ACCESSIBLE CATEGORIES AND HOMOTOPY THEORY

J. ROSICKÝ

Accessible categories have recently turned out to be useful in homotopy theory. This text is prepared as notes for a series of lectures at the Summer School on Contemporary Categorical Methods in Algebra and Topology which will be held in Haute Bodeux, June 3-10, 2007.

1. Combinatorial model categories

Model categories were introduced by Quillen [42] as the foundation of homotopy theory. In order to make their definition less complex, we will consider weak factorization systems at first.

Let $\mathcal{K}$ be a category and $f : A \to B$, $g : C \to D$ morphisms such that in each commutative square

$$\begin{array}{ccc} A & \xrightarrow{\;u\;} & C \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow[\;v\;]{} & D \end{array}$$

there is a diagonal $d : B \to C$ with $df = u$ and $gd = v$. Then we say that g has the *right lifting property* w.r.t. f and f has the *left lifting property* w.r.t. g. For a class $\mathcal{X}$ of morphisms of $\mathcal{K}$ we put

$\mathcal{X}^{\square} = \{g | g$ has the right lifting property w.r.t. each $f \in \mathcal{X}\}$ and

${}^{\square}\mathcal{X} = \{f | f$ has the left lifting property w.r.t. each $g \in \mathcal{X}\}$.

Definition 1.1. A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$ consists of two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms of $\mathcal{K}$ such that

(1) $\mathcal{R} = \mathcal{L}^{\square}$, $\mathcal{L} = {}^{\square}\mathcal{R}$ and
(2) any morphism h of $\mathcal{K}$ has a factorization $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

Definition 1.2. A *model category* is a complete and cocomplete category $\mathcal{K}$ together with three classes of morphisms $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{W}$ called *fibrations*, *cofibrations* and *weak equivalences* such that

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(1) $\mathcal{W}$ has the 2-out-of-3 property, i.e., with any two of f, g, gf belonging to $\mathcal{W}$ also the third morphism belongs to $\mathcal{W}$, and $\mathcal{W}$ is closed under retracts in the arrow category $\mathcal{K}^{\to}$,
(2) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems.

Morphisms from $\mathcal{F} \cap \mathcal{W}$ are called *trivial fibrations* while morphisms from $\mathcal{C} \cap \mathcal{W}$ *trivial cofibrations*. Each weak equivalence is a composition of a trivial cofibration followed by a trivial fibration. Hence weak equivalences are determined by cofibrations and fibrations. Conversely, cofibrations and weak equivalences determine fibrations and, analogously, fibrations and weak equivalences yield cofibrations. Weak equivalences serve for defining the *homotopy category* $\mathrm{Ho}(\mathcal{K})$ of $\mathcal{K}$ as the fraction category $\mathcal{K}[\mathcal{W}^{-1}]$. (Co)fibrations make possible to prove that the homotopy category is legitimate, i.e., that it does not has a proper class of morphisms between two objects. Moreover, weak equivalences are *saturated* in the sense that they coincide with morphisms going to isomorphisms in the projection

$$P : \mathcal{K} \to \mathrm{Ho}(\mathcal{K}).$$

We will briefly explain this and we recommend [29] or [28] for a full treatment.

A *cylinder object* $C(K)$ of an object K is given by a (cofibration, trivial fibration) factorization of the codiagonal

$$\nabla : K + K \xrightarrow{\gamma_K} C(K) \xrightarrow{\sigma_K} K$$

We denote by

$$\gamma_{1K}, \gamma_{2K} : K \to C(K)$$

the compositions of γ_K with the coproduct injections. Now, given two morphisms $f, g : K \to L$, we say that f and g are homotopic and write $f \sim g$ if there is a morphism $h : C(K) \to L$ such that the following diagram commutes

$$\begin{array}{ccc} K + K & \xrightarrow{(f,g)} & L \\ & {\scriptstyle\gamma_K}\searrow \quad \nearrow{\scriptstyle h} & \\ & C(K) & \end{array}$$

The homotopy relation $\sim$ is clearly reflexive, symmetric, compatible with the composition and does not depend on the choice of a cylinder object. But, it is not transitive in general and in order to form the quotient

$$Q : \mathcal{K} \to \mathcal{K}/\sim$$

we have to take its transitive hull. We have slightly modified the usual definition of a cylinder object which uses a (cofibration, weak equivalence) factorization in order to stress that the homotopy depends on cofibrations only. In fact, we have the concept of homotopy for any weak factorization system (see [36]).

The quotient category $\mathcal{K}/\sim$ is too big and it does not coincide with the homotopy category $\mathrm{Ho}(\mathcal{K})$. But $\mathrm{Ho}(\mathcal{K})$ is equivalent to the full subcategory of $\mathcal{K}/\sim$ consisting of objects which are both cofibrant and fibrant. An object K is called *cofibrant* if the unique morphism $0 \to K$ from an initial object is a cofibration and is called *fibrant* if $K \to 1$ is a fibration where 1 is a terminal object. We denote by $\mathcal{K}_c$ the full subcategory of $\mathcal{K}$ consisting of cofibrant objects. Analogously, $\mathcal{K}_f$ consists of fibrant objects and $\mathcal{K}_{cf} = \mathcal{K}_c \cap \mathcal{K}_f$. Using this notation, we get that $\mathrm{Ho}(\mathcal{K})$ is equivalent to $\mathcal{K}_{cf}/\sim$. This fundamental fact immediately implies that the homotopy category is legitimate (and that $\mathcal{W}$ is saturated). Moreover, on $\mathcal{K}_{cf}$, weak equivalences coincide with *homotopy equivalences*, i.e., with morphisms $f : K \to L$ having $g : L \to K$ such that both fg and gf are homotopic to the identity. Since the dual of a model category is a model category, we can define a homotopy starting from fibrations instead of cofibrations. Then we speak about a *path object* given by a (trivial cofibration, fibration) factorization of the diagonal $\Delta : K \to K \times K$. The both homotopies (also called left and right ones) coincide on $\mathcal{K}_{cf}$.

Cofibrations are *cofibrantly closed* in the sense that they contain all isomorphisms and are

(a) stable under pushout,

(b) closed under transfinite composition, and

(c) closed under retracts in comma categories $K \downarrow \mathcal{K}$.

The first condition says that if

$$\begin{array}{ccc} B & \xrightarrow{\ \overline{g}\ } & D \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \overline{f}} \\ A & \xrightarrow[\ g\ ]{} & C \end{array}$$

is a pushout and f a cofibration then $\overline{f}$ is a cofibration. The second condition means that cofibrations are closed under composition and if $(f_{ij} : A_i \to A_j)_{i \le j \le \lambda}$ is a smooth chain (i.e., λ is a limit ordinal, $(f_{ij} : A_i \to A_j)_{i<j}$ is a colimit for any limit ordinal $j \le \lambda$) and f_{ij} are cofibrations for each $i \le j < \lambda$ then $f_{0\lambda}$ is a cofibration. In fact, all conditions are satisfied by each class ${}^{\square}\mathcal{X}$. Hence trivial cofibrations are cofibrantly closed as well. Given a class of morphisms $\mathcal{X}$, $\operatorname{cof}(\mathcal{X})$ will denote its cofibrant closure, i.e., the smallest cofibrantly closed class containing $\mathcal{X}$. The main obstacle in showing that the classes $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ provide a model category structure is to verify the existence of the both weak factorizations of a morphism f. In homotopy theory, the following fundamental fact is called a small object argument and it follows from properties of injectivity classes in locally presentable categories. The basic observation is that $g : C \to D$ belongs to $\mathcal{X}^{\square}$ if and only iff it is injective, as an object of $\mathcal{K} \downarrow D$, to all morphisms from $\mathcal{X}$. Then a weak factorization $h = gf$ of a morphism $h : A \to D$ is given by a $\operatorname{cof}(\mathcal{X})$-morphism in $\mathcal{K} \downarrow D$ from h to an $\mathcal{X}$-injective g. One can explicitly find this result in [8] (see also [2]). We recommend [4] as a reference for locally presentable and accessible categories.

Theorem 1.3. *Let $\mathcal{K}$ be a locally presentable category and $\mathcal{X}$ a set of morphisms. Then* $(\operatorname{cof}(\mathcal{X}), \mathcal{X}^{\square})$ *is a weak factorization system.*

A cofibrantly closed class of morphisms $\mathcal{X}$ is called *cofibrantly generated* if it equals to $\operatorname{cof}(\mathcal{I})$ for a set $\mathcal{I}$. A model category $\mathcal{K}$ is called cofibrantly generated if both cofibrations and trivial cofibrations are cofibrantly generated. Following J. H. Smith, $\mathcal{K}$ is called *combinatorial* if it is both locally presentable and cofibrantly generated. One of consequences of 1.3 is that weak factorizations are functorial (cf. [2]). It means that there is a functor $F : \mathcal{K}^{\to} \to \mathcal{K}$ and natural transformations $\alpha : \operatorname{dom} \to F$ and $\beta : F \to \operatorname{cod}$ such that $f = \alpha_f \beta_f$ is the weak factorization of f. Here, $\operatorname{dom} : \mathcal{K}^{\to} \to \mathcal{K}$ assigns to each morphism its domain and cod analogously describes codomains. Hence a combinatorial model category has the both weak factorizations functorial. M. Hovey has put functoriality into the definition of a model category in [29]. We will call such a model category *functorial.* In a combinatorial model category, these factorizations are not only functorial but accessible. Let us recall that a functor is accessible if it preserves λ-filtered colimits for some regular cardinal λ. The following result is due to J. H. Smith and its proof is sketched in [22]; [44] contains more details.

Theorem 1.4. *Let $\mathcal{K}$ be a combinatorial model category. Then the functors $\mathcal{K}^{\to} \to \mathcal{K}$ giving the (cofibration, trivial fibration) and (trivial cofibration, fibration) factorizations are accessible.*

The *cofibrant replacement functor* $R_c : \mathcal{K} \to \mathcal{K}$ is given by the (cofibration, trivial fibration) factorization

$$0 \to R_c(K) \to K$$

of the unique morphism from 0 to K while *fibrant replacement functor* R_f is given by the (trivial cofibration, fibration) factorization of $K \to 1$. As a consequence of 1.4 we get that, in a combinatorial model category, the both functors are accessible. Hence their composition $R = R_f R_c$, which is called the *replacement functor*, is accessible too. Analogously, the *cylinder functor* $C : \mathcal{K} \to \mathcal{K}$ is accessible.

In the situation from 1.3, $\mathcal{X}^\square$ is closed under λ-filtered colimits for some regular cardinal λ (greater then presentability ranks of domains and codomains of morphims from $\mathcal{X}$). Hence, in a combinatorial model category, both fibrations and trivial fibrations are closed under λ-filtered colimits for some regular cardinal λ. Since (trivial cofibration, fibration) factorizations are accessible, weak equivalences are closed under λ-filtered colimits for some regular cardinal λ as well. This result is due to J. H. Smith and can be found in [22], 7.3.

Definition 1.5. *A combinatorial model category $\mathcal{K}$ will be called strongly combinatorial if the class $\mathcal{C}$ of cofibrations is closed under λ-filtered colimits in $\mathcal{K}^\to$ for some regular cardinal λ.*

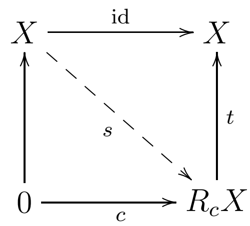
Theorem 1.6. *Let $\mathcal{K}$ be a strongly combinatorial model category. Then $\mathcal{K}_c$, $\mathcal{K}_f$ and $\mathcal{K}_{cf}$ are accessible categories.*

Proof. Let $\mathcal{K}$ be a strongly combinatorial model category. Since fibrant objects are precisely objects injective with respect to generating trivial cofibrations, it follows from [4], 4.7 that $\mathcal{K}_f$ is accessible and accessibly embedded; the latter means to be closed under λ-filtered colimits for some regular cardinal λ. Since $\mathcal{K}_c$ is closed under λ-filtered colimits in $\mathcal{K}$ for some regular cardinal λ, it remains to show that $\mathcal{K}_c$ is cone reflective in $\mathcal{K}$. Then it is accessible (see [4], 2.53) and thus $\mathcal{K}_{cf}$ is accessible too (by [4], 2.37).

Cone reflectivity of $\mathcal{K}_c$ means that, given an object K, there is a set $\mathcal{A}$ of morphisms $K \to A$ with A cofibrant such that each morphism $f : K \to X$ with X cofibrant factorizes through some morphism from $\mathcal{A}$. Consider the (cofibration, trivial fibration) factorization

$$0 \xrightarrow{c} R_c X \xrightarrow{t} X$$

where R_c is an accessible cofibrant replacement functor. Since X is cofibrant, t is split by diagonal s in



There is a regular cardinal λ such that $\mathcal{K}$ is locally λ-presentable, K is λ-presentable and R_c preserves λ-filtered colimits and λ-presentable objects (see [4], 2.19). We express X as a λ-directed colimit

$$(a_i : A_i \to X)_{i \in I}$$

of λ-presentable objects. We get a factorization $sf = R_c(a_i)g$ for some $i \in I$. Hence $f = tR_c(a_i)g$ and we can thus take for $\mathcal{A}$ the set of all morphisms having a λ-presentable cofibrant codomain. $\square$

A very useful tool for verifying that $\mathcal{C}$ and $\mathcal{W}$ provide a model category structure is the following theorem of J. H. Smith; its proof can be found in [8].

Theorem 1.7. *Let $\mathcal{K}$ be a locally presentable category, $\mathcal{I}$ a set of morphisms and $\mathcal{W}$ a class of morphisms containing $\mathcal{I}^{\square}$, closed under retracts in $\mathcal{K}^{\to}$, satisfying the 2-out-of-3 property and such that* $\operatorname{cof}(\mathcal{I}) \cap \mathcal{W}$ *is cofibrantly closed. Moreover, assume that $\mathcal{W}$ satisfies the solution set condition at $\mathcal{I}$.*

Then $\mathcal{C} = \operatorname{cof}(\mathcal{I})$ and $\mathcal{W}$ make $\mathcal{K}$ a combinatorial model category.

The solution set condition at $\mathcal{I}$ means that for every $f \in \mathcal{I}$ there is a subset $\mathcal{X}$ of $\mathcal{W}$ such that each morphism $f \to g$ in $\mathcal{K}^{\to}$ with $g \in \mathcal{W}$ factorizes through some $f \to h$ with $h \in \mathcal{X}$. Assuming Vopěnka's principle, every class $\mathcal{W}$ satisfies the solution set condition at some set of morphisms in a locally presentable category (it follows from [4], 6.6 and 1.57; see the proof of 2.2 in [46]). Recall that Vopěnka's principle is a set-theoretic hypothesis implying the existence of many large cardinals (cf. [4]). It yields the following, somewhat surprising corollary.

Corollary 1.8. *Let $\mathcal{K}$ be a locally presentable category, $\mathcal{I}$ a set of morphisms and $\mathcal{W}$ a class of morphisms containing $\mathcal{I}^{\square}$, closed under retracts in $\mathcal{K}^{\to}$, satisfying the 2-out-of-3 property and such that* $\operatorname{cof}(\mathcal{I}) \cap$ *$\mathcal{W}$ is cofibrantly closed. Then, under Vopěnka's principle, $\mathcal{C} = \operatorname{cof}(\mathcal{I})$ and $\mathcal{W}$ make $\mathcal{K}$ a combinatorial model category.*

Given a cofibrantly closed class $\mathcal{C}$ of morphisms, we can consider the smallest class $\mathcal{W}$ satisfying the closure conditions from 1.7, i.e.,

(i) $\mathcal{C}^{\square} \subseteq \mathcal{W}$,

(ii) $\mathcal{W}$ is closed under retracts and satisfies the 2-out-of-3 property,

(iii) $\mathcal{C} \cap \mathcal{W}$ is cofibrantly closed.

This construction is presented in [46] but it was known to J. H. Smith and was independently considered in [18]. We can even modify this construction by taking another class $\mathcal{Z}$ of morphisms and adding

(iv) $\mathcal{Z} \subseteq \mathcal{W}$.

Let us denote the resulting smallest class by $\overline{\mathcal{Z}}$.

Corollary 1.9. *Let $\mathcal{K}$ be a combinatorial model category and $\mathcal{Z}$ a class of morphisms of $\mathcal{K}$. Assuming Vopěnka's principle, $\mathcal{C}$ and $\overline{\mathcal{Z} \cup \mathcal{W}}$ make $\mathcal{K}$ a combinatorial model category.*

Of course, $\mathcal{C}$ denotes cofibrations and $\mathcal{W}$ weak equivalences in the original model category structure. Recall that a *left Quillen functor* $F : \mathcal{K} \to \mathcal{L}$ is a left adjoint functor between model categories preserving cofibrations and trivial cofibrations. A *left Bousfield localization* of $\mathcal{K}$ with respect to $\mathcal{Z}$ is a model category structure $\mathcal{K}\backslash\mathcal{Z}$ on $\mathcal{K}$ which has the same cofibrations as $\mathcal{K}$, weak equivalences containing both $\mathcal{Z}$ and the weak equivalences of $\mathcal{K}$ and each left Quillen functor $F : \mathcal{K} \to \mathcal{L}$ with FR_c sending $\mathcal{Z}$ to weak equivalences in $\mathcal{L}$ is the left Quillen functor $\mathcal{K}\backslash\mathcal{Z} \to \mathcal{L}$. A model category is called *left proper* if every pushout of a weak equivalence along a cofibrations is a weak equivalence. Left Bousfield localizations with respect a set of morphisms always exist in two main cases: when $\mathcal{K}$ is a left proper cellular model category (see [28], 4.1.1) and when $\mathcal{K}$ is a left proper combinatorial model category (J. H. Smith, unpublished). A consequence of the second fact is the following result (see [46], 2.3 and 2.4).

Corollary 1.10. *Let $\mathcal{K}$ be a left proper combinatorial model category and $\mathcal{Z}$ a class of morphisms. Assuming Vopěnka's principle, the left Bousfield localization of $\mathcal{K}$ with respect to $\mathcal{Z}$ exists and has $\overline{\mathcal{Z} \cup \mathcal{W}}$ as weak equivalences.*

The most of important model categories are combinatorial.

Examples 1.11. (1) The model category **SSet** of simplicial sets is combinatorial. Cofibrations are monomorphisms which are cofibrantly generated in every Grothendieck topos (see [8]). Trivial cofibrations are generated by horn inclusions (see [29], e.g.). The recently found *quasi-category model structure* is combinatorial too – cofibrations are unchanged and trivial cofibrations are generated by inclusions of inner horns (see [32]).

(2) Let $\mathbf{SSet}^{\mathcal{X}^{\mathrm{op}}}$ be the category of all functors $\mathcal{X}^{\mathrm{op}} \to \mathbf{SSet}$ where $\mathcal{X}$ is a small category. The *projective* model category structure is given by taking weak equivalences and fibrations pointwise, i.e., $\varphi : F \to G$ is a weak equivalence (fibration) if φ_X is a weak equivalence (fibration) for each object X of $\mathcal{X}$. It makes $\mathbf{SSet}^{\mathcal{X}^{\mathrm{op}}}$ a combinatorial model category because (trivial) cofibrations are generated by images of those from **SSet** in left adjoints to evaluation functors

$$\mathrm{ev}_X : \mathbf{SSet}^{\mathcal{X}^{\mathrm{op}}} \to \mathbf{SSet},$$

with $X \in \mathcal{X}$ (see [28], 14.2.1). We denote by

$$Y : \mathcal{X} \to \mathbf{Set}^{\mathcal{X}^{\mathrm{op}}} \subseteq \mathbf{SSet}^{\mathcal{X}^{\mathrm{op}}}$$

the Yoneda embedding; here, **Set** is taken as the category of discrete simplicial sets.

(3) Let us recall that a spectrum X is a sequence $(X_n)_{n=0}^{\infty}$ of pointed simplical sets equipped with morphisms $\sigma_n^X : \Sigma X_n \to X_{n+1}$ where Σ is the suspension functor (see 2.2(3)). The strict model category structure on the category **Sp** of spectra is induced by the projective model category structure on $\mathbf{SSet}_*^{\mathcal{X}}$ where $\mathcal{X}$ is the discrete category with non-negative integers as objects. It means that weak equivalences and fibrations are levelwise. The Bousfield-Friedlander model category structure on **Sp** is obtained by a left Bousfield localization of the strict structure with respect to a suitable set of morphisms. The both model category structures are combinatorial (see [48], A.3).

(4) Let R be a unital ring and R-**Mod** the category of left R-modules. Monomorphisms are cofibrantly generated (in fact, it it true in every Grothendieck abelian category, see [8]) and we take them as cofibrations. Trivial fibrations are then epimorphisms with an injective cokernel. We take homotopy equivalences as weak equivalences. It yields a combinatorial model category structure on R-**Mod** provided that R is a Frobenious ring, i.e., if injective R-modules coincide with projective ones (see [29]).

(5) Let $\mathbf{Ch}(R)$ denote the category of chain complexes of left R-modules. Monomorphisms are again cofibrantly generated and we take them as cofibrations. We take weak equivalences as homology isomorphisms, i.e., as morphisms f of chain complexes such that $H_n f$ are isomorphisms for all n. It yields a combinatorial model category structure on $\mathbf{Ch}(R)$ (see [29]). The resulting homotopy category is the derived category of R.

(6) The category **Top** of topological spaces is not locally presentable. J. H. Smith has proposed a combinatorial model category consisting of $\mathbf{\Delta}$-generated topological spaces (cf. [23]). A topological space X is

Δ-generated if a subset $Y \subseteq X$ is open if and only if $f^{-1}(Y)$ is open for each continuous map $f : \Delta_n \to X$. A (surprisingly delicate) proof that the category of **Δ**-generated spaces is locally presentable can be found in [25].

(7) The model category of chain complexes of abelian groups with cofibrations as dimensionwise split monomorphisms and weak equivalences as homotopy equivalences is locally presentable but not combinatorial (see [41], [16]).

(8) Each weak factorization system $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$ provide a model category structure by taking $\mathcal{L}$ as cofibrations and $\mathcal{K}$ as weak equivalences. Then $(Full, Top)$ consisting of full functors and topological functors is a weak factorization system on the category **Cat** of small categories which provides a non-combinatorial model category structure. Let us add that M. Hovey asked for examples of non-combinatorial model structures on locally presentable categories.

(9) In the category **Pos** of posets, take $\mathcal{C}$ consisting of split monomorphisms. It yields a weak factorization system $(\mathcal{C}, \mathcal{C}^{\square})$ cofibrantly generated by split monomorphisms between finite posets. The closure of split monomorphisms under λ-filtered colimits in $\mathbf{Pos}^{\to}$ precisely consists of λ-pure monomorphisms. It is easy to see that, for each regular cardinal λ, there is a λ-pure monomorphism which does not split. By putting $\mathcal{W} = \mathbf{Pos}$, we get a combinatorial model category which is not strongly combinatorial.

We finish this section with a very important result of D. Dugger [21] whose consequence is that **SSet** is a universal model category with one generator Δ_0. An elementary proof can be found in [46].

Theorem 1.12. *Let $\mathcal{K}$ be a functorial model category, $\mathcal{X}$ a small category and $H : \mathcal{X} \to \mathcal{K}$ a functor such that $H(\mathcal{X}) \subseteq \mathcal{K}_c$. Then there is a Quillen functor $H^* : \mathbf{SSet}^{\mathcal{X}^{\mathrm{op}}} \to \mathcal{K}$ such that $H^*Y = H$.*

2. Brown representability

The aim of this section is to show that, given a combinatorial model category $\mathcal{K}$, then its homotopy category $\mathrm{Ho}(\mathcal{K})$ is "weakly accessible". We will start with the following well known fact.

Proposition 2.1. *Let $\mathcal{K}$ be a model category. Then $\mathrm{Ho}(\mathcal{K})$ has (co)products and weak (co)limits.*

Proof. Since the dual of a model category is a model category, it suffices to show that $\mathcal{K}$ has coproducts and weak colimits. Consider objects K_i, $i \in I$ in $\mathrm{Ho}(\mathcal{K})$. Without any loss of generality, we can assume that they belong to $\mathcal{K}_{cf}$. Then their coproduct K in $\mathcal{K}$ is cofibrant and its fibrant replacement $R_f K$ is a coproduct of K_i in $\mathrm{Ho}(\mathcal{K})$. It remains to show that $\mathrm{Ho}(\mathcal{K})$ has weak pushouts. We will again use the fact that $\mathrm{Ho}(\mathcal{K})$ is equivalent to $\mathcal{K}_{cf}/\sim$ and consider

$$\begin{array}{ccc} B & & \\ {\scriptstyle f}\uparrow & & \\ A & \xrightarrow[g]{} & D \end{array}$$

in $\mathcal{K}_{cf}$. Then a pushout

$$\begin{array}{ccc} B_1 & \xrightarrow{\overline{g}} & E \\ {\scriptstyle f_1}\uparrow & & \uparrow{\scriptstyle \overline{f}} \\ A & \xrightarrow[g_1]{} & D_1 \end{array}$$

in $\mathcal{K}$ where $f = f_2 f_1$ and $g = g_2 g_1$ are (cofibration, trivial fibration) factorizations is called the *homotopy pushout* of the starting diagram and yields a weak pushout in $\mathrm{Ho}(\mathcal{K})$. □

Remark 2.2. (1) Since weak colimits are not unique, we have to be careful about their construction. In 2.1, we at first construct weak coequalizers from weak pushouts and then weak colimits from coproducts and weak coequalizers. In the both cases, it mimics the usual constructions of corresponding colimits. We will call the resulting weak colimits *standard*. In a functorial model category, they are functorial.

Given a diagram $D : \mathcal{D} \to \mathcal{K}$, we get a cone $(\delta_d : Dd \to K)$ such that $(P\delta_d : PDd \to PK)$ is a standard weak colimit of PD. If $(\overline{\delta}_d : Dd \to \overline{K})$ is a colimit in $\mathcal{K}$, we get a comparison morphism $p : K \to \overline{K}$.

(2) There is another construction of weak pushouts in $\mathrm{Ho}(\mathcal{K})$. Consider

$$\begin{array}{ccc} B & & \\ {\scriptstyle f}\uparrow & & \\ A & \xrightarrow[g]{} & D \end{array}$$

in $\mathcal{K}_{cf}$. Form the *double mapping cylinder* of f, g, i.e., the colimit

$$\begin{array}{ccccc} B & & \xrightarrow{\overline{g}} & & E \\ {\scriptstyle f}\uparrow & & & \nearrow{\scriptstyle t} & \\ A & \xrightarrow{\gamma_{1A}} & C(A) & & \uparrow{\scriptstyle \overline{f}} \\ & & \uparrow{\scriptstyle \gamma_{2A}} & & \\ & & A & \xrightarrow[g]{} & D \end{array}$$

of the diagram

$$\begin{array}{ccccc} B & & & & \\ {\scriptstyle f}\uparrow & & & & \\ A & \xrightarrow{\gamma_{1A}} & C(A) & & \\ & & \uparrow{\scriptstyle \gamma_{2A}} & & \\ & & A & \xrightarrow[g]{} & D \end{array}$$

where $C(A)$ is the cylinder object. Then

$$\begin{array}{ccc} QB & \xrightarrow{P\overline{g}} & PE \\ {\scriptstyle Qf}\uparrow & & \uparrow{\scriptstyle P\overline{f}} \\ QA & \xrightarrow[Qg]{} & QD \end{array}$$

is a weak pushout in $\mathrm{Ho}(\mathcal{K})$ (cf. [33]).

If $\mathcal{K}$ is left proper and functorial then the both constructions of weak pushouts in $\mathrm{Ho}(\mathcal{K})$ coincide. In fact, it is true in **SSet** (cf. [44], 3.1(iv)), thus in $\mathbf{SSet}^{\mathcal{X}^{\mathrm{op}}}$ and we can use 1.12 to transport it to $\mathcal{K}$. We need homotopy invariance of homotopy pushouts and pushouts along a cofibration in $\mathcal{K}$ (see [28], 13.3.3 and 13.3.4).

(3) Let $\mathcal{K}$ be a pointed model category. The double mapping cylinder

$$\begin{array}{ccc} 0 & \longrightarrow & \Sigma(A) \\ \uparrow & & \uparrow \\ A & \longrightarrow & 0 \end{array}$$

yields the suspension functor $\Sigma : \mathcal{K} \to \mathcal{K}$.

Given a small full subcategory $\mathcal{A}$ of a category $\mathcal{L}$ we get the functor

$$E_{\mathcal{L}} : \mathcal{L} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

such that $E_{\mathcal{A}}(L)$ is the restriction of the hom-functor $\hom(-, L)$ on $\mathcal{A}^{\mathrm{op}}$. For a regular cardinal λ, $\mathcal{L}_\lambda$ will denote the full subcategory consisting of λ-presentable objects of $\mathcal{L}$. If $\mathcal{L}$ is λ-accessible then $E_{\mathcal{L}_\lambda}$ is full and faithful and makes $\mathcal{L}$ equivalent with the full subcategory

$$\mathrm{Ind}_\lambda \, \mathcal{L}_\lambda \subseteq \mathbf{Set}^{\mathcal{L}_\lambda^{\mathrm{op}}}$$

consisting of λ-filtered colimits of hom-functors. In fact, the categories $\mathrm{Ind}_\lambda \, \mathcal{A}$, $\mathcal{A}$ small, coincide with λ-accessible categories. Moreover, $\mathrm{Ind}_\lambda \, \mathcal{A}$ is a free completion of $\mathcal{A}$ under λ-filtered colimits (see [4]).

Now, we consider the case when $\mathcal{L}$ is the homotopy category $\mathrm{Ho}(\mathcal{K})$ of a model category and $\mathcal{A}$ is its full subcategory consisting of λ-presentable objects in $\mathcal{K}$; we will denote it as $\mathrm{Ho}(\mathcal{K}_\lambda)$. The restriction of the canonical functor $P : \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ will be denoted by

$$P_\lambda : \mathcal{K}_\lambda \to \mathrm{Ho}(\mathcal{K}_\lambda).$$

We get the functor

$$E_{\mathrm{Ho}(\mathcal{K}_\lambda)} : \mathrm{Ho}(\mathcal{K}) \to \mathbf{Set}^{\mathrm{Ho}(\mathcal{K}_\lambda)^{\mathrm{op}}}$$

which will be shortly denoted by E_λ.

Proposition 2.3. *Let $\mathcal{K}$ be a combinatorial model category. Then there is a regular cardinal λ such that $E_\lambda P \cong Ind_\lambda(P_\lambda)$.*

It means that $\mathcal{K}$ is locally λ-presentable, i.e., $\mathcal{K} = \mathrm{Ind}_\lambda \, \mathcal{K}_\lambda$ and the composition $E_\lambda P$ preserves λ-filtered colimits. Then its image is contained in $\mathrm{Ind}_\lambda \mathrm{Ho}(\mathcal{K}_\lambda)$ and the codomain restriction of $E_\lambda P$ coincides with the Ind_λ-extension of P_λ. In particular, E_λ is the functor

$$\mathrm{Ho}(\mathcal{K}) \to \mathrm{Ind}_\lambda \mathrm{Ho}(\mathcal{K}_\lambda).$$

We say that E_λ is *essentially surjective* if (1) for each X in $\mathrm{Ind}_\lambda \mathrm{Ho}(\mathcal{K}_\lambda)$ there is K in $\mathrm{Ho}(\mathcal{K})$ with $E_\lambda K \cong X$ and (2) every morphism $f : X \to E_\lambda L$ allows a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & E_\lambda L \\ {\scriptstyle\cong}\downarrow & \nearrow_{E_\lambda(g)} & \\ E_\lambda K & & \end{array}$$

for some morphism $g : K \to L$. The following theorem (see [44]) can be interpreted as saying that $\mathrm{Ho}(\mathcal{K})$ is weakly accessible for every combinatorial model category $\mathcal{K}$.

Theorem 2.4. *Let $\mathcal{K}$ be a combinatorial model category. Then there is a regular cardinal λ such that*

$$E_\lambda : \mathrm{Ho}(\mathcal{K}) \to \mathrm{Ind}_\lambda \mathrm{Ho}(\mathcal{K}_\lambda)$$

is essentially surjective.

Remark 2.5. (1) A more subtle question is to describe regular cardinals λ satisfying 2.4. For example, for **Sp**, it is true for $\lambda = \omega$ and then for each $\omega_1 \triangleleft \lambda$. While the second claim follows from the proof of 2.4, the first one is due to [1] and is called a *Brown representability.* We will thus say that $\mathcal{K}$ is λ-*Brown* if it satisfies 2.4 for λ. **SSet** is ω_1-Brown (in fact, λ-Brown for each $\omega_1 \triangleleft \lambda$) but it is not known whether is ω-Brown.

Since $\mathrm{Ind}_\lambda(\mathcal{K}_\lambda)$ precisely consists of functors $\mathcal{K}_\lambda^{\mathrm{op}} \to \mathbf{Set}$ preserving weak λ-small limits, λ-Brown property says that any such functor on $\mathrm{Ho}(\mathcal{K}_\lambda)$ is representable. In case of $\mathbf{Ch}(R)$, our ω-Brown property is called a Brown representability for homology (see [17]).

(2) Consider a λ-filtered diagram $D : \mathcal{D} \to \mathrm{Ho}(\mathcal{K}_\lambda)$ and assume that $\mathcal{K}$ is λ-Brown. Let $(\varepsilon_d : E_\lambda Dd \to K)$ be a colimit of $E_\lambda D$ in $\mathrm{Ind}_\lambda \mathrm{Ho}(\mathcal{K}_\lambda)$. Since E_λ is full and essentially surjective on objects, there are morphisms $\widetilde{\delta}_d : Dd \to \widetilde{K}$ such that $E_\lambda \widetilde{K} = K$ and $E_\lambda \widetilde{\delta}_d = \varepsilon_d$ for each $d \in \mathcal{D}$. Since E_λ is faithful on $\mathrm{Ho}(\mathcal{K}_\lambda)$, $\widetilde{\delta}_d$ form a cone and, in fact, a weak colimit cone. We will call such weak λ-filtered colimits *minimal.*

(3) Assume that E_λ reflects isomorphisms. Then a minimal weak λ-filtered colimit $\widetilde{\delta}_d : Dd \to \widetilde{K}$ has the property that each endomorphism $t : \widetilde{K} \to \widetilde{K}$ satisfying $t\widetilde{\delta}_d = \widetilde{\delta}_d$ for each $d \in \mathcal{D}$ is an isomorphism. The consequence is that minimal weak λ-filtered colimits are unique up to an isomorphism. In **Sp**, this assumption is satisfied for $\lambda = \omega$ and for each $\omega_1 \triangleleft \lambda$. Recall that a pointed model category $\mathcal{K}$ is called *stable* if the suspension functor $\Sigma : \mathcal{K} \to \mathcal{K}$ is an isomorphism in $\mathrm{Ho}(\mathcal{K})$ (see [29]). Both **Sp** and $\mathbf{Ch}(R)$ are stable model categories. For a combinatorial stable model category $\mathcal{K}$, 2.4 can be strengthened by adding that E_λ also reflects isomorphisms (see [44], 6.4).

(4) It is very rare that E_λ is also faithful because, in this case, a combinatorial model category $\mathcal{K}$ would have $\mathrm{Ho}(\mathcal{K})$ accessible. For instance, neither $\mathrm{Ho}(\mathbf{SSet})$ nor $\mathrm{Ho}(\mathbf{Sp})$ are concrete (see [26]) and thus they cannot be accessible. An example of a combinatorial model category having $\mathrm{Ho}(\mathcal{K})$ accessible are truncated simplicial sets $\mathbf{SSet}_n$ for each natural number n (see [44]).

Corollary 2.6. *Let $\mathcal{K}$ be a combinatorial stable model category. Then every functor* $\mathrm{Ho}(\mathcal{K})^{\mathrm{op}} \to \mathbf{Set}$ *preserving weak limits is representable.*

The proof uses representability of functors $\mathrm{Ho}(\mathcal{K}_\lambda) \to \mathbf{Set}$ preserving weak λ-small limits with the fact that E_λ reflects isomorphisms. In [17], this property of $\mathrm{Ho}(\mathcal{K})$ is called a *Brown representability for cohomology.*

3. Homotopy accessibility

The second section was based on weak colimits in homotopy categories. Their disadvantage is that they are not uniquely determined and thus they depend on a particular construction. This is true not only for general colimits but also for colimits of strict diagrams in $\mathrm{Ho}(\mathcal{K})$, i.e., for those given by diagrams $D : \mathcal{D} \to \mathcal{K}$. In this case, there is another colimit concept called a homotopy colimit. Intuitively, it is given by an initial homotopy coherent cone. Both for coproducts and pushouts, homotopy colimits and standard weak colimits coincide. But it is not true in general because a homotopy colimit need not be a weak colimit. Unlike weak colimits, homotopy colimits yield a very reasonable and important concept of "homotopy accessibility" (see [38]). There are various approaches to this concept, [38] uses quasi-categories of Joyal (see [31], or [32]), [49] and [50] are based on Segal categories while [43] uses model categories. We will follow [45], which uses simplicial categories.

A *simplicial category* is a category enriched over **SSet**. We recommend [11] as a reference for enriched categories in general. Like any enriched category, a simplicial category $\mathcal{K}$ has the underlying category $\mathcal{K}_0$ whose hom-sets $\hom_0(K, L)$ are sets of points (i.e., 0-simplices) of the simplicial hom-set $\hom(K, L)$. It is induced by the functor

$$\hom(\Delta_0, -) : \mathbf{SSet} \to \mathbf{Set}$$

which has a left adjoint

$$D : \mathbf{Set} \to \mathbf{SSet}$$

sending a set X to the discrete simplicial set having X as the set of points. What is specific for simplicial sets is that D has a left adjoint as well. It is the functor

$$\pi_0 : \mathbf{SSet} \to \mathbf{Set}$$

sending a simplicial set A to its set $\pi_0 A$ of connected components. This makes possible to develop homotopy theory of simplicial categories – the homotopy category $\mathrm{Ho}(\mathcal{K})$ of a simplicial category $\mathcal{K}$ has the same objects as $\mathcal{K}$ and the hom-sets

$$\hom_{\mathrm{Ho}(\mathcal{K})}(K, L) = \pi_0 \hom_{\mathcal{K}}(K, L).$$

A morphism $f : K \to L$ is now called a *homotopy equivalence* if it induces an isomorphism in $\mathrm{Ho}(\mathcal{K})$. We will use the notation $K \simeq L$ for homotopy equivalent object while $K \cong L$ is kept for isomorphic ones. In **SSet**, these homotopy equivalences coincide with the usual ones. But Ho(**SSet**) is not the usual homotopy category of simplicial sets because isomorphisms in the latter are weak equivalences and not homotopy equivalences. The both homotopy categories coincide for the full category **S** of **SSet** consisting of fibrant simplicial sets, i.e., of Kan complexes. Since **S** is neither complete nor cocomplete, it cannot serve as a base for enriched category theory. Instead, we will consider simplicial categories $\mathcal{K}$ whose simplicial hom-sets $\hom(K, L)$ are fibrant; we will call such simplicial categories *fibrant*. If $\mathcal{K}$ is a model category then $\mathcal{K}_{cf}$ is a fibrant simplicial category.

Every simplicial functor $F : \mathcal{K} \to \mathcal{L}$ induces the functor

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{K}) \to \mathrm{Ho}(\mathcal{L}).$$

Definition 3.1. A simplicial functor $F : \mathcal{K} \to \mathcal{L}$ is called a *weak equivalence* if

(1) the induced morphisms $\hom(K_1, K_2) \to \hom(F(K_1), F(K_2))$ are weak equivalences for all objects K_1 and K_2 of $\mathcal{K}$ and
(2) each object L of $\mathrm{Ho}(\mathcal{L})$ is isomorphic in $\mathrm{Ho}(\mathcal{L})$ to $\mathrm{Ho}(F)(K)$ for some object K of $\mathcal{K}$.

These weak equivalences are often called DK-equivalences because they were first described by Dwyer and Kan in [24]. They are a part of a model category structure on the category **SCat** of small simplicial categories and simplicial functors (see [10]). Fibrant objects in this model category are our fibrant simplicial categories.

Ordinary limits (and colimits) are not homotopy invariant in simplicial categories. It means that given a natural transformation

$$\varphi : D_1 \to D_2$$

of diagrams $D_1, D_2 : \mathcal{D} \to \mathcal{K}$ such that δ_d is a homotopy equivalence for each object d of $\mathcal{D}$ then the induced morphism $\lim D_1 \to \lim D_2$ does not need to be a homotopy equivalence. The ordinary limit of a diagram $D : \mathcal{D} \to \mathcal{K}$ in a simplicial category $\mathcal{K}$ is the limit weighted by the constant functor $G : \mathcal{D} \to \mathbf{SSet}$ at the point Δ_0. In simplicial model categories, there is a concept of a homotopy limit going back to Bousfield and Kan [15] (see [28]) which can be expressed as a limit weighted by $B(\mathcal{D} \downarrow -) : \mathcal{D} \to \mathbf{SSet}$ (see [14]). We will call these homotopy limits simplicial. Dually, one has simplicial homotopy colimits.

Definition 3.2. Let $\mathcal{K}$ be a simplicial category, $\mathcal{D}$ a small category and $D : \mathcal{D} \to \mathcal{K}$ a functor. Then the *simplicial homotopy limit* $\operatorname{holim}_s D$ of D is defined as the limit of D weighted by

$$B(\mathcal{D} \downarrow -) : \mathcal{D} \to \mathbf{SSet}\,.$$

The *simplicial homotopy colimit* $\operatorname{hocolim}_s D$ of D is defined as the colimit of D weighted by

$$B((- \downarrow \mathcal{D})^{op}) : \mathcal{D}^{op} \to \mathbf{SSet}\,.$$

Even in simplicial model categories, simplicial homotopy (co)limits are not fully homotopy invariant. Moreover, they only depend on the simplicial structure and thus, taking a left Bousfield localization $\mathcal{K}/\mathcal{Z}$ of a simplicial model category $\mathcal{K}$, the both model categories have the same simplicial homotopy (co)limits. The following definition has been recently proposed in [45].

Definition 3.3. Let $\mathcal{K}$ be a fibrant simplicial category, $\mathcal{D}$ a category and consider a diagram $D : \mathcal{D} \to \mathcal{K}$. We say that $\operatorname{holim}_f D$ is a *fibrant homotopy limit* of D if there are homotopy equivalences

$$\delta_A : \hom(A, \operatorname{holim}_f D) \to \operatorname{holim}_s \hom(A, D)$$

which are simplicially natural in A.

Analogously, we define *fibrant homotopy colimit* $\operatorname{hocolim}_f D$ of D by the existence of homotopy equivalences

$$\delta_A : \hom(\operatorname{hocolim}_f D, A) \to \operatorname{holim}_s \hom(D, A).$$

which are simplicially natural in A.

In particular, we have the formulas

$$\hom(A, \operatorname{holim}_f D) \simeq \operatorname{holim}_s \hom(A, D)$$

and

$$\hom(\operatorname{hocolim}_f D, A) \simeq \operatorname{holim}_s \hom(D, A).$$

It was shown in [45] that fibrant homotopy (co)limits are determined uniquely up to a homotopy equivalence and that they are homotopy invariant. Moreover, δ yields the morphism

$$\tilde{\delta} : B(\mathcal{D} \downarrow -) \to \hom(\operatorname{holim}_f D, D)$$

which can be understood as an analogy of the limit cone for a usual limit. Dually, one gets

$$\tilde{\delta} : B(- \downarrow \mathcal{D})^{\mathrm{op}} \to \hom(D, \operatorname{hocolim}_f D).$$

Then a simplicial functor $F : \mathcal{K} \to \mathcal{L}$ *preserves* the fibrant homotopy limit of a diagram $D : \mathcal{D} \to \mathcal{K}$ if $(F \operatorname{holim}_f D, F\tilde{\delta})$ is a fibrant homotopy limit of FD. Dually, we define the preservation of fibrant homotopy colimits.

Assume that $\mathcal{K}$ is a simplicial model category and consider a diagram $D : \mathcal{D} \to \mathcal{K}_{cf}$. Then

$$\operatorname{holim}_f D \simeq R_c \operatorname{holim}_s D$$

and

$$\operatorname{hocolim}_f D \simeq R_f \operatorname{hocolim}_s D.$$

Let $\mathcal{C}$ be a small simplicial category and consider the simplicial category $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ of simplicial functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{SSet}$. We have the Yoneda embedding

$$Y_{\mathcal{C}} : \mathcal{C} \to \mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$$

given by $Y(C) = \hom(-, C)$. The category $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ has all weighted colimits and all weighted limits and the Yoneda embedding $Y_{\mathcal{C}}$ makes it the free completion of $\mathcal{C}$ under weighted colimits. It also preserves all existing weighted limits (cf. [34]). Like in 1.11(2), there is the projective model category structure on $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$. We put

$$\operatorname{Pre}(\mathcal{C}) = (\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}})_{cf}.$$

Since hom-functors $\hom(-, C)$ are always cofibrant, they belong to $\operatorname{Pre}(\mathcal{C})$ provided that $\mathcal{C}$ is fibrant. Thus, for a small fibrant simplicial category $\mathcal{C}$, we have the Yoneda embedding

$$Y_{\mathcal{C}} : \mathcal{C} \to \operatorname{Pre}(\mathcal{C}).$$

Since discrete simplicial sets are fibrant, every ordinary category is a fibrant simplicial category. The following basic result was proved in [45]. For an ordinary category $\mathcal{C}$, it follows from Dugger's proof of 1.12.

Theorem 3.4. *Let $\mathcal{C}$ be a small fibrant simplicial category. Then every object of* $\operatorname{Pre}(\mathcal{C})$ *is homotopy equivalent to a fibrant homotopy colimit of hom-functors.*

An object K of a fibrant simplicial category $\mathcal{K}$ is *homotopy absolutely presentable* if its hom-functor $\hom(K, -) : \mathcal{K} \to \mathbf{S}$ preserves fibrant homotopy colimits. Since fibrant homotopy colimits in $\operatorname{Pre}(\mathcal{C})$ are pointwise, hom-functors $\hom(-, C)$ are homotopy absolutely presentable in $\operatorname{Pre}(\mathcal{C})$.

Theorem 3.5. *A fibrant simplicial category $\mathcal{K}$ is weakly equivalent to* $\operatorname{Pre}(\mathcal{C})$ *for some small fibrant simplicial category $\mathcal{C}$ if and only if it has fibrant homotopy colimits and has a set $\mathcal{A}$ of homotopy absolutely presentable objects such that every object of $\mathcal{K}$ is a fibrant homotopy colimit of objects from $\mathcal{A}$.*

Proof. We consider the simplicial functor

$$E : \mathcal{K} \to \mathrm{Pre}(\mathcal{A})$$

such that $E(K)$ is the restriction of $\hom(-, K)$ on $\mathcal{A}^{\mathrm{op}}$. By [45], E preserves fibrant homotopy colimits and satisfies condition (1) from 3.1. The second condition follows from 3.4. □

Remark 3.6. In the proof of 3.5, we could not use $F : \mathrm{Pre}(\mathcal{A}) \to \mathcal{K}$ such that $FY_{\mathcal{A}}$ is the inclusion of $\mathcal{A}$ into $\mathcal{K}$ and F preserves fibrant homotopy colimits because we do not know in our framework that fibrant homotopy colimits are functorial. For instance, if $\mathcal{K}$ is a non-functorial model category then there is no reason for having fibrant homotopy colimits functorial. An important example of a non-functorial model category is the projective model category $\mathcal{M}(\mathcal{C})$ of small simplicial functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{SSet}$, i.e., small weighted colimits of hom-functors, on a large category $\mathcal{C}$ (see [19]). Given a fibrant simplicial category $\mathcal{K}$, $Y_{\mathcal{K}}$ embeds $\mathcal{K}$ into $\mathcal{M}(\mathcal{K})_{cf}$ and our fibrant homotopy (co)limits are homotopy equivalent to those calculated in $\mathcal{M}(\mathcal{K})_{cf}$.

As a consequence, we get the following model theoretic result.

Theorem 3.7. *A simplicial model category $\mathcal{K}$ is Quillen equivalent to the model category $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ for some small fibrant simplicial category $\mathcal{C}$ if and only $\mathcal{K}_{cf}$ has a set $\mathcal{A}$ of homotopy absolutely presentable objects such that every object of $\mathcal{K}_{cf}$ is a fibrant homotopy colimit of objects from $\mathcal{A}$.*

Proposition 3.8. *In* $\mathbf{S}$*, filtered fibrant homotopy colimits are homotopy equivalent to filtered colimits.*

Proof. Since $\mathbf{S}$ is closed under filtered colimits, it follows from the fact that filtered simplicial homotopy colimits in $\mathbf{SSet}$ are weakly equivalent to filtered colimits (see [15], XII., 3.5(ii)). □

Since filtered colimits in $\mathbf{SSet}$ commute with finite weighted limits (see [12]), the consequence of 3.8 is that filtered fibrant homotopy colimits commute with finite fibrant homotopy limits in $\mathbf{S}$. An object K of a fibrant simplicial category $\mathcal{K}$ is *homotopy* λ-presentable (λ is a regular cardinal) if its hom-functor $\hom(K, -) : \mathcal{K} \to \mathbf{S}$ preserves fibrant homotopy λ-filtered colimits.

Definition 3.9. A fibrant simplicial category $\mathcal{K}$ is called *homotopy locally λ-presentable* (λ is a regular cardinal) provided that it has fibrant homotopy colimits and has a set $\mathcal{A}$ of homotopy λ-presentable objects such that every object of $\mathcal{K}$ is a λ-filtered fibrant homotopy colimit of objects from $\mathcal{A}$.

$\mathcal{K}$ is *homotopy locally presentable* if it is homotopy locally λ-presentable for some regular cardinal λ.

Recall that a category is λ-small if it has less than λ morphisms.

Definition 3.10. A *homotopy λ-limit theory* is defined as a small fibrant simplicial category $\mathcal{T}$ having all λ-small fibrant homotopy limits.

A *homotopy $\mathcal{T}$-model* is a simplicial functor $A : \mathcal{T} \to \mathbf{S}$ belonging to $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$ and preserving λ-small fibrant homotopy limits.

We will denote by $\mathrm{HMod}(\mathcal{T})$ the full subcategory of $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$ consisting of all homotopy $\mathcal{T}$-models.

Theorem 3.11. *Let λ be a regular cardinal. A fibrant simplicial category $\mathcal{K}$ is homotopy locally λ-presentable if and only if it is weakly equivalent to* $\mathrm{HMod}(\mathcal{T})$ *for some homotopy λ-limit theory $\mathcal{T}$.*

Let us mention that, given a λ-limit theory $\mathcal{T}$, $\mathrm{HMod}(\mathcal{T})$ is weakly equivalent to the closure $\mathrm{HInd}_\lambda(\mathcal{T}^{\mathrm{op}})$ of hom-functors under λ-filtered fibrant homotopy colimits in $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$. We also get morphisms

$$m_D : \mathrm{hocolim}_f \hom(D, -) \to \hom(\mathrm{holim}_f D, -)$$

in $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$ for each λ-small diagram $\mathcal{D} \to \mathcal{T}$. The left Bousfield localization of $\mathbf{SSet}^{\mathcal{T}}$ with respect to the set $\mathcal{Z}$ of all morphisms m_D is called the *model category for homotopy $\mathcal{T}$-models*.

Corollary 3.12. *A simplicial model category $\mathcal{K}$ is Quillen equivalent to the model category for homotopy $\mathcal{T}$-models for some homotopy λ-limit theory $\mathcal{T}$ if and only if $\mathcal{K}_{cf}$ is homotopy locally λ-presentable.*

Corollary 3.13. *Let $\mathcal{K}$ be a simplicial combinatorial model category. Then $\mathcal{K}_{cf}$ is homotopy locally presentable.*

Proof. We know that $\mathcal{K}_{cf}$ has fibrant homotopy colimits. By 1.6, there is a regular cardinal λ such that $\mathcal{K}_{cf}$ is λ-accessible and closed under λ-filtered colimits in $\mathcal{K}$. Let $\mathcal{A}$ be a representative set of λ-presentable objects in $\mathcal{K}_{cf}$. By 3.8, λ-filtered fibrant homotopy colimits are homotopy equivalent to λ-filtered colimits in $\mathcal{K}_{cf}$. Hence $\mathcal{A}$ makes $\mathcal{K}_{cf}$ homotopy locally λ-presentable. □

It follows from 3.12 and 3.13 that every simplicial combinatorial model category is Quillen equivalent to a left Bousfield localization $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}/\mathcal{Z}$ for some small category $\mathcal{C}$ and a set $\mathcal{Z}$. Dugger [22] proved this for every combinatorial model category.

In [45], there are also considered *homotopy varieties* which are given by simplicial algebraic theories. They form a special case of homotopy locally finitely presentable categories and there are proved characterizations of the kind of 3.11 and 3.12 for them. The situation is more delicate here because the emerging homotopy sifted colimits do not coincide with sifted ones. One has to replace reflexive coequalizers by homotopy colimits of simplicial objects. It makes possible to extend [5] and [3] to the homotopy context. It remains to develop the theory of *homotopy exact* categories which has been started in [37], [39], [49], [50] and [43]. On the other hand, one can consider theories specified by both homotopy limits and homotopy colimits and relate them to homotopy accessible categories. These categories are considered in [38].

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J. Rosický
Department of Mathematics and Statistics
Masaryk University, Faculty of Sciences
Janáčkovo nám. 2a, 60200 Brno, Czech Republic
rosicky@math.muni.cz