Periods in arithmetic geometry

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Consider the *cartesian defining equation* of the unit circle, i.e., $C : f(x, y) = 0$ where $f(x, y) := x^2 + y^2 - 1 \in \mathbb{Q}[x, y]$. Note that the two variables polynomial f has rational coefficients, i.e., $f \in \mathbb{Q}[x, y]$. The object C is an example of an *affine algebraic variety* defined over $\mathbb{Q}$. If one takes the real points of C, i.e., if one considers the set

$$S^1 = C(\mathbb{R}) := \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\},$$

then one recovers the usual unit circle viewed as a subset of $\mathbb{R}^2$. Consider now the *algebraic differential form* $\omega = \frac{dx}{y}$ on $C(\mathbb{R})$. One should think of a algebraic differential form on S^1 as an "algebraic expression" that one can integrate on the space S^1. Note that the differential form ω is again defined over $\mathbb{Q}$. We have

$$\frac{1}{2}\int_{S^1} \omega = \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} = \pi, \tag{0.1}$$

where π is the half circumference of S^1. A famous theorem of Lindemann (1882) proves that π is transcendental over $\mathbb{Q}$. In particular, π is irrational. A priori, the irrationality of π seems to be a bit surprising, since after all, the polynomial f and the differential form ω were *both initially* defined over $\mathbb{Q}$. This last observation may be summarized by saying that, in general, the "integral operator $\int$" is a transcendental operation (contrary to finite symbolic expressions involving only the classical binary operators $+, -, \cdot, \div$ which give rise to algebraic operators). The number π is the simplest example of a **period** which is not an algebraic number.

Consider now the Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \tag{0.2}$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. A famous theorem of Euler states that for all $k \in \mathbb{Z}_{\geq 1}$,

$$\zeta(2k) \in \pi^{2k} \cdot \mathbb{Q}. \tag{0.3}$$

For example $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$ etc. Similarly to the integral operator $\int$, the infinite summation in (0.2) may again be viewed (in a sense which we do not want to make precise) as a transcendental operation.

(i) One of the goals of this series of lecture is to provide a proof of (0.3) which clarifies the relationship between the formulas (0.1) and (0.3). Moreover, we shall present some generalizations of (0.3). For example, we intend to prove that

$$\sum_{m,n\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m+n\,\mathrm{i})^{4k}} \in \Omega^{4k}\cdot\mathbb{Q},$$

where $\mathrm{i} = \sqrt{-1}$ and $\Omega = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^4}}$. The real number Ω is again a period which was proved to be transcendental by Schneider (1937).

(ii) More generally, if one replaces the unit circle $C : x^2 + y^2 - 1 = 0$ by the Fermat curve $C_N : x^N + y^N - 1 = 0$, then one may consider similar constructions as in (i). In particular, we will show that the real number $\Gamma(\frac{1}{N})^N$ can be written as a certain product of periods associated to the Fermat curve C_N.

(iii) Let $c \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ be a coset. One may consider the partial zeta function

$$\zeta(c;s) := \sum_{0\neq n\in c} \frac{1}{|n|^s},$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. One may show that $s \mapsto \zeta(c;s)$ admits a meromorphic continuation to all of $\mathbb{C}$. Among other things, we shall prove that the numbers $\zeta'(c;0)$ are periods which satisfy certain distribution relations.

(iv) As before, let $c \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ be a coset. One may also consider the truncated partial zeta functions

(1) $\zeta^+(c;s) := \sum_{\substack{n\in c\\ n>0}} \frac{1}{n^s}$,

(2) $\zeta^-(c;s) := \sum_{\substack{n\in c\\ n<0}} \frac{1}{|n|^s}$,

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. In particular, note that $\zeta^+(-c;s) = \zeta^-(c;s)$ and that $\zeta^+(c;s) + \zeta^-(c;s) = \zeta(c;s)$. For $c \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ and $c \neq \overline{0}$, a famous formula from the Czech mathematician Matyáš Lerch states that

$$\zeta'(c;0) = \log\frac{\Gamma(c_0)}{\sqrt{2\pi}}, \tag{0.4}$$

where $c_0 \in c$ is the unique representative such that $0 < c_0 < 1$ and Γ corresponds to the Gamma function. We will try to provide some relationships between the formula (0.4) and the formulas which appear in (ii) and (iii).

(v) Finally, if time permits, we would like to explain from the point of view of (iv) how some of the work of R. Kučera can sometimes be used to predict unexpected algebraic identities over $\overline{\mathbb{Q}}$ between roots of the periods considered in (ii). For example, we intend to provide a conceptual explanation for the relation

$$\frac{\Gamma(\frac{1}{24}) \cdot \Gamma(\frac{11}{24})}{\Gamma(\frac{5}{24}) \cdot \Gamma(\frac{7}{24})} \in \overline{\mathbb{Q}}.$$

More precisely, one can show that

$$\frac{\Gamma(\frac{1}{24}) \cdot \Gamma(\frac{11}{24})}{\Gamma(\frac{5}{24}) \cdot \Gamma(\frac{7}{24})} = \sqrt{3} \cdot \sqrt{2 + \sqrt{3}}.$$

Prerequisites: Real analysis, calculus, complex analysis and some familiarities with line integrals on Riemann surfaces.