# Class-combinatorial model categories

J. Rosický*

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*Joint work with B. Chorny

**Definition.** A category $\mathcal{K}$ is called $\lambda$-*class-accessible*, where $\lambda$ is a regular cardinal, provided that

(1) $\mathcal{K}$ has $\lambda$-filtered colimits,

(2) $\mathcal{K}$ has a class $\mathcal{A}$ of $\lambda$-presentable objects such that every object of $\mathcal{K}$ is a $\lambda$-filtered colimit of objects from $\mathcal{A}$.

B. Banaschewski and H. Herrlich introduced these categories in 1976 – it means long ago the appearence of Makkai and Paré.

A category is *class-accessible* if it is $\lambda$-class-accessible for some regular cardinal $\lambda$. A complete and cocomplete $\lambda$-class-accessible category is called *locally* $\lambda$-*class-presentable.* A category is *locally class-presentable* is it is locally $\lambda$-class-presentable for some regular cardinal $\lambda$.

Each $\lambda$-accessible category is $\lambda$-class-accessible. Since each locally presentable category is complete, each locally $\lambda$-presentable category is locally $\lambda$-class-presentable.

Given a category $\mathcal{A}$, $\mathcal{P}(\mathcal{A})$ will denote the category of *small presheaves* on $\mathcal{A}$, i.e. functors $F : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ which are small colimits of hom-functors. Of course, for a small category $\mathcal{A}$, each $F$ is small.

The category $\mathcal{P}(\mathcal{A})$ is always finitely class-accessible because each small presheaf is a small filtered colimit of finite colimits of hom-functors and the latter are finitely presentable. $\mathcal{P}(\mathcal{A})$ is always cocomplete but not necessarily complete. For instance, it does not have a terminal object in the case when $\mathcal{A}$ is a large discrete category (it means that it has a proper class of objects and the only morphisms are the identities). It explains why we added completeness into the definition of a locally class-presentable category.

Given a category $\mathcal{A}$, let $\mathbf{Ind}_\lambda(\mathcal{A})$ be the full subcategory of $\mathcal{P}(\mathcal{A})$ consisting of small $\lambda$-filtered colimits of hom-functors. $\mathbf{Ind}_\lambda(\mathcal{A})$ is always $\lambda$-class-accessible. In fact, each $\lambda$-class-accessible category $\mathcal{K}$ is equivalent to $\mathbf{Ind}_\lambda(\mathcal{A})$ for $\mathcal{A}$ being the full subcategory of $\mathcal{K}$ consisting of $\lambda$-presentable objects.

A functor $F : \mathcal{K} \to \mathcal{L}$ is called $\lambda$-*class-accessible* (where $\lambda$ is a regular cardinal) if $\mathcal{K}$ and $\mathcal{L}$ are $\lambda$-class-accessible categories and $F$ preserves $\lambda$-filtered colimits. A $\lambda$-class-accessible functor preserving $\lambda$-presentable objects is called *strongly* $\lambda$-*class-accessible.* $F$ is called *(strongly) class-accessible* if it is (strongly) $\lambda$-class-accessible for some regular cardinal $\lambda$.

Each accessible functor is strongly accessible but this does not generalize to class-accessible functors.

**Proposition.** Let $\lambda$ be a regular cardinal and $F : \mathcal{K} \to \mathcal{M}$ and $G : \mathcal{L} \to \mathcal{M}$ (strongly) $\lambda$-class-accessible functors. Then their pseudopullback

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\bar{F}} & \mathcal{L} \\ {\scriptstyle \bar{G}}\downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{K} & \xrightarrow[F]{} & \mathcal{M} \end{array}$$

is a $\lambda^+$-class-accessible category and the functors $\bar{F}, \bar{G}$ are (strongly) $\lambda^+$-class accessible.

The analogous result holds for equifiers and I believe that class-accessible categories are closed under lax limits.

**Definition.** A class $\mathcal{C}$ of morphisms of a category $\mathcal{K}$ is called *locally small* if, for each morphism $f$ in $\mathcal{K}$, there is a subset $\mathcal{S}$ of $\mathcal{C}$ such that each morphism $g \to f$ in $\mathcal{K}^{\to}$ with $g \in \mathcal{C}$ factorizes as

$$g \to h \to f$$

with $h \in \mathcal{S}$.

Each set $\mathcal{C}$ of morphisms is locally small.

**Theorem.** Let $\mathcal{K}$ be a locally class-presentable category, $\mathcal{C}$ a locally small class of morphisms of $\mathcal{K}$ and assume that there is a regular cardinal $\lambda$ such that each morphism from $\mathcal{C}$ has a $\lambda$-presentable domain. Then $(\mathrm{cof}(\mathcal{C}), \mathcal{C}^{\square})$ is a weak factorization system in $\mathcal{K}$.

Given a weak factorization system $(\mathcal{L}, \mathcal{R})$, then the class $\mathcal{L}$ is always locally small.

**Corollary.** Let $\mathcal{K}$ be a locally class-presentable category, $\mathcal{C}$ a locally small class of morphisms of $\mathcal{K}$. Let $\lambda$ be a regular cardinal such that each morphism from $\mathcal{C}$ has a $\lambda$-presentable domain. Then $\mathbf{Inj}(\mathcal{C})$ is weakly reflective and closed under $\lambda$-filtered colimits in $\mathcal{K}$.

But $\mathbf{Inj}(\mathcal{C})$ does not need to be class-accessible.

**Theorem.** Let $\mathcal{K}$ be a locally class-presentable category, $\mathcal{C}$ a locally small class of morphisms of $\mathcal{K}$ and assume that there is a regular cardinal $\lambda$ such that each morphism from $\mathcal{C}$ has a $\lambda$-presentable domain and codomain. Then $(\mathrm{colim}(\mathcal{C}), \mathcal{C}^{\perp})$ is a factorization system in $\mathcal{K}$.

**Corollary.** Let $\mathcal{K}$ be a locally $\lambda$-class-presentable category, $\mathcal{C}$ a locally small class of morphisms of $\mathcal{K}$ such that each morphism from $\mathcal{C}$ has a $\lambda$-presentable domain and codomain. Then $\mathbf{Ort}(\mathcal{C})$ is reflective and closed under $\lambda$-filtered colimits in $\mathcal{K}$. Moreover, $\mathbf{Ort}(\mathcal{C})$ is locally $\lambda$-class-presentable.

**Definition** Let $\mathcal{K}$ be a locally class-presentable category. A weak factorization system $(\mathcal{L}, \mathcal{R})$ in $\mathcal{K}$ is called *cofibrantly class-generated* if $\mathcal{L} = \mathrm{cof}(\mathcal{C})$ for a locally small class $\mathcal{C}$ of morphisms having $\lambda$-presentable domains and codomains (for some regular cardinal $\lambda$).

**Definition.** A model category is called *class-combinatorial* if its underlying category $\mathcal{K}$ is locally class-presentable and both (cofibrations, trivial fibrations) and (trivial cofibrations, fibrations) are cofibrantly class-generated weak factorization systems.

Each combinatorial model category is class-combinatorial.

Given a simplicial category $\mathcal{A}$, then the category of small simplicial presheaves $\mathcal{A}^{\mathrm{op}} \to \mathbf{SSet}$ with the projective model category structure is class-combinatorial.

**Theorem.** Let $\mathcal{K}$ be a class-combinatorial model category and $\mathcal{W}$ its class of weak equivalences. Then the inclusion of $\mathcal{W}$ in $\mathcal{K}^{\to}$ is a class-accessible functor. In particular, $\mathcal{W}$ is a class-accessible category.

For combinatorial model categories, this result was claimed by J. H. Smith. Last year, I gave a proof based on the fact that homotopy equivalences are the full image of an acccessible functor and using a pseudopullback

$$\begin{array}{ccc}
\mathcal{K}^{\to} & \xrightarrow{R} & \mathcal{K}^{\to} \\
\uparrow & & \uparrow \\
\mathcal{W} & \longrightarrow & \mathcal{H}
\end{array}$$

where $R$ is the replacement functor.

A. Stanculescu informed me that J. Lurie gave a simpler proof based on a pseudopullback

$$\begin{array}{ccc} \mathcal{K}^{\to} & \xrightarrow{F} & \mathcal{K}^{\to} \\ \uparrow & & \uparrow \\ \mathcal{W} & \longrightarrow & \mathcal{F} \end{array}$$

where $F$ gives the fibration part in the (trivial cofibration, fibration) factorization and $\mathcal{F}$ denotes trivial fibrations.

Since $\mathcal{F}$ is not necessarily class-accessible in a class-combinatorial setting, I have to combine both mine and Lurie's proof here. Mine yields (2) and Lurie's (1) in the definition of a class-accessible category.

Our goal has been to develop the theory of left Bousfield localizations in the class-combinatorial setting. We have some particular result for simplicial class-combinatorial model categories.