{t_0}^{t_k}\widetilde{\tau}_i'(s)\int_{0}^{t_k}\widetilde{\tau}_j'(\xi)\,\mathrm{d}\xi\,\mathrm{d}s|x|_C \le\\
&\le \sum_{i=1}^{\mu} B_i\tau_i^0(T)|x|_C +\\
&\qquad + |\Lambda_2^{-1}|\sum_{k=1}^{\nu}|A_k|\sum_{i=1}^{\mu}\sum_{j=1}^{\mu} B_iB_j(\tau_i^0(t_k)-\tau_i^0(t_0))\tau_j^0(t_k)|x|_C =\\
&= \Big[\sum_{i=1}^{\mu} B_i\tau_i^0(T) + |\Lambda_2^{-1}|\sum_{k=1}^{\nu}\sum_{i=1}^{\mu}\sum_{j=1}^{\mu}|A_k|B_iB_j(\tau_i^0(t_k)-\tau_i^0(t_0))\tau_j^0(t_k)\Big]|x|_C =\\
&= S_2|x|_C.
\end{aligned}$$

Whence, together with the assumption $r(S_2) < 1$, we get $x(t) \equiv 0$. □

Corollary 5.1. *Let, for every* $i \in \{1,\dots,\mu\}$, $P_i \in L(I,\mathbb{R}^{n\times n})$, $\tau_i \in \widetilde{C}(I,\mathbb{R})$ *be nondecreasing delays and let there exist matrices* $B_i \in \mathbb{R}_+^{n\times n}$ *such that the condition*

(5.2) *holds for* $i = 1, \dots, \mu$. *Let, moreover,* $r(S) < 1$, *where*

$$S = \sum_{i=1}^{\mu} B_i \tau_i^0(T).$$

Then the problem (5.1), (2.7) *has a unique solution.*

Proof. The validity of the corollary follows immediately from Theorem 5.2 with $\nu = 1, A_1 = E$, and $t_1 = t_0$ because, in this case, the matrix Λ_1 given by (4.1) is the unit matrix. □

Corollary 5.2. *Let, for every* $i \in \{1, \dots, \mu\}$, $P_i \in L(I, \mathbb{R}^{n\times n})$, $\tau_i \in \widetilde{C}(I, \mathbb{R})$ *be nondecreasing delays and let there exist matrices* $B_i \in \mathbb{R}_+^{n\times n}$ *such that the condition* (5.2) *holds for* $i = 1, \dots, \mu$. *Let, moreover,*

$$\Lambda = \sum_{i=1}^{\mu} \int_0^T \chi_I(\tau_i(t)) P_i(t)\, \mathrm{d}t$$

be a nonsingular matrix and $r(S) < 1$, *where*

$$S = \sum_{i=1}^{\mu} B_i \tau_i^0(T) + |\Lambda_2^{-1}| \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} B_i B_j \tau_i^0(T) \tau_j^0(T).$$

Then the problem (5.1), (2.8) *has a unique solution.*

Proof. The validity of the corollary follows immediately from Theorem 5.2 with $\nu = 2, A_1 = -E, A_2 = E, t_1 = 0$, and $t_2 = T$, because, in this case, $t_0 = 0$ and the matrices Λ_1 and Λ_2 given by (4.1) and (4.3), respectively, satisfy $\Lambda_1 = \Theta$ and $\Lambda_2 = \Lambda$. □

Remark. Further criteria can be derived analogously to the theorems stated in Chapter 4.

5.3. Linear System with Constant and Proportional Delays

Let $\tau_i(t) = q_i t - \Delta_i$ for $t \in I$, where $q_i, \Delta_i \in \mathbb{R}_+$ are such that $(q_i - 1)T \leq \Delta_i$ ($i = 1, \dots, \mu$). Then we get the following criteria of the solvability of the problem (5.1), (2.4) with linear delays.

Corollary 5.3. *Let* $t_0 = \min\{t_k : k = 1, 2, \dots, \nu\}$, $\tau_i(t) = q_i t - \Delta_i$ *for* $t \in I$, *where* $q_i, \Delta_i \in \mathbb{R}_+$ *are such that* $q_i \neq 0, (q_i - 1)T \leq \Delta_i$ $(i = 1, \dots, \mu)$, *and let there exist matrices* $B_i \in \mathbb{R}_+^{n \times n}$ $(i = 1, \dots, \mu)$ *such that the inequality* (5.2) *holds for* $i = 1, \dots, \mu$. *Put for* $i = 1, \dots, \mu$ *and* $k = 0, 1, \dots, \nu$

$$\delta_i = \chi_{[0,T]}(\frac{\Delta_i}{q_i})(q_i T - \Delta_i) \text{ and } \delta_{ik} = \chi_{[0,t_k]}(\frac{\Delta_i}{q_i})(q_i t_k - \Delta_i).$$

Let, moreover, either the matrix Λ_1 *given by* (4.1) *be nonsingular and* $r(S_1) < 1$, *where*

$$S_1 = \sum_{i=1}^{\mu} B_i \delta_i + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{\mu} |A_k| B_i (\delta_{ik} - \delta_{i0}), \tag{5.6}$$

or the matrix Λ_2 *given by* (4.3) *be nonsingular and* $r(S_2) < 1$, *where*

$$S_2 = \sum_{i=1}^{\mu} B_i \delta_i + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} |A_k| B_i B_j (\delta_{ik} - \delta_{i0}) \delta_{jk}. \tag{5.7}$$

Then the problem (5.1), (2.4) *has a unique solution.*

For the system with constant and proportional delays we get the following corollaries.

Corollary 5.4. *Let* $t_0 = \min\{t_k : k = 1, 2, \dots, \nu\}$, $\tau_i(t) = t - \Delta_i$ *for* $t \in I$, *where* $\Delta_i \in \mathbb{R}_+$ $(i = 1, \dots, \mu)$, *and let there exist matrices* $B_i \in \mathbb{R}_+^{n \times n}$ $(i = 1, \dots, \mu)$ *such that the inequality* (5.2) *holds for* $i = 1, \dots, \mu$. *Put* $\delta_i = \chi_{[0,T]}(\Delta_i)(T - \Delta_i)$ *and* $\delta_{ik} = \chi_{[0,t_k]}(\Delta_i)(t_k - \Delta_i)$ *for* $i = 1, \dots, \mu, k = 0, 1, \dots, \nu$. *Let, moreover, either the matrix* Λ_1 *given by* (4.1) *be nonsingular and* $r(S_1) < 1$, *where the matrix* S_1 *is defined by* (5.6), *or the matrix* Λ_2 *given by* (4.3) *be nonsingular and* $r(S_2) < 1$, *where the matrix* S_2 *is defined by* (5.7).

Then the problem (5.1), (2.4) *has a unique solution.*

Example 5.1. *Consider the problem*

$$x'(t) = -10^{-1} x(t) + 10^{-1} x(t - 1/2) + q_0(t), \quad t \in [0, 1],$$

$$x(0) = x(1), \quad x(t) = u(t) \text{ for } t \in [-1/2, 0[,$$

where $q_0 \in L([0, 1], \mathbb{R}^n)$ *and* $u : [-1/2, 0[\to \mathbb{R}^n$ *is a continuous and bounded vector function.*

Then $\nu = 2, \mu = 2$, $t_0 = t_1 = 0$, $t_2 = 1$, $\tau_1(t) \equiv t$, $\tau_2(t) = t - 1/2$, $P_1 = -10^{-1}E$, $P_2 = 10^{-1}E$, $A_1 = E$, $A_2 = -E$ *and we get* $\Lambda_1 = \Theta$, *nonsingular matrix* $\Lambda_2 = \frac{1}{20}E$ *and* $S_2 = \frac{12}{20}E$ *with* $r(S_2) < 1$. *According to Definition 1.3, the solution of the problem is an absolutely continuous vector function on the interval* $[0,1]$. *It may not generally be a continuous extension of the function* u *defined outside the interval* $[0,1]$.

From Corollary 5.4 it follows that the considered problem has a unique solution.

Corollary 5.5. *Let* $t_0 = \min\{t_k : k = 1, 2, \dots, \nu\}$, $\tau_i(t) = q_i t$ *for* $t \in I$, *where* $q_i \in]0,1]$ $(i = 1, \dots, \mu)$, *and let there exist matrices* $B_i \in \mathbb{R}_+^{n\times n}$ $(i = 1, \dots, \mu)$ *such that* (5.2) *holds for* $i = 1, \dots, \mu$. *Let, moreover, either the matrix* Λ_1 *given by* (4.1) *be nonsingular and* $r(S_1) < 1$, *where*

$$S_1 = \sum_{i=1}^{\mu} B_i q_i T + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{\mu} |A_k| B_i q_i (t_k - t_0),$$

or the matrix Λ_2 *given by* (4.3) *be nonsingular and* $r(S_2) < 1$, *where*

$$S_2 = \sum_{i=1}^{\mu} B_i q_i T + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} |A_k| B_i B_j q_i q_j (t_k - t_0) t_k.$$

Then the problem (5.1), (2.4) *has a unique solution.*

Proof of Corollaries 5.3 - 5.5. The validity of corollaries follows from Theorem 5.2 when, for $t \in I$, $\tau_i(t) = q_i t - \Delta_i$, $\tau_i(t) = t - \Delta_i$ and $\tau_i(t) = q_i t$, respectively.

Remark. From Corollaries 5.3 - 5.5 we can easily derive analogous criteria for the solvability of the Cauchy problem (5.1), (2.7) and periodic problem (5.1), (2.8).

Remark. We can establish special criteria for unique solvability of the problem with an integral boundary condition the same way as in Chapter 4.

6. On Construction of Solutions

6.1. Statement of the Problem

On the bounded interval $I = [0, T]$, consider the system of linear differential equations with deviating arguments (5.1), i.e.,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{\mu} P_i(t)x(\tau_i(t)) + q_0(t), \quad x(t) = u(t) \text{ for } t < 0$$

with the multi-point boundary condition (2.4), i.e.,

$$\sum_{k=1}^{\nu} A_k x(t_k) = c_0,$$

where $T > 0$, $P_i \in L(I, \mathbb{R}^{n\times n})$, $\tau_i : I \to \mathbb{R}$ are absolutely continuous nondecreasing delays (i.e. $\tau_i(t) \le t$ for every $t \in I$) for $i = 1, \ldots, \mu$, $q_0 \in L(I, \mathbb{R}^n)$, $t_k \in I$, $A_k \in \mathbb{R}^{n\times n}$ $(k = 1, \ldots, \nu)$, $c_0 \in \mathbb{R}^n$ and $u :]-\infty, 0[\to \mathbb{R}^n$ is a continuous and bounded vector function.

For the construction of solutions of the boundary value problem (5.1), (2.4) on the segment I we use the method of professor Kiguradze. The solution of the considered problem is obtained as a limit of a sequence of solutions of certain auxiliary problems. Methods and results are illustrated by examples created in Maple 10.

In this part, we show one method of constructing the solution of multi-point boundary value problem (5.1), (2.4) and its special cases. This method was used for constructing the solution of the Cauchy-Nicolleti's linear boundary value problem for linear systems of ordinary and functional differential equations (see [3, 6]).

Particular cases of the system (5.1) are the system of differential equations with one deviating argument

$$x'(t) = P(t)x(\tau(t)) + q_0(t) \tag{6.1}$$

and the system of ordinary differential equations

$$x'(t) = P(t)x(t) + q_0(t). \tag{6.2}$$

Important cases of deviations, which are discussed in literature together with the pantograph equation, are proportional delay

$$\tau_i(t) = q_i t, \tag{6.3}$$

where $q_i \in]0, 1]$, and constant delay

$$\tau_i(t) = t - \Delta_i, \tag{6.4}$$

where $\Delta_i \geq 0$.

6.2. Method of Successive Approximation

According to the previous chapter, for any $i = 1, \dots, \mu$ and $t \in I$, we put

$$\tau_i^0(t) = \begin{cases} 0 & \text{if } \tau_i(t) < 0 \\ \tau_i(t) & \text{if } 0 \leq \tau_i(t) \end{cases}$$

and let us recall, in the following proposition, Theorem 5.2 dealing with the unique solvability of the problem (5.1), (2.4).

Proposition 6.1. *Let, for every $i \in \{1, \dots, \mu\}$, $P_i \in L(I, \mathbb{R}^{n\times n})$, $\tau_i \in \widetilde{C}(I, \mathbb{R})$ be nondecreasing delays, $t_0 = \min\{t_j : j = 1, 2, \dots, \nu\}$ and let there exist matrices $B_i \in \mathbb{R}_+^{n\times n}$ $(i = 1, \dots, \mu)$ such that*

$$\chi_I(\tau_i(t))|P_i(t)| \leq B_i \tau_i'(t) \quad \text{almost everywhere on } I.$$

Let, moreover, either

1. *the matrix*
$$\Lambda_1 = \sum_{k=1}^{\nu} A_k$$
be nonsingular and $r(S_1) < 1$, where
$$S_1 = \sum_{i=1}^{\mu} B_i \tau_i^0(T) + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{\mu} |A_k| B_i(\tau_i^0(t_k) - \tau_i^0(t_0)),$$
or

2. the matrix

$$\Lambda_2 = \Lambda_1 + \sum_{k=1}^{\nu}\sum_{i=1}^{\mu} A_k \int_{t_0}^{t_k} \chi_I(\tau_i(t))P_i(t)\,\mathrm{d}t$$

be nonsingular and $r(S_2) < 1$, *where*

$$S_2 = \sum_{i=1}^{\mu} B_i\tau_i^0(T) + |\Lambda_2^{-1}| \sum_{k=1}^{\nu}\sum_{i=1}^{\mu}\sum_{j=1}^{\mu} |A_k| B_i B_j(\tau_i^0(t_k) - \tau_i^0(t_0))\tau_j^0(t_k).$$

Then the problem (5.1), (2.4) *has a unique solution.*

To describe the method of successive approximation of solutions we define operators $T_1, T_2 : C(I;\mathbb{R}^n) \to C(I;\mathbb{R}^n)$ (for the case when Λ_1 is nonsingular matrix and the case when Λ_2 is nonsingular matrix) by setting

$$T_1(x)(t) = \int_{t_0}^{t} p(x)(s)\,\mathrm{d}s - \Lambda_1^{-1}\sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s,$$
$$T_2(x)(t) = \int_{t_0}^{t} p(x)(s)\,\mathrm{d}s - \Lambda_2^{-1}\sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p\Big(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\,\mathrm{d}s,$$

where $t_0 = \min\{t_j : j = 1, 2, \dots, \nu\}$, $p(x)(t) = \sum_{j=1}^{\mu} \chi_I(\tau_j(t))P_j(t)x(\tau_j^0(t))$,
and functions

$$q_1(t) = \Lambda_1^{-1}\left[c_0 - \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} q(s)\,\mathrm{d}s\right] + \int_{t_0}^{t} q(s)\,\mathrm{d}s,$$
$$q_2(t) = \Lambda_2^{-1}\left\{c_0 - \sum_{k=1}^{\nu} A_k \left[\int_{t_0}^{t_k} p\left(\int_{t_0}^{\cdot} q(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s - \int_{t_0}^{t_k} q(s)\,\mathrm{d}s\right]\right\} + \int_{t_0}^{t} q(s)\,\mathrm{d}s,$$

where

$$q(t) = q_0(t) + \sum_{j=1}^{\mu}(1 - \chi_I(\tau_j(t)))P_j(t)u(\tau_j(t)).$$

We construct the solution of the problem (5.1), (2.4) in the following theorem using the method of successive approximation.

Theorem 6.1. *Let the first or the second assumption of Proposition 6.1 be fulfilled. Then the problem* (5.1), (2.4) *has a unique solution* x *such that*

$$\|x - x_m\|_C \to 0 \text{ for } m \to \infty, \tag{6.5}$$

where $\{x_m\}_{m=1}^{\infty} \subset C(I;\mathbb{R}^n)$ *is a sequence of continuous n-dimensional vector functions, which are solutions of the following sequence of problems*

$$x_1 \in C(I;\mathbb{R}^n) \text{ is arbitrary and}$$

$$x_m(t) = T_i(x_{m-1})(t) + q_i(t) \text{ for } t \in I, m \in \mathbb{N} \setminus \{1\}, \tag{6.6}$$

where $i = 1$ *in the first case and* $i = 2$ *in the second one.*

Proof. The unique solvability of the problem (5.1), (2.4) follows from Proposition 6.1. Let x be the unique solution of the problem (5.1), (2.4) and $t_0 = \min\{t_k : k = 1, 2, \dots, \nu)$. Integrate the equation (5.1) from t_0 to t. We get the equation

$$x(t) = c + \int_{t_0}^{t} p(x)(s)\,\mathrm{d}s + \int_{t_0}^{t} q(s)\,\mathrm{d}s, \tag{6.7}$$

where $c = x(t_0)$. Since the solution x of the equation (5.1) fulfils the condition (2.4), this vector function fulfils the integral equation

$$\begin{aligned} x(t) = \left[E + \int_{t_0}^{t} p(E)(s)\,\mathrm{d}s\right] c + \int_{t_0}^{t} p\left(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s + \\ + \int_{t_0}^{t} p\left(\int_{t_0}^{\cdot} q(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s + \int_{t_0}^{t} q(s)\,\mathrm{d}s \end{aligned} \tag{6.8}$$

as well. In the first case, when Λ_1 is nonsingular, with use of (2.4) and (6.7), we get

$$c = \Lambda_1^{-1}\left[c_0 - \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s - \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} q(s)\,\mathrm{d}s\right]$$

and in the second case, when Λ_2 is nonsingular, with use of (2.4) and (6.8), we get

$$\begin{aligned} c = \Lambda_2^{-1}\left[c_0 - \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p\left(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s\right] - \\ -\Lambda_2^{-1}\sum_{k=1}^{\nu} A_k \left[\int_{t_0}^{t_k} p\left(\int_{t_0}^{\cdot} q(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s + \int_{t_0}^{t_k} q(s)\,\mathrm{d}s\right]. \end{aligned}$$

This implies that the solution x of the boundary problem (5.1), (2.4) is also the solution of the integral equation

$$x(t) = T_i(x)(t) + q_i(t), \quad t \in I. \tag{6.9}$$

Let now $x_1 \in C(I;\mathbb{R}^n)$ be arbitrary and $\{x_m\}_{m=2}^{\infty}$ be the sequence of n-dimensional vector functions given by (6.6). We proof the uniform convergence of the sequence

$\{x_m\}_{m=1}^{\infty}$ to the solution x, i.e., we show, that (6.5) holds. According to the proof of Theorem 5.2, we get

$$|T_i(z)|_C \le S_i|z|_C \text{ for } z \in C(I;\mathbb{R}^n).$$

Therefore,

$$|x - x_m|_C = |x - (T_i(x_{m-1}) + q_i)|_C = |T_i(x - x_{m-1})|_C \le$$
$$\le S_i|x - x_{m-1}|_C \le \ldots \le S_i^{m-1}|x - x_1|_C.$$

Since $r(S_i) < 1$ we get $S_i^m \to \Theta$ for $m \to \infty$. Hence

$$\lim_{m\to\infty} |x - x_m|_C = 0,$$

i.e., the sequence $\{x_m\}$ is uniformly convergent to the vector function x on the interval I. □

Remark. The numerical stability of the convergency process in this case can be proved using the methods published in [7] for boundary problems for ordinary differential equations. This problem is solved by Lukáš Maňásek and and in our collective work [14].

6.3. Examples

We apply the method of successive approximation introduced above to a simple problem. We consider the system (5.1), where $\mu = 2, \tau_1 \equiv t, \tau_2 = \tau$, and the boundary condition (2.4), where $\nu = 2, A_1 = \lambda_0 E, A_2 = \lambda_T E, t_1 = 0, t_2 = T$. Thus we solve the following problem

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = P_1(t)x(t) + P_2(t)x(\tau(t)) + q_0(t), \quad x(t) = u(t) \text{ for } t < 0, \tag{6.10}$$

$$\lambda_0 x(0) + \lambda_T x(T) = c_0 \tag{6.11}$$

which is known as the pantograph equation.

Example 6.1. *Consider the problem* (6.10), (6.11), *where* $P_1(t) = 1, P_2(t) = 1, \tau(t) = \frac{t}{2}, q_0(t) = 1, \lambda_0 = -1, \lambda_T = 1, T = 1, c_0 = 0$, *i.e., the problem*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t) + x\left(\frac{t}{2}\right) + 1,$$

$$x(1) = x(0).$$

In Fig. 1 we can see starting function $x_0(t) = 1$, $t \in [0, 1]$ *and the first iteration and in Fig. 2 the same starting function and iterations 1 – 40.*

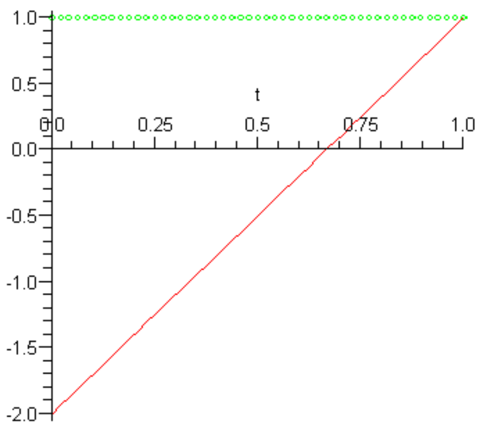

Figure 1: Iteration: 1

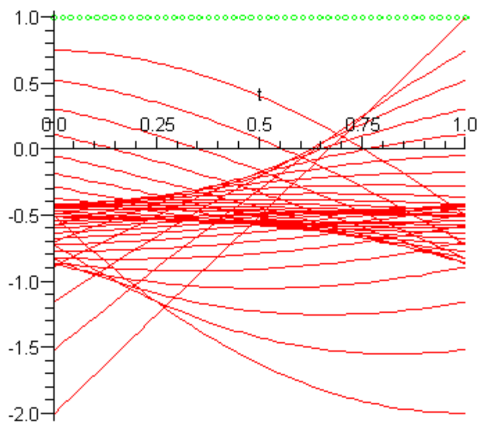

Figure 2: Iterations: 1 - 40

In Fig. 3 we can see starting function $x_0(t) = -t - 1$, $t \in [0, 1]$ *and the first iteration and in Fig. 4 the same starting function and iterations 1 – 40.*

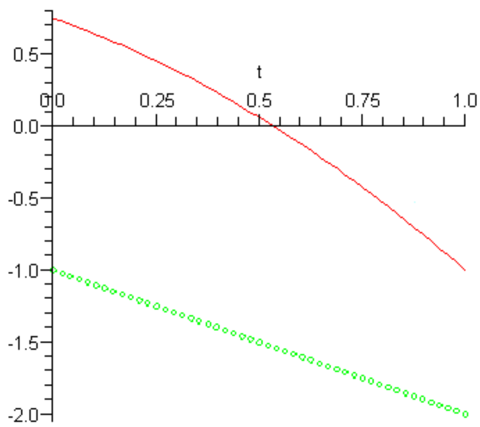

Figure 3: Iteration: 1

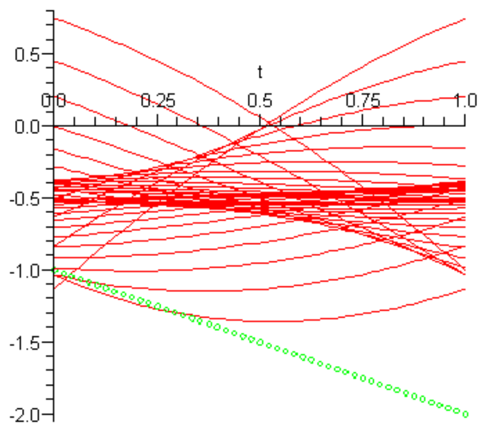

Figure 4: Iterations: 1 - 40

Example 6.2. *Consider the problem* (6.10), (6.11), *where* $P_1(t) = 0, P_2(t) = 1, \tau(t) = t - \frac{1}{3}, q_0(t) = t, \lambda_0 = -1, \lambda_T = 1, T = 2, c_0 = 0$, *i.e., the problem*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x\left(t - \frac{1}{3}\right) + t,$$
$$x(2) = x(0).$$

In Fig. 5 we can see starting function $x_0(t) = 1,\ t \in [0,1],\ u(t) = 1$ for $t < 0$ *and first iteration and in Fig. 6 we can see the same starting function, function u and iterations 1 – 40.*

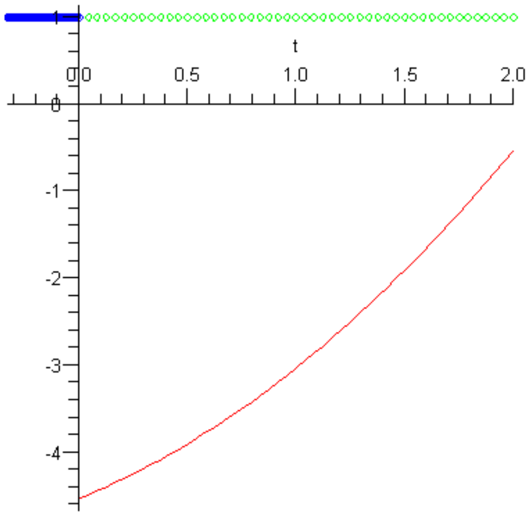

Figure 5: Iteration: 1

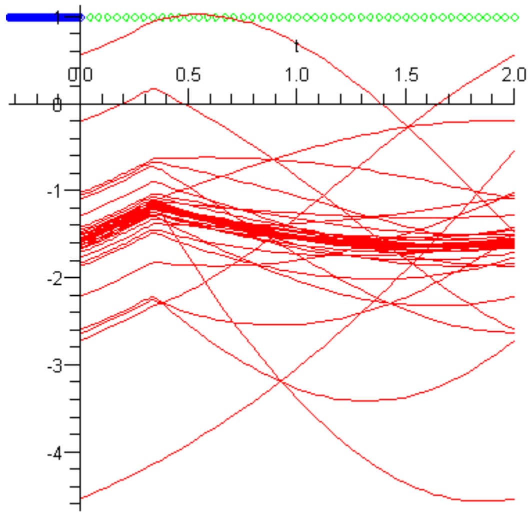

Figure 6: Iterations: 1 - 40

In Fig. 7 we can see starting function $x_0(t) = 1,\ t \in [0,1],\ u(t) = t - 1$ for $t < 0$ *and first iteration and in Fig. 8 we can see the same starting function, function u and iterations 1 – 40.*

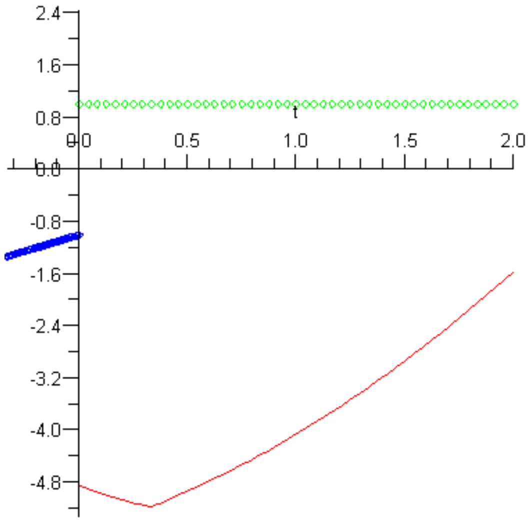

Figure 7: Iteration: 1

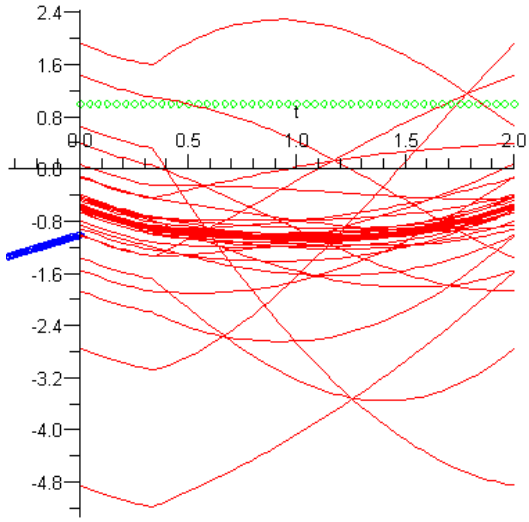

Figure 8: Iterations: 1 - 40

Previous examples were computed in Maple 10. In pictures are plotted starting function x_0, the first iteration (eventually more iterations) and function u (if it is necessary). From the first example follows that the solution really does not depend on the starting function x_0. Second example shows that we receive different solutions for different functions u.

Conclusion

We have studied the question on solvability of the multi-point and integral boundary value problems for systems of functional differential equations, especially the systems with more deviating arguments. During our studies, we have concentrated on linear systems and also on systems with small parameter. The reason is huge number of results we gained in this course of study, which is important in practice (see generalized pantograph equation).

There still remain, however, many open problems, for example:

– the construction of a solution for other boundary value problems,

– the nonnegativeness of a solution,

– other effective criteria for unique solvability for linear boundary value problems,

– effective criteria for unique solvability for nonlinear boundary value problems of systems of functional differential equations and systems with more deviating arguments.

More open problems we can find in theory and applications of generally nonlinear and singular boundary value problems for systems of functional differential equations.

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