A guide to recent progress in the profinite front

Jorge Almeida (University of Porto)

For their intimate connections with the theory of rational languages, relatively free profinite semigroups have become a well-established tool in many aspects of the theory and its applications, particularly in the framework of Eilenberg-type algebraic classification schemes. They serve as a powerful descriptive tool and, despite their rather general tendency to be uncountable, they often provide a route to decidability results. By focusing our attention on rich structural entities, they also fit well into the solid ground of classical Mathematics, opening at the same time doors to toolboxes and drawing analogies with other areas, such as group theory, topology, and dynamical systems.

The purpose of this talk is to introduce and put into (the authors') perspective the results presented in more detail in talks given by A. Costa, J. C. Costa, B. Steinberg, and M. Zeitoun. They are all concerned with various structural aspects of free profinite semigroups over 'large' pseudovarieties. Until recently, not much was known in such cases.

Two contributions from symbolic dynamics to the comprehension of Green's relations in relatively free profinite semigroups

Alfredo Costa (University of Coimbra)

The importance of relatively free profinite semigroups as a tool used in formal language theory motivates the study of their structure. For some pseudovarieties of semigroups, complete characterizations of the structure is known, but for pseudovarieties containing **LSl**, even very partial knowledge is usually considered a difficult issue. In recent years, symbolic dynamics played an important role in the study of maximal subgroups of such semigroups. In this talk we apply symbolic dynamics in the "orthogonal" direction (i.e., we forget group structure). The results are about Green's relations, and they are subdivided in two groups:

1. Existence of $\mathcal{R}$-chains of regular elements with continuum cardinality — joint work with J. Almeida, J.C. Costa and M. Zeitoun.

2. Structure of $\mathcal{J}$-classes associated to subshifts, as compact partial semigroups — joint work with J. Almeida.

A new approach to McCammond's solution of the ω-word problem for finite aperiodic semigroups

José Carlos Cruz Costa (University of Minho)
(joint work with Jorge Almeida and Marc Zeitoun)

An ω-term is an expression obtained from letters of an alphabet using the operations of concatenation and ω-power. These expressions can be naturally viewed as (implicit) operations on finite semigroups. The omega-word problem for a pseudovariety of semigroups $\mathbf{V}$ is the problem of determining whether two ω-terms represent the same operation on the elements of $\mathbf{V}$.

This problem was solved by McCammond for the pseudovariety $\mathbf{A}$ of finite aperiodic semigroups, by defining a normal form for ω-terms and by showing that two ω-terms define the same operation over $\mathbf{A}$ if and only if they can be transformed into the same ω-term in normal form. He does that by appealing to his solution of the word problem for certain Burnside semigroups.

In this talk, we describe an alternative proof of McCammond's algorithm. Our approach consists in associating to each ω-term x and positive integer n a certain language $L_n(x)$, and showing that the uniqueness of McCammond's normal forms is a consequence of two properties of such languages: for ω-terms x and y in normal form and a sufficiently large n
(1) $L_n(x)$ and $L_n(y)$ are star free languages
(2) if $L_n(x)$ and $L_n(y)$ are not disjoint then $x = y$.

Recent progress on the structure of free profinite monoids

Benjamin Steinberg (Carleton University)

As all equational descriptions of classes of regular languages rely on the expressive power of free profinite monoids, it is imperative to understand their structure. This talk will survey some recent progress in this direction focusing on:

1. Free clopen submonoids
2. Ideal structure
3. Finite subsemigroups
4. Maximal subgroups

This talk encompasses joint work with Jorge Almeida and with John Rhodes.

The Pin-Reutenauer algorithm for aperiodic semigroups

Marc Zeitoun (University of Bordeaux)
(joint work with Jorge Almeida and José Carlos Costa)

The Pin-Reutenauer algorithm gives a method to compute the closure in the free group of a rational language, for the Hall topology. We state and prove a similar algorithm for the pseudovariety **A** of aperiodic semigroups instead of that of groups. We also show that this algorithm can be transferred from a pseudovariety to a subpseudovariety if both of them enjoy the property of being full. We show that **A**, as well as its subpseudovariety **R** of $\mathcal{R}$-trivial semigroups, are both full. This yields the validity of the algorithm for **R** as well.

Noncommutative p-adic analysis

Pedro V. Silva (University of Porto)
(joint work with Jean-Éric Pin)

A function f from A^* to B^* preserves a certain variety of languages $\mathcal{V}$ (through f^{-1}) if and only if it is uniformly continuous for the pro-$\mathbf{V}$ (quasi)metric, where $\mathbf{V}$ denotes the pseudovariety of finite monoids associated to $\mathcal{V}$. We present in this talk part of the output of a project that involves the study of $\mathbf{V}$-uniformly continuous and $\mathbf{V}$-hereditarily continuous functions between free (commutative) monoids/groups, for pseudovarieties such as $\mathbf{G}_p$, $\mathbf{G}$, $\mathbf{A}$ or $\mathbf{M}$.

One of the most curious and unlikely results is the generalization to the realm of words of the celebrated Mahler's Theorem of p-adic analysis. The original theorem states that a function from $\mathbb{N}$ to $\mathbb{Z}$ is uniformly continuous for the p-adic metric if and only if the coefficients of its (unique) binomial series decomposition converge to 0. In our generalization, $\mathbb{N}$ is replaced by an arbitrary free monoid, and the p-adic metric by the pro-$\mathbf{G}_p$ metric.

The theory of stabilization monoids and regular cost functions

Thomas Colcombet (University Paris Diderot)

This work continues the long history of limitedness problems on distance, desert, distance desert and nested distance desert automata. We develop a quantitative extension of the theory of regular languages of finite words.

Algebraic side:
We introduce stabilization monoids: This algebraic object extends the notion of monoid by the ability to specify what happens when iterating 'many times' some element. We show how to associate a unique semantics to each stabilization monoid, and we define the corresponding notion of recognizable mappings from words to $\omega + 1$ (up to a certain equivalence relation).

Automata side:
We use cost automata: finite automata that associate a value in $\omega + 1$ to each input word. Those come in two dual variants B- and S-automata. We prove that the functions computed by B-automata, computed by S-automata, and recognizable by stabilization monoids are equivalent. We also prove the equivalence with history-deterministic B- and S-automata: a weakened form of determinism that is suitable for the further development of the theory. We name such functions regular.

Regular cost functions:
From the above results we derive closure properties and decidability results for regular cost functions that are strict extension of the closure under union, intersection, projection, dual of projection, and the decidability of inclusion for regular languages. Those results entail decidability results for a suitable variant of MSO.