Abstract

At present the expansion of computation technology affected also the teaching of mathematics. Innovative teaching methods are often based on using computers. Since in our conditions texts devoted to teaching with help of computers have not been published, this thesis presents an attempt of creation of such texts. The purpose of this thesis is to show, how we can support teaching the differential calculus functions of several variables by means of computer.

At the start of the work it was necessary to map out existing materials available on the computer network Internet. In this period the publications and [1], [2], [3], [4] and the serial of papers [5], [6], [7] and [8] were created. All of them are dedicated to the problematics of Internet and to its services for mathematics and for teaching of mathematical analysis. Some results closely connected to the contents of this thesis were published in the Journal Pokroky MFA in the paper *Technology in teaching calculus: Local extrema of functions of two variables* ([9], in Czech). The chapter The graph of a function of several variables was modified according to the demand of the journal MapleTech and submitted to be published with the name *Graphing a Function of Two Variables with Maple* ([10]).

The traditional theme multivariable calculus was chosen for the computer processing. It was shown, that this theme was complicated for the students (it posses higher requirements on the abstraction and geometrical imagination). The use of graphic and computational resources of Maple is especially effective here. The computer algebra system Maple was chosen for practical implementation. All instructions of the system Maple are stored in computer form on an enclosed floppy disk and they are also accessible through the computer network Internet (`http://www.math.muni.cz/~plch/difer/difer.html`).

A lot of examples mentioned in this thesis was already checked-out while working with students. At the same time it is shown, that suitable usage of computers in teaching can lead not only to deeper understanding of the taught problematic, but also to revival of the teaching process and to the enhancement of the interest in mathematical analysis at all.

However, some problems are also connected with using computers in teaching. Not always the computer program gives result corresponding to the reality. In the text we notice such cases and we try to use these cases as a motivation for a deeper study.

Introduction: This part is dedicated to methodical questions of computer supported teaching, emphases is setting especially on answering questions: “Where, why and how to use computer in teaching mathematical analysis”?

Individual chapters deal with examples of illustrative graphic and examples solved by

means of a computer (on the whole 62 examples and 113 pictures). Whole text thematicly follows the text of Došla Z., Došlý 0.: "The multivariable calculus", which is dedicated to the students of teachers study (these students compose the biggest group and the understanding of studied terms is more complicated here comparing mathematics students). This textbook contains the theory, which is necessary for the work.

Now let us have a closer look on individual chapters of this thesis.

Chapter 1 "The function of several variables": This chapter deals with the problems of creating mathematical graphics with Maple. To distinct the reality (the graph of the function of two variables) from the computer output we indicate the idea of PC-graph. The difference between the graph and the PC-graph we illustrate e.g. by the function $g(x, y) = \sin(2\pi x)\sin(2\pi y)$.

Example 1. Let us make a PC-graph of the function $g(x, y) = \sin(2\pi x)\sin(2\pi y)$ for x and y from the interval $[0, 25]$ without the change in the implicit setting of options (see Fig. 1).

```
> plot3d(sin(2*Pi*x)*sin(2*Pi*y), x=0..25, y=0..25, axes=boxed,
> labels=[x,y,z]);
```

However, a more detailed analysis of the above function shows that the resulting PC-graph (Fig. 1) does not correspond to reality. The functions $\sin(2\pi x)$ and $\sin(2\pi y)$ are periodic with period 1 and the PC-graph in Fig. 1 is not. Making the grid denser we obtain a result more closer to the real behavior of the considered function (Fig. 2).

```
> plot3d(sin(2*Pi*x)*sin(2*Pi*y), x=0..25, y=0..25, axes=boxed,
> grid=[60,60], labels=[x,y,z]);
```

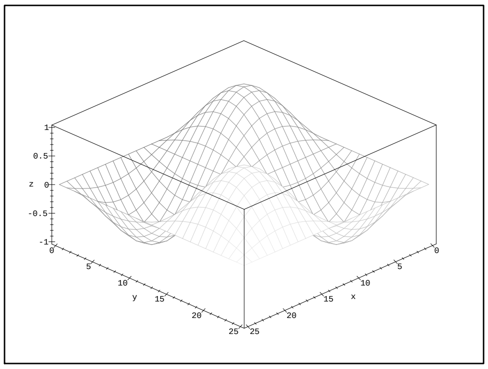


Fig. 1

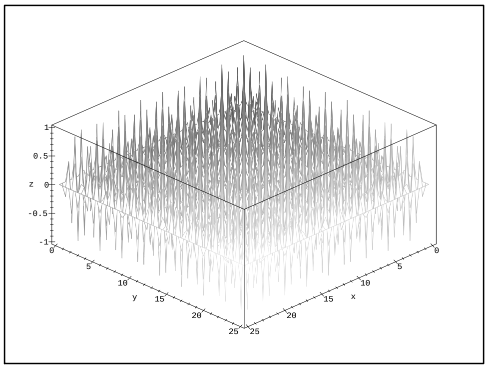


Fig. 2

Further problems arise when plotting graphs of discontinuous functions. The situation is much more complicated here, as in the case of functions of one real variable, where the generation of the PC-graph is simplified by the option `discont=true`. Here we must find the suitable options for the PC-graph by ourselves. When we are not able choose such a grid, that the discontinuity points differ from the grid points, the best way to generate the PC-graph is to divide it into several parts. At the end we merge them into the final PC-graph.

Example 2. Let us make a PC-graph of the function

$$f(x,y) = \frac{2}{x^2+y^2-9}.$$

```
> f:=2/(x^2+y^2-9);
```

$$f := 2\,\frac{1}{x^2+y^2-9}$$

Using the parametric definition of the function $x = u\cos v$, $y = u\sin v$, $z = \frac{2}{u^2-9}$ we generate the PC-graph of this function. We divide the generation of the PC-graph into two parts with u attaining values from intervals $[0, 2.999]$, $[3.001, 6]$:

```
> s1:=plot3d([u*cos(v), u*sin(v), subs({x=u*cos(v), y=u*sin(v)}, f)],
> v=0..2*Pi, u=0..2.999):
> s2:=plot3d([u*cos(v), u*sin(v), subs({x=u*cos(v), y=u*sin(v)}, f)],
> v=0..2*Pi, u=3.001..6):
> display3d({s1,s2}, view=-8..8);
> display3d({s1,s2}, view=-8..8, orientation=[40,102]);
```

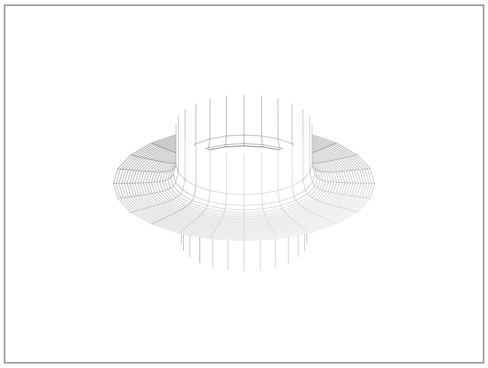


Fig. 3

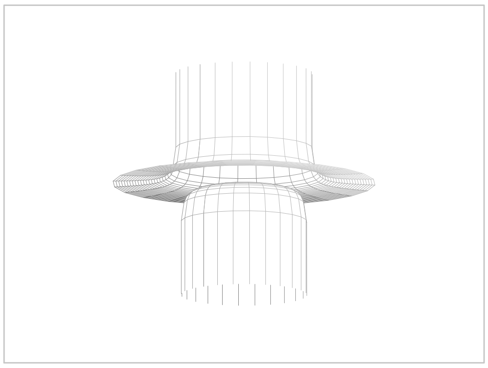


Fig. 4

The last part of this chapter is devoted to explaining of the term "level curve" and to the technique of displaying level curves in Maple.

Chapter 2 "The limit": We give examples of using Maple's graphical and computational capabilities in exploring limits of functions of two variables. PC-graphs play an important role in creating the geometrical imagination about examined functions. In most cases there is no possibility to compute the limit, but the computer may help us find directional limits and transform the computation to the polar coordinates. The graphical and computational capabilities of Maple are, in the best way, shown by the following example:

Example 3. Let us consider the function

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2}, & [x,y] \neq [0,0], \\ 0, & [x,y] = [0,0]. \end{cases}$$

Find the limit $\lim_{(x,y)\to(0,0)} f(x, y)$.

```
> f:=proc(x,y) if x=0 and y=0 then 0 else x^2*y/(x^4+y^2) fi end:
```

Setting $y = kx$, i.e. we approach to the limit point by lines, we get:

```
> Limit(f(x,k*x), x=0)=limit(f(x,k*x),x=0);
```

$$\lim_{x\to 0} \frac{x^3\,k}{x^4 + k^2\,x^2} = 0$$

But if we set $y = kx^2$, i.e. we approach to the limit point by parabolas, we get:

```
> Limit(f(x,k*x^2), x=0)=limit(f(x,k*x^2),x=0);
```

$$\lim_{x\to 0} \frac{x^4\,k}{x^4 + k^2\,x^4} = \frac{k}{1 + k^2}$$

The last limit depends on the constant k and that is why the function f has no limit at the point $(0, 0)$ (Fig. 5).

```
> plot3d(f, -2..2, -2..2,  grid=[100,100], style=patchcontour,
> orientation=[-46,35], contours=12,axes=boxed,labels=[x,y,'z']);
```

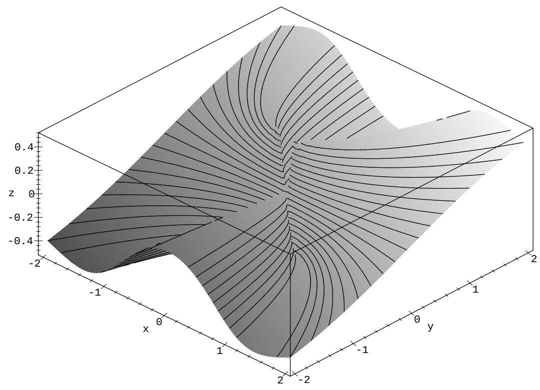
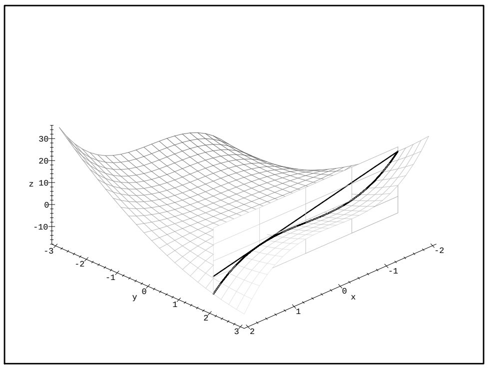


Fig. 5

Chapter 3 "Partial derivatives": The most important part from the educational point of view is the part devoted to the geometrical interpretation of the partial derivatives (Figs. 6 and 7).



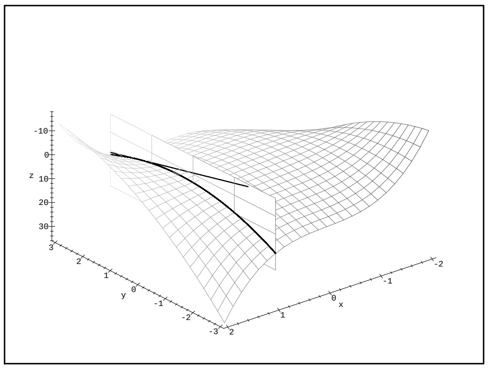


Fig. 6 Fig. 7

Similarly one can illustrate the geometrical interpretation of the directional derivatives.

Chapter 4 "Differentials": We use PC-graphs to illustrate the terms "differentiable function" and "tangent plane" (Fig. 8) and to illustrate the relations between differentiability, continuity and the existence of the partial derivatives.

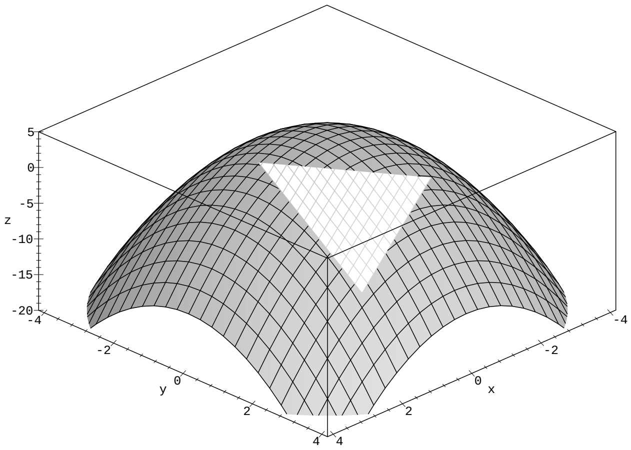


Fig. 8

The computational aspects of using Maple for finding the differential, numerical computations with the differential and for solving the exact equation are shown in the next part of this chapter. The program steps for the above computations are available in new functions `difer`, `difern`, `GraphTan` and `kmen`.

Chapter 5, "The chain rule for partial derivatives and Taylor polynomial": We show very affective use of Maple to compute the partial derivatives of a composite function. The possibilities of the procedures `dzd1`, `dzd2`, `dzdd1`, `dzdd2` and `dzdd12` are well illustrated in the next example.

Example 4. Transform the equation

$$x^2 z_{xx} + y^2 z_{yy} - 2xyz_{xy} + xz_x + yz_y = 0$$

to the new independent variables $u = xy$ and $v = x/y$.

```
> zx:=dzd1(z, x*y,x/y,u,v,x,y);
```

$$zx := \left(\frac{\partial}{\partial u} z\right) y + \frac{\frac{\partial}{\partial v} z}{y}$$

```
> zy:=dzd2(z, x*y,x/y,u,v,x,y);
```

$$zy := \left(\frac{\partial}{\partial u} z\right) x - \frac{\left(\frac{\partial}{\partial v} z\right) x}{y^2}$$

```
> zxx:=dzdd1(z, x*y,x/y,u,v,x,y);
```

$$zxx := \left(\frac{\partial^2}{\partial u^2} z\right) y^2 + 2\left(\frac{\partial^2}{\partial v\,\partial u} z\right) + \frac{\frac{\partial^2}{\partial v^2} z}{y^2}$$

```
> zyy:=dzdd2(z, x*y,x/y,u,v,x,y);
```

$$zyy := \left(\frac{\partial^2}{\partial u^2} z\right) x^2 - 2\,\frac{\left(\frac{\partial^2}{\partial v\,\partial u} z\right) x^2}{y^2} + \frac{\left(\frac{\partial^2}{\partial v^2} z\right) x^2}{y^4} + 2\,\frac{\left(\frac{\partial}{\partial v} z\right) x}{y^3}$$

```
> zxy:=dzdd12(z, x*y,x/y,u,v,x,y);
```

$$zxy := \left(\frac{\partial^2}{\partial u^2} z\right) y\,x - \frac{\left(\frac{\partial^2}{\partial v^2} z\right) x}{y^3} + \left(\frac{\partial}{\partial u} z\right) - \frac{\frac{\partial}{\partial v} z}{y^2}$$

```
> tr:=expand(simplify(x^2*zxx+y^2*zyy-2*x*y*zxy+x*zx+y*zy=0));
```

$$tr := 4\,\frac{x^2\left(\frac{\partial^2}{\partial v^2} z\right)}{y^2} + 4\,\frac{x\left(\frac{\partial}{\partial v} z\right)}{y} = 0$$

```
> tr:=factor(student[powsubs](x/y=v,tr));
```

$$tr := 4\,v\left(\left(\frac{\partial^2}{\partial v^2} z\right) v + \left(\frac{\partial}{\partial v} z\right)\right) = 0$$

```
> tr/(4*v);
```

$$\left(\frac{\partial^2}{\partial v^2} z\right) v + \left(\frac{\partial}{\partial v} z\right) = 0$$

The second part of this chapter deals with examples of computer graphics and computations, which may be used in explanation of the Taylor polynomial of a function of two variables.

Example 5. Find the second order Taylor polynomial with the center at $(0,0)$ for the function $f(x,y) = \sin x \cos y$.

```
>  f:=(x,y)->sin(x)*sin(y);
```

$$f := (x, y) \to \sin(x)\sin(y)$$

```
> fx:=D[1](f);fy:=D[2](f);
```

$$fx := (x, y) \to \cos(x)\sin(y)$$

$$fy := (x, y) \to \sin(x)\cos(y)$$

```
> fxx:=D[1,1](f);fxy:=D[1,2](f);fyy:=D[2,2](f);
```

$$fxx := (x, y) \to -\sin(x)\sin(y)$$

$$fxy := (x, y) \to \cos(x)\cos(y)$$

$$fyy := (x, y) \rightarrow -\sin(x)\sin(y)$$

```
> T2:=f(0,0)+fx(0,0)*(x-0)+fy(0,0)*(y-0)+
> (1/2)*((fxx(0,0)*(x-0)^2+2*fxy(0,0)*(x-0)*(y-0)+fyy(0,0)*(y-0)^2));
```

$$T2 := x\,y$$

```
> plot3d(f(x,y), x=-Pi..Pi, y=-Pi..Pi, orientation=[45,60],
> style=patchcontour);
> plot3d(T2, x=-Pi..Pi, y=-Pi..Pi, orientation=[45,60],
> style=patchcontour);
```

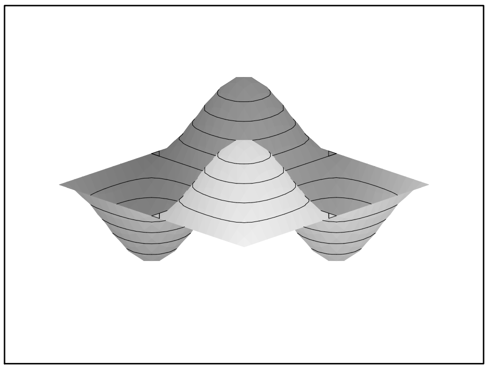


Fig. 9

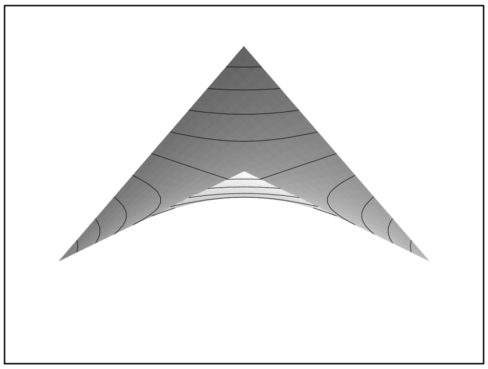


Fig. 10

Chapter 6 "The local and global extrema": In this chapter we focus on the problems connected with using Maple to find the local and global extrema of functions of two variables. The examples in this chapter show not only the advantages of using Maple to solve them, but also notice the situations, when the computer gives incomplete or not exact results. Frequently used calculations are stored in new functions `sing` and `mvextrem`. Here we show one example of a standard solution.

Example 6. Find the local extrema of the function

$$f(x, y) = xy\mathrm{e}^{(-\frac{x^2+y^2}{2})}.$$

```
> f := (x,y) -> x*y*exp(-(x^2+y^2)/2);
```

$$f := (x,\, y) \rightarrow x\,y\,e^{(-1/2\,x^2-1/2\,y^2)}$$

Let us generate the PC-graph of the function f (Fig. 11) and the PC-graph with the level curves of f (Fig. 12):

```
> plot3d(f(x,y),x=-3..3,y=-3..3,style =patch,orientation=[70,65],
> axes=framed, grid=[40,30], labels=[x,y,z]);
> with(plots):
> contourplot(f(x,y),x=-3..3,y=-3..3, grid=[50,50], axes=boxed);
```

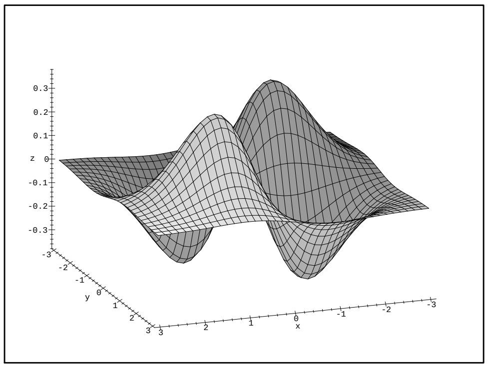


Fig. 11

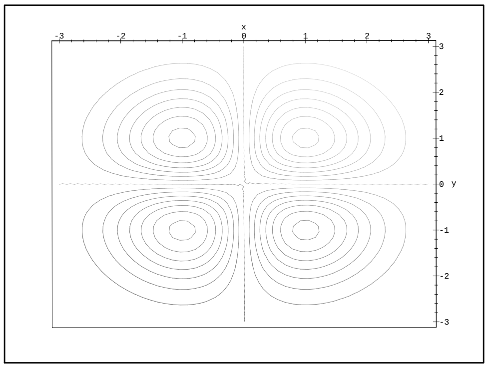


Fig. 12

From the stated PC-graphs we can estimate the local extrema in each of four quadrants, namely the local minima in the first and third quadrants and local maxima in the second and fourth quadrants. We check the correctness of our consideration by the calculation:

```
> cp:=solve({diff(f(x,y),x)=0, diff(f(x,y),y)=0}, {x,y});
```

$$cp := \{\, x=0, y=0 \,\}, \{\, y=1, x=1 \,\}, \{\, y=-1, x=1 \,\}, \{\, x=-1, y=1 \,\}, \{\, x=-1, y=-1 \,\}$$

```
> fxx := factor(diff(f(x,y),x,x));
```

$$fxx := x\,y\,\mathrm{e}^{(-1/2\,x^2-1/2\,y^2)}\,(-3+x^2)$$

```
> fxy := factor(diff(f(x,y),x,y));
```

$$fxy := \mathrm{e}^{(-1/2\,x^2-1/2\,y^2)}\,(y-1)\,(y+1)\,(x-1)\,(x+1)$$

```
> fyy := factor(diff(f(x,y),y,y));
```

$$fyy := x\,y\,\mathrm{e}^{(-1/2\,x^2-1/2\,y^2)}\,(-3+y^2)$$

```
> Delta:=factor(fxx*fyy-fxy^2);
```

$$\Delta := -(\,\mathrm{e}^{(-1/2\,x^2-1/2\,y^2)}\,)^2\,(-5\,x^2\,y^2+y^4\,x^2+x^4\,y^2+1-2\,x^2-2\,y^2+x^4+y^4)$$

```
> eval(subs(x=0,y=0,Delta));
```

$$-1$$

```
> subs(x=1,y=1,[Delta,fxx]);
```

$$\left[4\,(\,\mathrm{e}^{(-1)}\,)^2, -2\,\mathrm{e}^{(-1)}\right]$$

```
> subs(x=-1,y=1,[Delta,fxx]);
```

$$\left[4\,(\,\mathrm{e}^{(-1)}\,)^2, 2\,\mathrm{e}^{(-1)}\right]$$

```
> subs(x=-1,y=-1,[Delta,fxx]);subs(x=1,y=-1,[Delta,fxx]);
```

$$\left[4\,(\,\mathrm{e}^{(-1)}\,)^2, -2\,\mathrm{e}^{(-1)}\right]$$

$$\left[4\,(\,\mathrm{e}^{(-1)}\,)^2, 2\,\mathrm{e}^{(-1)}\right]$$

At the end we determine the functional values for the extrema:

```
> zip(f,[1,-1,-1,1],[1,1,-1,-1]);
```

$$[\,\mathrm{e}^{(-1)}, -\mathrm{e}^{(-1)}, \mathrm{e}^{(-1)}, -\mathrm{e}^{(-1)}\,]$$

There is a local maximum $1/\mathrm{e}$ at the points $(1,1)$ and $(-1,-1)$ and there is a local minimum $-1/\mathrm{e}$ at the points $(1,-1)$ and $(-1,1)$.

Chapter 7 "The implicit function": The first part of this chapter shows that the algorithms for generating PC-graphs for curves defined implicitly may give incorrect results (Figs. 13 and 14):

```
> plots[implicitplot](2*x^4+y^4-3*x^2*y-2*y^3+y^2, x=-5/2..5/2,
> y=-5/2..5/2);
> Digits := 80:
> plots[implicitplot]((3*x*y)/(x^3+y^3)=1,x=-4..4,y=-4..4,grid=[40,40]);
```

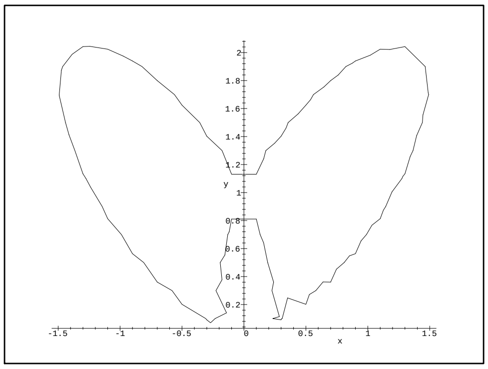


Fig. 13

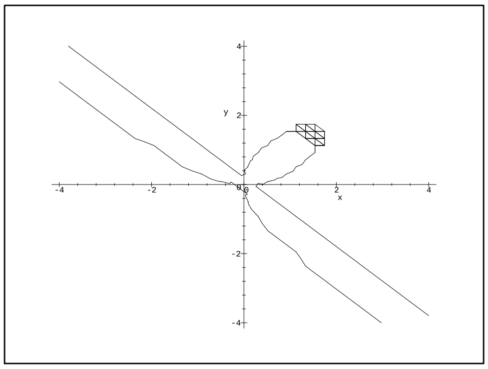


Fig. 14

One the other hand, there are advantages of using Maple for the computation of derivatives for functions defined implicitly, and this is discussed in the second part of this chapter. New procedures `implicitdiff` and `graf_t` are given: `implicitdiff` for computing the first derivative of a function defined implicitly, `graf_t`for finding the equation of the tangent line of a function defined implicitly.

Appendix: In the final part of the thesis, the program Maple is described and another CAS systems, which can be used for preparation similar texts, are presented. Commercial CAS systems can be in some cases supplied or partly replaced by public domain software. One of appendixes is therefore dedicated to best-known archives of such programs on Internet and to the possibility of their browsing. Last appendix is dedicated to world's archives of materials for computer supported teaching of the multivariable calculus and teaching of mathematical analysis.