Homotopy accessible categories

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*Joint work with S. Lack

$\mathcal{V}$ combinatorial monoidal model category

One can do homotopy theory for $\mathcal{V}$-categories.

$$P : \mathcal{V} \to \mathrm{Ho}\,\mathcal{V} \xrightarrow{\mathrm{Ho}(I,-)} \mathbf{Set}$$

is a strong monoidal functor.

Hence, for each $\mathcal{V}$-category $\mathcal{K}$, we have its homotopy category $\mathrm{ho}\,\mathcal{K}$ given by

$$\mathrm{ho}\,\mathcal{K}(A,B) = P\mathcal{K}(A,B)$$

$f : A \to B$ *homotopy equivalence* if $P(f)$ is isomorphism

$A \simeq B$ denotes homotopy equivalent objects

Example. $\mathcal{V} = \mathbf{SSet}$

$$\operatorname{ho}\mathcal{K}(A, B) = \pi_0\mathcal{K}(A, B)$$

is the set of connected components of a simplicial category $\mathcal{K}$

$$\operatorname{ho}\mathbf{SSet} \neq \operatorname{Ho}\mathbf{SSet}$$

isomorphisms are homotopy equivalences in $\operatorname{ho}\mathbf{SSet}$ and weak equivalences in $\operatorname{Ho}\mathbf{SSet}$

$$\operatorname{ho}\mathbf{Kan} = \operatorname{Ho}\mathbf{Kan}$$

where **Kan** is the simplicial category of Kan complexes, i.e., of fibrant simplicial sets

Other important examples are

$$\mathcal{V} = \mathbf{Sp}$$

the category of spectra, or

$$\mathcal{V} = \mathbf{Ch}_R$$

the category of unbounded chain complexes over a ring R.

In the first case we get homotopy theory of spectral categories and, in the second case, of dg-categories.

$F : \mathcal{K} \to \mathcal{L} \qquad \mathcal{V}$-functor

$\operatorname{ho} F : \operatorname{ho} \mathcal{K} \to \operatorname{ho} \mathcal{L}$ the induced functor

Definition. F is called a *weak equivalence* if

(1) $\mathcal{K}(A, B) \to \mathcal{L}(FA, FB)$ is a weak equivalence for each A, B,

(2) for each L there is K such that $L \simeq FK$.

In the simplicial case, these functors are called Dwyer-Kan equivalences. They are weak equivalences in a model category structure on small simplicial categories. We will not need this model category structure but we will need its fibrant objects.

Definition. $\mathcal{V}$-category $\mathcal{K}$ is called *fibrant* if $\mathcal{K}(A, B)$ is fibrant in $\mathcal{V}$ for each A, B.

Fibrant $\mathcal{V}$ categories are homotopy correct ones. For example, **Kan** is fibrant.

Lemma. Let $\mathcal{K}$ be a fibrant $\mathcal{V}$-category and $f : A \to B$ a morphism. Then the following conditions are equivalent

(i) f is a homotopy equivalence,

(ii) $\mathcal{K}(C, f)$ is a weak equivalence for all C in $\mathcal{K}$,

(iii) $\mathcal{K}(f, C)$ is a weak equivalence for all C in $\mathcal{K}$.

In fact, it suffices to take C equal to A or B.

$\mathcal{M}$ model $\mathcal{V}$-category, $\operatorname{Int}\mathcal{M}$ its full subcategory consisting of objects which are both fibrant and cofibrant.

Then $\operatorname{Int}\mathcal{M}$ is a fibrant $\mathcal{V}$-category.

$\mathbf{Kan} = \operatorname{Int}\mathbf{SSet}$

$\mathcal{C}$ small $\mathcal{V}$-category, $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ $\mathcal{V}$-category of $\mathcal{V}$-functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ is a model category w.r.t. the projective model structure where weak equivalences and fibrations are pointwise.

Recall that a *colimit* $\mathrm{colim}_G\, D$ of $D : \mathcal{D} \to \mathcal{K}$ *weighted* by $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{V}$ is given by the formula

$$\mathcal{K}(\mathrm{colim}_G\, D, -) \cong [\mathcal{D}^{\mathrm{op}}, \mathcal{V}](G, \mathcal{K}(D, -))$$

A weight $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{V}$ is called *cofibrant* if it is a cofibrant object in $[\mathcal{D}^{\mathrm{op}}, \mathcal{V}]$. Cofibrant weights form a saturated class in the sense that they are closed under weighted colimits with a cofibrant weight.

Definition. Let $\mathcal{K}$ be a fibrant $\mathcal{V}$-category, $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{V}$ a cofibrant weight and $D : \mathcal{D} \to \mathcal{K}$ a diagram. Then $\operatorname{hocolim}_G D$ is called a *homotopy weighted colimit* if there is a natural transformation

$$\delta : \mathcal{K}(\operatorname{hocolim}_G D, -) \to [\mathcal{D}^{\mathrm{op}}, \mathcal{V}](G, \mathcal{K}(D, -))$$

whose components are weak equivalences.

By Yoneda, to give δ is the same as to give a natural transformation

$$\eta : G \to \mathcal{K}(D, \operatorname{hocolim}_G D)$$

in $[\mathcal{D}^{\mathrm{op}}, \mathcal{V}]$.

Proposition 1. Homotopy weighted colimits are defined uniquely up to homotopy equivalence.

Proof. Let K and K' be two such homotopy colimits. Let $J : \mathcal{X} \to \mathcal{K}$ be the inclusion of the small full subcategory of $\mathcal{K}$ consisting of K, K' and the image of D. The induced maps

$$\mathcal{K}(K, J-) \to [\mathcal{D}^{\mathrm{op}}, \mathcal{V}](G, \mathcal{K}(D, J-)) \leftarrow \mathcal{K}(K', J-)$$

are pointwise weak equivalences in $[\mathcal{X}, \mathcal{V}]$, so

$$\mathcal{X}(K, -) = \mathcal{K}(K, J-)$$

and

$$\mathcal{X}(K', -) = \mathcal{K}'(K', J-)$$

are weakly equivalent in $[\mathcal{X}, \mathcal{V}]$. Since they are both cofibrant (as hom-functors) and fibrant (because $\mathcal{K}$ is fibrant), they are homotopy equivalent. So, K and K' are homotopy equivalent.

Similarly, any object homotopy equivalent to a homotopy colimit can be itself used as a homotopy colimit.

Example. Let $\mathcal{K}$ be a simplicial category and $D : \mathcal{D} \to \mathcal{K}$ a diagram. Then the *simplicial homotopy colimit* of D is defined as the colimit of D weighted by

$$B((- \downarrow \mathcal{D})^{\mathrm{op}}) : \mathcal{D}^{\mathrm{op}} \to \mathbf{SSet}\,.$$

This weight is a cofibrant approximation of the constant diagram $\mathcal{D}^{\mathrm{op}} \to \mathbf{SSet}$ at 1. **Kan** in particular and $\operatorname{Int}\mathcal{M}$ in general are not closed under simplicial homotopy colimits. In order to take homotopy colimits living in $\operatorname{Int}\mathcal{M}$ one has to take fibrant replacements of simplicial homotopy colimits. They are automatically cofibrant because cofibrant objects are closed under simplicial homotopy colimits. In this way, we arrive to our general definition above.

All what I said up to now is a joint work with Steve Lack. The simplicial case was done in my paper [OHV] On homotopy varieties.

Proposition 2. Let $\mathcal{M}$ be a model $\mathcal{V}$-category. Then $\operatorname{Int}\mathcal{M}$ has homotopy weighted colimits and homotopy weighted limits.

Of course, homotopy weighted limits are defined dually. Emphasize that weights remain to be cofibrant.

Proposition 3. Let $\mathcal{C}$ be a small fibrant $\mathcal{V}$-category. Then $\operatorname{Int}[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is a free completion of $\mathcal{C}$ under weighted homotopy colimits.

Proof. It suffices to add to Proposition 2 that

$$F = \operatorname{colim}_F Y$$

for each F in $\operatorname{Int}[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$.

I do not claim that there is a universal property

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "Y"] \arrow[dr, "H"'] & & {[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]} \arrow[dl, "H^*"] \\ & \mathcal{K} & \end{tikzcd}$$

assigning to a $\mathcal{V}$-functor H a $\mathcal{V}$-functor H^* preserving homotopy weighted colimits. The reason is that homotopy weighted colimits are not functorial in general. For example, given a complete $\mathcal{V}$-category $\mathcal{K}$, the $\mathcal{V}$-category of small $\mathcal{V}$-functors form a model category with the projective model structure (Chorny, Dwyer). But this model structure is not functorial and, consequently, homotopy weighted colimits in its Int are not functorial.

Theorem 1. Let $\mathcal{C}$ be a small fibrant simplicial category. Then $\mathrm{Int}[\mathcal{C}^{\mathrm{op}}, \mathbf{SSet}]$ is a free completion of $\mathcal{C}$ under homotopy colimits.

This is much deeper than Proposition 3 and I proved it in [OHV]. In the special case of an ordinary category $\mathcal{C}$ (which is a fibrant simplicial category), it was proved by Dugger in his work on universal model categories.

simplicial nerve

$$\mathbf{SSet}\text{-}\mathbf{Cat} \to \mathbf{SSet}$$

with a left adjoint (simplicial realization)

$$R : \mathbf{SSet} \to \mathbf{SSet}\text{-}\mathbf{Cat}$$

fibrant simplicial categories go to quasicategories

our homotopy (co)limits correspond to (co)limits of quasicategories

Let $\mathcal{D}$ be a category, $\mathcal{K}$ a simplicial category and $D : \mathcal{D} \to \mathcal{K}$ a functor such that the weight

$$B(\mathcal{D} \downarrow -) \to \mathbf{SSet}$$

is finite (i.e., $B(\mathcal{D} \downarrow d)$ is finitely presentable for each d in $\mathcal{D}$. The the homotopy limit $\operatorname{holim} D$ is called *finite*.

Proposition 4. Filtered homotopy colimits commute in **Kan** with finite homotopy limits.

Proof. Filtered homotopy colimits are homotopy equivalent with filtered colimits. The latter commute with finite weighted limits.

This is not true for any finite diagram, i.e., having $\mathcal{D}$ finite (Daniel Davis, T. Torii, L. Vokřínek).

An object A of a fibrant simplicial category $\mathcal{K}$ is called *homotopy finitely presentable* if $\hom(A, -) : \mathcal{K} \to \mathbf{SSet}$ preserves filtered homotopy colimits.

Definition. A fibrant simplicial category $\mathcal{K}$ is called *homotopy locally finitely presentable* if

(1) it is homotopy cocomplete and

(2) it has a set $\mathcal{A}$ of homotopy finitely presentable objects such that each object is a homotopy colimit of objects from $\mathcal{A}$.

Homotopy locally finitely presentable categories are precisely categories $\mathrm{HInd}(\mathcal{C})$ where $\mathcal{C}$ is a small fibrant simplicial category.
$\mathrm{HInd}(\mathcal{C})$ is a full subcategory of $\mathrm{Int}[\mathcal{C}^{\mathrm{op}}, \mathbf{SSet}]$ consisting of filtered homotopy colimits of hom-functors.

Homotopy locally finitely presentable categories are precisely categories $\mathrm{HMod}(\mathcal{C})$ where $\mathcal{C}$ is a small fibrant simplicial category having finite homotopy limits.

$\mathrm{HMod}(\mathcal{C})$ is a full subcategory of $\mathrm{Int}[\mathcal{C}, \mathbf{SSet}]$ consisting of simplicial functors preserving finite homotopy limits

finite homotopy limit sketches, homotopy orthogonality

Definition. A small category $\mathcal{D}$ is called *homotopy sifted* if homotopy colimits over $\mathcal{D}$ commute with finite products in **Kan**.

Theorem 2. $\mathcal{D}$ is homotopy sifted iff $\triangle : \mathcal{D} \to \mathcal{D}\times\mathcal{D}$ is homotopy final.

filtered $\Rightarrow$ homotopy sifted $\Rightarrow$ sifted

reflexive pairs are not homotopy sifted, Δ^{op} is homotopy sifted

homotopy sifted = totally coaspherical (in the sense of Maltsiniotis)

An object A of a fibrant simplicial category $\mathcal{K}$ is called *homotopy projectively finitely presentable* if

$$\hom(A, -) : \mathcal{K} \to \mathbf{SSet}$$

preserves homotopy sifted homotopy colimits.

Definition. A fibrant simplicial category $\mathcal{K}$ is called *homotopy algebraic* if

(1) it is homotopy cocomplete and

(2) it has a set $\mathcal{A}$ of homotopy projectively finitely presentable objects such that each object is a homotopy colimit of objects from $\mathcal{A}$.

Homotopy algebraic categories are precisely categories $\mathrm{HSInd}(\mathcal{C})$ where $\mathcal{C}$ is a small fibrant simplicial category.

$\mathrm{HSInd}(\mathcal{C})$ is a full subcategory of $\mathrm{Int}[\mathcal{C}^{\mathrm{op}}, \mathbf{SSet}]$ consisting of homotopy sifted homotopy colimits of hom-functors

Homotopy algebraic categories are precisely categories $\mathrm{HAlg}(\mathcal{C})$ where $\mathcal{C}$ is a small fibrant simplicial category having finite products.

$\mathrm{HAlg}(\mathcal{C})$ is a full subcategory of $\mathrm{Int}[\mathcal{C}, \mathbf{SSet}]$ consisting of simplicial functors preserving finite products.

Definition. A fibrant simplicial category $\mathcal{K}$ is called *homotopy finitely accessible* if

(1) it has filtered homotopy colimits and

(2) it has a set $\mathcal{A}$ of homotopy finitely presentable objects such that each object is a filtered homotopy colimit of objects from $\mathcal{A}$.

homotopy accessible categories

They should be given by (homotopy limit, homotopy colimit) sketches.