The Pin-Reutenauer algorithm for classes of aperiodic semigroups

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Outline

A. Profinite semigroups and closures: some notation.

B. The Pin-Reutenauer algorithm.

C. Proof ideas and main ingredients.

Pseudovarieties

- Pseudovariety: class of finite semigroups closed under
  - finite direct products,
  - subsemigroup,
  - quotient.

- S: all finite semigroups.
- G: all finite groups.
- A: all finite aperiodic (group-free) semigroups.
- R: all finite $\mathcal{R}$-trivial semigroups.
- V: a generic pseudovariety.

Relatively V-free profinite semigroups

- X: fixed finite alphabet.
- A semigroup S separates $u, v \in X^+$ if there is a homomorphism $\varphi : X^+ \to S$ such that $\varphi(u) \neq \varphi(v)$.
- Define a pseudo-metric d_V:

$$\begin{cases} r_V(u, v) & = \min\{|S| : S \in V \text{ and S separates u and v}\}. \\ d_V(u, v) & = 2^{-r_V(u,v)}. \end{cases}$$

- $u \sim_V v$ if and only if $d_V(u, v) = 0$ defines a congruence.
- Relatively V-free profinite semigroup $\overline{\Omega}_X V$: completion of $(X^+/\sim_V, d_V)$. Elements of $\overline{\Omega}_X S$ are called pseudowords.

Implicit signatures

- Implicit signature σ: set of elements of pseudowords containing the multiplication.
- Example: $\kappa = \{\_.\_ \, , \, \_^{\omega-1}\}$.
- Each element of σ can be interpreted on a profinite semigroup.
- Given σ, a profinite semigroup S has a structure of "σ-semigroup" obtained by evaluating each operation of σ in S.
- $\Omega_X^\sigma \mathsf{V}$ is the σ-subsemigroup of $\overline{\Omega}_X \mathsf{V}$ generated by X.

Notation: Closures for profinite topologies

- $L \subseteq S$ topological semigroup: $\mathrm{cl}_S(L)$ denotes the closure of L in S.

$$\mathrm{cl}(L) \stackrel{\text{def}}{=} \mathrm{cl}_{\overline{\Omega}_X \mathsf{S}}(L) \qquad \mathrm{cl}_\sigma(L) \stackrel{\text{def}}{=} \mathrm{cl}_{\Omega_X^\sigma \mathsf{S}}(L)$$

$$\mathrm{cl}_{\mathsf{V}}(L) \stackrel{\text{def}}{=} \mathrm{cl}_{\overline{\Omega}_X \mathsf{V}}(L) \qquad \mathrm{cl}_{\sigma,\mathsf{V}}(L) \stackrel{\text{def}}{=} \mathrm{cl}_{\Omega_X^\sigma \mathsf{V}}(L)$$

- The topology on $\Omega_X^\sigma \mathsf{V}$ is the induced topology in $\overline{\Omega}_X \mathsf{V}$:

$$\mathrm{cl}_{\sigma,\mathsf{V}}(L) = \mathrm{cl}_{\mathsf{V}}(L) \cap \Omega_X^\sigma \mathsf{V}.$$

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- We abusively use the above notation for $L \subseteq X^+$: eg, we write $\mathrm{cl}_{\sigma,\mathsf{V}}(L)$ instead of $\mathrm{cl}_{\sigma,\mathsf{V}}(p_{\mathsf{V}}(\iota(L)))$, where $p_{\mathsf{V}} : \overline{\Omega}_X \mathsf{S} \to \overline{\Omega}_X \mathsf{V}$ is the canonical projection and $\iota : X^+ \to \overline{\Omega}_X \mathsf{S}$ the canonical embedding.

Notation: algebraic closures

- Let σ be an implicit signature, S be a σ-semigroup, and $L \subseteq S$.

$$\langle L\rangle_\sigma = \sigma\text{-subsemigroup of } S \text{ generated by } L.$$

(in practice in $L \subseteq \Omega_X^\sigma \mathsf{S}$)

$$\langle L\rangle_{\sigma,\mathsf{V}} = \langle p_{\mathsf{V}}(L)\rangle_\sigma$$

The Pin-Reutenauer algorithm

- The Pin-Reutenauer algorithm holds for V and σ if, for all rational languages $K, L \subseteq X^+$, the following equations hold:

$$\begin{aligned} \mathrm{cl}_{\sigma,\mathsf{V}}(KL) &= \mathrm{cl}_{\sigma,\mathsf{V}}(K) \cdot \mathrm{cl}_{\sigma,\mathsf{V}}(L), \\ \mathrm{cl}_{\sigma,\mathsf{V}}(L^+) &= \langle \mathrm{cl}_{\sigma,\mathsf{V}}(L) \rangle_\sigma. \end{aligned}$$

- Makes it possible to "compute" the closure of any rational language in the relatively V-free σ-semigroup $\Omega_X^\sigma \mathsf{V}$.

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- Makes it possible to “compute” the closure of any rational language in the relatively V-free σ-semigroup $\Omega_X^\sigma \mathsf{V}$.
- Note: $\mathrm{cl}_{\sigma,\mathsf{V}}(KL) \supseteq \mathrm{cl}_{\sigma,\mathsf{V}}(K) \cdot \mathrm{cl}_{\sigma,\mathsf{V}}(L)$ always hold true (multiplication is continuous).

The Pin-Reutenauer algorithm holds for G and κ

- In the free group $\Omega_X^\kappa G$ endowed with the profinite topology, for $K, L \subseteq X^+$ regular:

$$\begin{aligned} \mathrm{cl}_{\kappa,G}(KL) &= \mathrm{cl}_{\kappa,G}(K) \cdot \mathrm{cl}_{\kappa,G}(L), \\ \mathrm{cl}_{\kappa,G}(L^+) &= \langle L \rangle_\kappa. \end{aligned} \tag{1}$$

  It is actually not necessary to propagate the closure in (1).
- Conjectured by Pin and Reutenauer, reduced to another conjecture proved by Ribes and Zalesskiĭ.
- Equivalent to Rhodes' type II conjecture, proved by Ash.

The Pin-Reutenauer algorithm holds for A and κ

Theorem [Almeida, JC. Costa, Z.]

The Pin-Reutenauer procedure holds for A and κ:

$$\mathrm{cl}_{\kappa,\mathsf{A}}(KL) = \mathrm{cl}_{\kappa,\mathsf{A}}(K) \cdot \mathrm{cl}_{\kappa,\mathsf{A}}(L), \tag{2}$$

$$\mathrm{cl}_{\kappa,\mathsf{A}}(L^+) = \langle \mathrm{cl}_{\kappa,\mathsf{A}}(L) \rangle_\kappa. \tag{3}$$

Proof ideas and ingredients: σ-fullness (Almeida, Steinberg '00)

- The following always hold:

$$\mathrm{cl}_{\sigma,\mathsf{V}}(L) = p_{\mathsf{V}}(\mathrm{cl}(L)) \cap \Omega_X^\sigma \mathsf{V}.$$

- A pseudovariety V is σ-full if for every regular $L \subseteq X^+$:

$$\mathrm{cl}_{\sigma,\mathsf{V}}(L) = p_{\mathsf{V}}\Big(\mathrm{cl}(L) \cap \Omega_X^\sigma \mathsf{S}\Big)$$

- One can show this is equivalent to: for every regular $L \subseteq X^+$,

$$\mathrm{cl}_{\sigma,\mathsf{V}}(L) = p_{\mathsf{V}}(\mathrm{cl}_\sigma(L)).$$

- To compute the closure in $\Omega_X^\sigma \mathsf{V}$, one can compute it in $\Omega_X^\sigma \mathsf{S}$ and project onto the free pro-V semigroup.

σ-fullness and inheritance of the PR-algorithm

Proposition [ACZ]

Let V and W be pseudovarieties such that

1. $\mathsf{V} \subseteq \mathsf{W}$,
2. Both V and W are σ-full,
3. The Pin-Reutenauer algorithm holds for W.

Then the Pin-Reutenauer algorithm also holds for V.

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Easy proof. Eg, if product and closure commute for W:

$$\begin{aligned}
\mathrm{cl}_{\sigma,\mathsf{V}}(KL) &= p_{\mathsf{V}}(\mathrm{cl}_\sigma(KL)) && \text{since V is } \sigma\text{–full} \\
&= p_{\mathsf{W},\mathsf{V}}[p_{\mathsf{W}}(\mathrm{cl}_\sigma(KL))] \\
&= p_{\mathsf{W},\mathsf{V}}[\mathrm{cl}_{\sigma,\mathsf{W}}(KL)] && \text{since W is } \sigma\text{–full} \\
&= p_{\mathsf{W},\mathsf{V}}[\mathrm{cl}_{\sigma,\mathsf{W}}(K) \cdot \mathrm{cl}_{\sigma,\mathsf{W}}(L)] && \text{by hypothesis} \\
&= p_{\mathsf{W},\mathsf{V}}[\mathrm{cl}_{\sigma,\mathsf{W}}(K)] \cdot p_{\mathsf{W},\mathsf{V}}[\mathrm{cl}_{\sigma,\mathsf{W}}(L)]
\end{aligned}$$

and back to $\mathrm{cl}_{\sigma,\mathsf{V}}(K)\mathrm{cl}_{\sigma,\mathsf{V}}(L)$.

The Pin-Reutenauer for A: Case of the product

- $\mathrm{cl}_{\kappa,\mathsf{A}}(KL) \supseteq \mathrm{cl}_{\kappa,\mathsf{A}}(K) \cdot \mathrm{cl}_{\kappa,\mathsf{A}}(L)$ by continuity of multiplication.
- For the reverse implication, use the fact that A is κ-factorial.
  Every factor in $\overline{\Omega}_X\mathsf{A}$ of an element of $\Omega_X^\kappa\mathsf{A}$ is again in $\Omega_X^\kappa\mathsf{A}$.

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  - By compactness, one can assume (x_n) and (y_n) convergent to $x \in \mathrm{cl}_{\mathsf{A}}(K)$ and $y \in \mathrm{cl}_{\mathsf{A}}(L)$.

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  - There exists $w_n \in KL$ converging to w in $\overline{\Omega}_X\mathsf{A}$.
  - Write $w_n = x_n y_n$ with $x_n \in K$ and $y_n \in L$.
  - By compactness, one can assume (x_n) and (y_n) convergent to $x \in \mathrm{cl}_{\mathsf{A}}(K)$ and $y \in \mathrm{cl}_{\mathsf{A}}(L)$.
  - Since A is κ-factorial and $w = xy$, we get $x, y \in \Omega_X^\kappa\mathsf{A}$.
  - So $x \in \mathrm{cl}_{\kappa,\mathsf{A}}(K)$, and $y \in \mathrm{cl}_{\kappa,\mathsf{A}}(L)$, whence $w \in \mathrm{cl}_{\kappa,\mathsf{A}}(K).\mathrm{cl}_{\kappa,\mathsf{A}}(L)$.

Another ingredient: star-free languages separating elements of $\Omega_X^\kappa A$.

Theorem (McCammond'2001)

Using the rewriting following system, there is a procedure to transform any ω-word into a normal form: two ω-words are equal over $\overline{\Omega}_X A$ if and only if they have the same normal form.

1. $(x^\omega)^\omega \longleftrightarrow x^\omega$;
2. $(x^k)^\omega \longleftrightarrow x^\omega$ for $k \geqslant 2$;
3. $x^\omega x^\omega \longleftrightarrow x^\omega$;
4. $x^\omega x \longleftrightarrow x^\omega \longleftrightarrow x x^\omega$;
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5. $(xy)^\omega x \longleftrightarrow x(yx)^\omega$.

The rank of $w \in \Omega_X^\kappa A$ is the maximum nesting of ω-powers in the term in normal form representing w.

Neighborhood bases of star-free languages

- For $L \subseteq X^+$, let $L^{>n} = L^n L^+$.
- Given an ω-term w (term built from X using concatenation and ω-power), let $L_n(w)$ be the (regular) language obtained from w by replacing all "ω" by "$> n$".
- Example:

$$L_2(a^\omega abb^\omega) = a^2a^+abb^2b^+,$$
$$L_2((a^\omega b)^\omega) = (a^2a^+b)^2(a^2a^+b)^+.$$

- Informally, $L_n(w)$ is obtained from w by replacing ω-powers by large iterations (more than n times).

Key properties of the languages $L_n(w)$

Theorem [ACZ]

If w is in normal form, then

1. $L_n(w)$ is star-free for n large enough, depending only on w.
2. $p_{\mathsf{A}}^{-1}(w) = \bigcap_n \mathrm{cl}(L_n(w))$

Families $L_n(w)$ separate ω-terms, in the sense that for two ω-terms u, v:

$$(\forall n\ L_n(u) \cap L_n(v) \neq \varnothing) \Longrightarrow p_{\mathsf{A}}(u) = p_{\mathsf{A}}(v).$$

and

$$p_{\mathsf{A}}\Big(\bigcap_n \mathrm{cl}(L_n(u))\Big) = \{p_{\mathsf{A}}(u)\} = \bigcap_n p_{\mathsf{A}}(\mathrm{cl}(L_n(u))).$$

PR algorithm for A: handling iteration: $\mathrm{cl}_{\kappa,\mathsf{A}}(L^+) = \langle \mathrm{cl}_{\kappa,\mathsf{A}}(L) \rangle_{\kappa,\mathsf{A}}$

- The inclusion $\langle \mathrm{cl}_{\kappa,\mathsf{A}}(L) \rangle_{\kappa,\mathsf{A}} \subseteq \mathrm{cl}_{\kappa,\mathsf{A}}(L^+)$ is easy: since $\mathrm{cl}_{\kappa,\mathsf{A}}(L^+)$ contains $\mathrm{cl}_{\kappa,\mathsf{A}}(L)$, it suffices to show that $\mathrm{cl}_{\kappa,\mathsf{A}}(L^+)$ is a κ-semigroup.

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- We want to represent $w \in \mathrm{cl}_{\kappa,\mathsf{A}}(L^+)$ by a κ-term on $\mathrm{cl}_{\kappa,\mathsf{A}}(L)$.
- Use induction on the rank and "length" of w.
- Proof sketch for a normal form $w = v^\omega$ of rank $n \geqslant 1$.

PR algorithm for A: handling iteration: inclusion $\mathrm{cl}_{\kappa,\mathsf{A}}(L^+) \subseteq \langle \mathrm{cl}_{\kappa,\mathsf{A}}(L) \rangle_{\kappa,\mathsf{A}}$

- $w = v^\omega \in \mathrm{cl}_{\kappa,\mathsf{A}}(L^+)$.
- Since $L_n(w)$ is star-free for n large enough, $\mathrm{cl}_{\mathsf{A}}(L_n(w))$ is clopen.
- Since $w \in \mathrm{cl}_{\kappa,\mathsf{A}}(L^+)$, there exists $w_n \in L_n(w) \cap L^+$.
- Since $w_n \in L_n(w)$, the sequence $(w_n)_n$ converges to w.
- Easy case: there is a subsequence $(w_{i_n})_n$ of w_n and a fixed N such that $w_{i_n} \in L^N$. Then use the product case:

$$w \in \mathrm{cl}_{\kappa,\mathsf{A}}(L^N) \subseteq \left(\mathrm{cl}_{\kappa,\mathsf{A}}(L)\right)^N \subseteq \langle \mathrm{cl}_{\kappa,\mathsf{A}}(L) \rangle_{\kappa,\mathsf{A}}.$$

PR algorithm for A: handling iteration: inclusion $\mathrm{cl}_{\kappa,\mathsf{A}}(L^+) \subseteq \langle \mathrm{cl}_{\kappa,\mathsf{A}}(L) \rangle_{\kappa,\mathsf{A}}$

- Otherwise: write

$$w_n = w_{1,n} w_{2,n} \cdots w_{k_n,n}, \qquad w_{j,n} \in L.$$

(with k_n unbounded.)
Main problem: reduce to a bounded number of factors, while still converging to w.

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  - Group factors of L.
  - If necessary, use periodic repetitions: replace w_n by

$$\tilde{w}_n = \tilde{w}_{1,n} \tilde{w}_{2,n} \cdots (\tilde{w}_{i,n} \cdots \tilde{w}_{j,n})^{\omega} \cdots \tilde{w}_{K,n}, \qquad \tilde{w}_{j,n} \in L.$$

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- $w_n \in L_n(w) = L_n(v^{\omega}) = [L_n(v)]^{>n}$, so we get another factorization

$$w_n = v_{1,n} v_{2,n} \cdots v_{p_n,n}, \qquad p_n > n \text{ and } v_{j,n} \in L_n(v)$$

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$$w_n = v_{1,n} v_{2,n} \cdots v_{p_n,n}, \qquad v_{j,n} \in L_n(v), \text{ etc.}$$

- Consider a morphism $\varphi : X^* \to M$ recognizing L and $\{1\}$.
- Build a finite graph Γ_n as follows:
  - Vertices: $\{\hat{}, \$\} \cup \{(s,t) \in M \times M : L_n(v) \cap \varphi^{-1}(s) L^* \varphi^{-1}(t) \neq \varnothing\}$
  - Edge $\hat{} \to (s,t)$ if $\big(L_n(v)\big)^* \varphi^{-1}(s) \cap L \neq \varnothing$.
  - Edge $(s,t) \to \$$ dually.
  - Edges $(s_1,t_1) \to (s_2,t_2)$ if $\varphi^{-1}(t_1)\big(L_n(v)\big)^* \varphi^{-1}(s_2) \cap L \neq \varnothing$.
- The 2 factorizations define a path γ_n from $\hat{}$ to $\$$ in the graph.

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- Since the number of vertices is fixed, one can assume that the set of vertices and edges ("support") used by the paths γ_n is constant.
- First case: this support if paths γ_n has no cycle. In this case, all paths γ_n are the same simple path from ^ to \$.
- We deduce for each n sequences of the length of that path $(x_{i,n})_i$ and $(y_{i,n})_i$ corresponding to edges and vertices of the path.
- $x_{i,n} \in L$ so $\lim_n x_{i,n} = x_i \in \mathrm{cl}_{\mathsf{A}}(L)$,
- $y_{i,n} \in L^* \cap X^* L_n(v) X^*$, so it converges to $y_i \in \mathrm{cl}_{\mathsf{A}}(L^*)$ and has rank less than that of w. Induction: $y_i \in \langle \mathrm{cl}_{\kappa,\mathsf{A}}(L) \rangle_{\kappa,\mathsf{A}}$
- Therefore $w = x_{1,n} y_{1,n} x_{2,n} y_{2,n} \cdots$ is also in $\langle \mathrm{cl}_{\kappa,\mathsf{A}}(L) \rangle_{\kappa,\mathsf{A}}$.

PR algorithm for A: handling iteration: inclusion $\mathrm{cl}_{\kappa,\mathsf{A}}(L^{+}) \subseteq \langle \mathrm{cl}_{\kappa,\mathsf{A}}(L) \rangle_{\kappa,\mathsf{A}}$

- Since the number of vertices is fixed, one can assume that the set of vertices and edges ("support") used by the paths γ_n is constant.
- Second case: this support has a loop. Extracting if necessary, one can assume that all γ_n have the same prefix up to the same simple loop.

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- The definition of vertices/edges makes it possible to
  - cut other loops while staying in L^+ $\Rightarrow$ bounded number of factors.

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  - pump this loop with all factors staying in L^+: replace the loop by its ω-power. $\Rightarrow$ as many copies of words of $L_n(v)$ as wanted, ensuring convergence to w.

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Proposition

The pseudovariety R is κ-full.

Proof by induction on X, using the algebraic structure of $\overline{\Omega}_X \mathsf{R}$.

Corollary

The Pin-Reutenauer algorithm holds for the pseudovariety R and the canonical signature κ.

Proof using the inheritance theorem for κ-full pseudovarieties.

Two natural questions

1. Automata for term languages.
   - (Henckell's algorithm) Given regular $K, L \subseteq X^+$, one can decide whether
     $$\mathrm{cl}_{\mathsf{A}}(K) \cap \mathrm{cl}_{\mathsf{A}}(L) = \varnothing.$$
   - By a weak form of κ-reducibility for A, this is equivalent
     $$\mathrm{cl}_{\kappa,\mathsf{A}}(K) \cap \mathrm{cl}_{\kappa,\mathsf{A}}(L) = \varnothing.$$

     Is it possible to test it using automata accepting languages in $\Omega_X^\kappa \mathsf{A}$?
2. The pseudovariety S of all finite semigroups is σ-full, for every σ. Does the Pin-Reutenauer algorithm hold for S and κ?