Noncommutative p-adic analysis

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The pro-$\mathbf{V}$ quasimetric

Let M be a (f.g.) monoid and $\mathbf{V}$ a pseudovariety of finite monoids.

Given $u, v \in M$, let

$$r_{\mathbf{V}}(u,v) = \min\{|N| \mid N \in \mathbf{V} \text{ and separates } u \text{ and } v\}$$

$$d_{\mathbf{V}}(u,v) = 2^{-r_{\mathbf{V}}(u,v)}$$

with the usual conventions. Then $\min \emptyset = -\infty$ and $p^{-\infty} = 0$. Then $d_{\mathbf{V}}$ is a quasi(ultra)metric on M.

$\mathbf{V}$-uniform continuity

Let $f : M \to N$ be a function between (f.g.) monoids.

We say that f is

- $\mathbf{V}$-uniformly continuous if it is uniformly continuous with respect to the $\mathbf{V}$-quasimetric;
- $\mathbf{V}$-hereditarily continuous if it is $\mathbf{W}$-uniformly continuous for every subvariety $\mathbf{W}$ of $\mathbf{V}$.

Connection to languages

Theorem

Let $f : M \to N$ *be a function between (f.g.) monoids. TFCAE:*

(1) f *is* $\mathbf{V}$*-uniformly continuous;*

(2) *if* $X \subseteq N$ *is* $\mathbf{V}$*-recognizable, then* $f^{-1}(X)$ *is also* $\mathbf{V}$*-recognizable.*

Who's who

Our results involve:

- free monoids and free commutative monoids;
- the pseudovarieties

  $\mathbf{G}_p$ – finite p-groups
  $\mathbf{G}$ – finite groups
  $\mathbf{A}$ – finite aperiodic monoids
  $\mathbf{M}$ – finite monoids

Characterizations obtained:

$\mathbf{G}_p$-uniform continuity: $\mathbb{N}^k \to \mathbb{Z}$, $A^* \to \mathbb{Z}$

$\mathbf{G}_p$-hereditary continuity: $\mathbb{N}^k \to \mathbb{Z}$, $A^* \to \mathbb{Z}$

$\mathbf{G}$-uniform continuity: $\mathbb{N}^k \to \mathbb{Z}$

$\mathbf{G}$-hereditary continuity: $\mathbb{N}^k \to \mathbb{Z}$, $A^* \to \mathbb{Z}$

$\mathbf{A}$-uniform continuity: $\mathbb{N}^k \to \mathbb{N}$

$\mathbf{A}$-hereditary continuity: $\mathbb{N}^k \to \mathbb{N}$

$\mathbf{M}$-hereditary continuity: $\mathbb{N}^k \to \mathbb{N}$

Types of characterizations:

Combinatorial/number-theoretic

e.g., $f : \mathbb{N} \to \mathbb{N}$ is $\mathbf{A}$-uniformly continuous if and only if, for every $n \in \mathbb{N}$, $f^{-1}(n)$ is either finite or cofinite.

Analytic (???)

Mahler's expansions

For each function $f : \mathbb{N} \to \mathbb{Z}$, there exists a unique family a_k of integers such that, for all $n \in \mathbb{N}$,

$$f(n) = \sum_{k=0}^{\infty} a_k \binom{n}{k}$$

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This family is given by

$$a_k = (\Delta^k f)(0)$$

where Δ is the difference operator, defined by

$$(\Delta f)(n) = f(n+1) - f(n)$$

Examples

Fibonacci sequence: $f(0) = f(1) = 1$ and $f(n) = f(n-1) + f(n-2)$ for $(n \geqslant 2)$. Then

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Let $f(n) = r^n$. Then

$$f(n) = \sum_{k=0}^{\infty} (r-1)^k \binom{n}{k}$$

The p-adic valuation

Let p be a prime number. The p-adic valuation of a non-zero integer n is

$$\nu_p(n) = \max\left\{k \in \mathbb{N} \mid p^k \text{ divides } n\right\}$$

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Finally, the metric d_p can be defined by

$$d_p(u, v) = |u - v|_p$$

Mahler's theorem

Theorem (Mahler)

Let $f(n) = \sum_{k=0}^{\infty} a_k \binom{n}{k}$ *be the Mahler's expansion of a function* $f : \mathbb{N} \to \mathbb{Z}$*. TFCAE:*

(1) f *is uniformly continuous for the* p*-adic norm,*

(2) *the polynomial functions* $n \to \sum_{k=0}^{m} a_k \binom{n}{k}$ *converge uniformly to* f*,*

(3) $\lim_{k\to\infty} |a_k|_p = 0$.

(2) means that $\lim_{m\to\infty} \sup_{n\in\mathbb{N}} \left|\sum_{k=m}^{\infty} a_k \binom{n}{k}\right|_p = 0$.

Mahler's theorem (2)

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f *is uniformly continuous iff its Mahler's expansion converges uniformly to* f.

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f is uniformly continuous iff its Mahler's expansion converges uniformly to f.

The most remarkable part of the theorem is the fact that any uniformly continuous function can be approximated by polynomial functions, in contrast to Stone-Weierstrass approximation theorem, which requires much stronger conditions.

Examples

- The Fibonacci function is not uniformly continuous (for any p) since $f(n) = \sum_{k=0}^{\infty} (-1)^{k+1} f(k) \binom{n}{k}$.

- The function $f(n) = r^n$ is uniformly continuous iff $p \mid r - 1$ since $f(n) = \sum_{k=0}^{\infty} (r-1)^k \binom{n}{k}$.

Extension to words

Is it possible to obtain similar results for functions from A^* to $\mathbb{Z}$?

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Questions to be solved:

(1) Extend binomial coefficients to words and difference operators to word functions.

(2) Find a Mahler expansion for functions from A^* to $\mathbb{Z}$.

(3) Find a metric on A^* which generalizes d_p.

(4) Extend Mahler's theorem.

Binomial coefficients (see Eilenberg or Lothaire)

Let $u = a_1 \cdots a_n$ and v be two words of A^*. Then u is a subword of v if there exist $v_0, \ldots, v_n \in A^*$ such that $v = v_0 a_1 v_1 \ldots a_n v_n$. The binomial coefficient of u and v is

$$\binom{v}{u} = |\{(v_0, \ldots, v_n) \mid v = v_0 a_1 v_1 \ldots a_n v_n\}|$$

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If a is a letter, then $\binom{u}{a} = |u|_a$. If $u = a^n$ and $v = a^m$, then

$$\binom{v}{u} = \binom{m}{n}$$

Pascal triangle

Let $u, v \in A^*$ and $a, b \in A$. Then

(1) $\binom{u}{1} = 1$,

(2) $\binom{u}{v} = 0$ if $|u| \leqslant |v|$ and $u \neq v$,

(3) $$\binom{ua}{vb} = \begin{cases} \binom{u}{vb} & \text{if } a \neq b \\ \binom{u}{vb} + \binom{u}{v} & \text{if } a = b \end{cases}$$

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Example

$\binom{abab}{ab} = 3$

Difference operator

Let $f : A^* \to \mathbb{Z}$ be a function. We define inductively an operator Δ^w for each word $w \in A^*$ by setting $(\Delta^1 f)(u) = f(u)$, and for each $a \in A$:

$$(\Delta^a f)(u) = f(ua) - f(u),$$

$$(\Delta^{aw} f)(u) = (\Delta^a(\Delta^w f))(u).$$

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(Equivalent) direct definition of Δ^w:

$$\Delta^w f(u) = \sum_{0 \leqslant |x| \leqslant |w|} (-1)^{|w|+|x|} \binom{w}{x} f(ux)$$

Mahler's expansion of word functions

Theorem (cf. Lothaire)

For each function $f : A^* \to \mathbb{Z}$*, there exists a* ***unique family*** $\langle f, v\rangle_{v\in A^*}$ *of integers such that, for all* $u \in A^*$,

$$f(u) = \sum_{v\in A^*} \langle f, v\rangle \binom{u}{v}$$

This family is given by

$$\langle f, v\rangle = (\Delta^v f)(1) = \sum_{0\leqslant |x|\leqslant |v|} (-1)^{|v|+|x|} \binom{v}{x} f(x)$$

An example

Let $f : \{0,1\}^* \to \mathbb{N}$ the function mapping a binary word onto its value: $f(010111) = f(10111) = 23$.

$$(\Delta^v f) = \begin{cases} f + 1 & \text{if } v \in 1\{0,1\}^* \\ f & \text{otherwise} \end{cases}$$

$$(\Delta^v f)(\varepsilon) = \begin{cases} 1 & \text{if } v \in 1\{0,1\}^* \\ 0 & \text{otherwise} \end{cases}$$

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Thus, if $u = 01001$, then
$f(u) = \binom{u}{1} + \binom{u}{10} + \binom{u}{11} + \binom{u}{100} + \binom{u}{101} + \binom{u}{1001} = 2+2+1+1+2+1 = 9$.

Mahler's expansion of the product of two functions

An interesting question is to compute the Mahler's expansion of the product of two functions.

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Proposition

Let f and g be two word functions. The coefficients of the Mahler's expansion of fg are given by

$$\langle fg, x\rangle = \sum_{v_1, v_2 \in A^*} \langle f, v_1\rangle \langle g, v_2\rangle \langle v_1 \uparrow v_2, x\rangle$$

where $v_1 \uparrow v_2$ denotes the infiltration product.

Infiltration product (Chen, Fox, Lyndon)

Intuitively, the coefficient $\langle u \uparrow v, x\rangle$ is the number of pairs of subsequences of x which are respectively equal to u and v and whose union gives the whole sequence x.

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Intuitively, the coefficient $\langle u \uparrow v, x\rangle$ is the number of pairs of subsequences of x which are respectively equal to u and v and whose union gives the whole sequence x. For instance,

$$ab \uparrow ab = ab + 2aab + 2abb + 4aabb + 2abab$$

($4aabb$ since $aabb = aabb = aabb = aabb = aabb$)

$$ab \uparrow ba = aba + bab + abab + 2abba + 2baab + baba$$

Mahler polynomials

A function $f : A^* \to \mathbb{Z}$ is a Mahler polynomial if its Mahler's expansion has finite support, that is, if the number of nonzero coefficients $\langle f, v \rangle$ is finite.

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Proposition

Mahler polynomials form a subring of the ring of all functions from A^* to $\mathbb{Z}$ for addition and multiplication.

Proposition

Any pair of distinct words can be separated by a p*-group.*

p-groups

Proposition

Any pair of distinct words can be separated by a p-group.

Proposition

If $|A| = 1$, $d_{\mathbf{G}_p}$ is uniformly equivalent to the p-adic metric.

Mahler's theorem for word functions

Theorem

Let $f(n) = \sum_{v \in A^*} \langle f, v \rangle \binom{u}{v}$ *be the Mahler's expansion of a function* $f : A^* \to \mathbb{Z}$*. TFCAE:*

(1) f *is uniformly continuous for* d_p*,*

(2) *the partial sums* $\sum_{0 \leqslant |v| \leqslant n} \langle f, v \rangle \binom{u}{v}$ *converge uniformly to* f*,*

(3) $\lim_{|v| \to \infty} |\langle f, v \rangle|_p = 0$.

Further results

Other Mahler type characterizations can be obtained for:

$\mathbf{G}_p$-uniform continuity:	$\mathbb{N}^k \to \mathbb{Z}$ (*)
$\mathbf{G}_p$-hereditary continuity:	$\mathbb{N}^k \to \mathbb{Z}$, $A^* \to \mathbb{Z}$
$\mathbf{G}$-hereditary continuity:	$\mathbb{N}^k \to \mathbb{Z}$, $A^* \to \mathbb{Z}$
$\mathbf{A}$-uniform continuity:	$\mathbb{N} \to \mathbb{N}$

(*) alternative proof to Amice's Theorem

An example

Theorem

Let $f(n) = \sum_{k=0}^{\infty} a_k \binom{n}{k}$ *be the Mahler's expansion of a function* $f : \mathbb{N} \to \mathbb{Z}$. *TFCAE:*

(1) f *is* $\mathbf{G}$*-hereditarily continuous,*

(2) *if* $1 \leqslant j \leqslant k$, *then* $j | a_k$.

Further motivations

The Wadge hierarchy classifies topological spaces through continuous reductions: given two sets X and Y, Y reduces to X if there exists a continuous function f such that $X = f^{-1}(Y)$. Let us call p-reduction a uniformly continuous function between the metric spaces (A^*, d_p) and (B^*, d_p). These p-reductions define a hierarchy of regular languages that we would like to explore.