Equational Theory of regular languages

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Summary

Foreword

The idea of profinite topologies goes back at least to Birkhoff's paper Moore-Smith convergence in general topology (1937).

In this paper, Birkhoff introduces topologies defined by congruences on abstract algebras and states that, if each congruence has finite index, then the completion of the topological algebra is compact.

Further, he explicitly mentions three examples: p-adic numbers, Stone's duality of Boolean algebras and topologization of free groups.

Part I

The profinite world

Separating words

A deterministic finite automaton (DFA) separates two words if it accepts one of the words but not the other one.

A monoid M separates two words u and v of A^* if there exists a monoid morphism $\varphi : A^* \to M$ such that $\varphi(u) \neq \varphi(v)$.

Proposition

One can always separate two distinct words by a finite automaton (respectively by a finite monoid).

The profinite metric

Let u and v be two words. Put

$$r(u,v) = \min\Big\{|M| \;\Big|\; M \text{ is a finite monoid that separates } u \text{ and } v\Big\}$$

$$d(u,v) = 2^{-r(u,v)}$$

Then d is an ultrametric, that is, for all $x, y, z \in A^*$,

(1) $d(x,y) = 0$ iff $x = y$,

(2) $d(x,y) = d(y,x)$,

(3) $d(x,z) \leqslant \max\{d(x,y), d(y,z)\}$

Another profinite metric

Let

$$r'(u,v) = \min\{\#\ \text{states}(\mathcal{A}) \mid \mathcal{A} \text{ is a finite DFA separating } u \text{ and } v\}$$

$$d'(u,v) = 2^{-r'(u,v)}$$

The metric d' is uniformly equivalent to d:

$$2^{-\frac{1}{d'(u,v)}} \leqslant d(u,v) \leqslant d'(u,v)$$

Therefore, a function is uniformly continuous for d iff it is uniformly continuous for d'.

Main properties of d

Intuitively, two words are close for d if one needs a large monoid to separate them.

A sequence of words u_n is a Cauchy sequence iff, for every morphism φ from A^* to a finite monoid, the sequence $\varphi(u_n)$ is ultimately constant.

A sequence of words u_n converges to a word u iff, for every morphism φ from A^* to a finite monoid, the sequence $\varphi(u_n)$ is ultimately equal to $\varphi(u)$.

The free profinite monoid

The completion of the metric space (A^*, d) is the free profinite monoid on A and is denoted by $\widehat{A^*}$. It is a compact space, whose elements are called profinite words.

The concatenation product is uniformly continuous on A^* and can be extended by continuity to $\widehat{A^*}$.

Any morphism $\varphi : A^* \to M$, where M is a (discrete) finite monoid extends in a unique way to a uniformly continuous morphism $\hat{\varphi} : \widehat{A^*} \to M$.

Regular languages and clopen sets

The maps $L \mapsto \overline{L}$ and $K \mapsto K \cap A^*$ are inverse isomorphisms between the Boolean algebras $\mathrm{Reg}(A^*)$ and $\mathrm{Clopen}(\widehat{A^*})$. For all regular langauges L, L_1, L_2 of A^*:

(1) $\overline{L^c} = (\overline{L})^c$,

(2) $\overline{L_1 \cup L_2} = \overline{L_1} \cup \overline{L_2}$,

(3) $\overline{L_1 \cap L_2} = \overline{L_1} \cap \overline{L_2}$,

(4) for all $x, y \in A^*$, then $\overline{x^{-1}Ly^{-1}} = x^{-1}\overline{L}y^{-1}$.

(5) If $\varphi : A^* \to B^*$ is a morphism and $L \in \mathrm{Reg}(B^*)$, then $\hat{\varphi}^{-1}(\overline{L}) = \overline{\varphi^{-1}(L)}$.

Part II

Uniform spaces

J.-E. Pin et P. Weil, Uniformities on free semigroups, *IJAC* **9** (1999), 431–453.

Metric, ultrametrics and qu-metrics

A metric on a set X is a mapping $d : X \times X \to \mathbb{R}^+$ satisfying, for all $x, y, z \in X$,

(1) $d(x, y) = 0$ iff $x = y$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, z) \leqslant d(x, y) + d(y, z)$.

A quasimetric satisfies (3) and (1'):

(1′) For all $x \in X$, $d(x, x) = 0$.

A qu-metric satisfies (1') and (3'):

(3′) For each $x, y, z \in X$,
$d(x, z) \leqslant \max(d(x, y), d(y, z))$.

Notations

Let X be a set. The subsets of $X \times X$ are viewed as relations on X. Given $U, V \subseteq X \times X$, UV denotes the composition of U and V

$$UV = \big\{(x, y) \in X \times X \mid \text{there exists } z \in X, \\ (x, z) \in U \text{ and } (z, y) \in V\big\}.$$

The transposed relation of U is the relation

$${}^tU = \big\{(x, y) \in X \times X \mid (y, x) \in U\big\}$$

Quasi-uniformities

A quasi-uniformity on a set X is a nonempty set $\mathcal{U}$ of subsets of $X \times X$, called entourages, satisfying:

(1) Any superset of an entourage is an entourage,

(2) The intersection of two entourages is an entourage,

(3) Each entourage contains the diagonal of $X \times X$,

(4) For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $VV \subseteq U$.

A uniformity satisfies the additional condition:

(5) For each $U \in \mathcal{U}$, ${}^tU \in \mathcal{U}$.

Transitive quasi-uniformities

Abstract notion of ultrametric:

An entourage U is transitive if $UU \subseteq U$. A basis is said to be transitive if all its elements are transitive.

A quasi-uniformity is transitive if it has a transitive basis.

Bases of quasi-uniform spaces

A basis of $\mathcal{U}$ is a subset $\mathcal{B}$ of $\mathcal{U}$ such that each element of $\mathcal{U}$ contains an element of $\mathcal{B}$. Then $\mathcal{U}$ consists of all the relations on X containing an element of $\mathcal{B}$ and is said to be generated by $\mathcal{B}$.

A set $\mathcal{B}$ of subsets of $X \times X$ is a basis of some quasi-uniformity iff it satisfies (2), (3) and (4).

Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be quasi-uniform spaces. A mapping $\varphi : X \to Y$ is said to be uniformly continuous if, for each $V \in \mathcal{V}$, $(\varphi \times \varphi)^{-1}(V) \in \mathcal{U}$.

Topology associated to a quasi-uniformity

For each $x \in X$, let $\mathcal{U}(x) = \{U(x) \mid U \in \mathcal{U}\}$.
There exists a unique topology on X for which $\mathcal{U}(x)$ is the filter of neighborhoods of x for each $x \in X$.

In general, the intersection of all the entourages of $\mathcal{U}$ is a quasi-order $\leqslant$ and the closure of x is the set $\{y \in X \mid x \leqslant y\}$.

This topological space is Hausdorff iff the intersection of all the entourages of $\mathcal{U}$ is equal to the diagonal of $X \times X$.

Filters

Let X be a set. A filter on X is a set $\mathcal{F}$ of nonempty subsets of X closed under supersets and finite intersections. A filter is convergent to $x \in X$ if it contains all the neighborhoods of x.

A filter is Cauchy if, for each entourage $U \in \mathcal{U}$, there exists $F \in \mathcal{F}$ such that $F \times F \subseteq U$. For instance, the neighborhood filter of each point is a minimal Cauchy filter.

If $f : E \to F$ is uniformly continuous and $\mathcal{F}$ is a Cauchy filter, then the set $\{f(X) \mid X \in \mathcal{F}\}$ is also a Cauchy filter.

Cauchy filters and completions

A space is complete if every Cauchy filter is convergent.

The completion $\widehat{X}$ of X and the uniformly continuous mapping $\imath : X \to \widehat{X}$ are uniquely defined by the following universal property:

Every uniformly continuous mapping φ from X into a complete uniform space $(Y, \mathcal{V})$ induces a unique uniformly continuous mapping $\hat{\varphi} : \widehat{X} \to Y$ such that $\hat{\varphi} \circ \imath = \varphi$.

Totally bounded spaces and completions

A basis is totally bounded if, for each entourage U, there exist finitely many subsets $B_1, \ldots, B_n$ of X such that $X = \bigcup_i B_i$ and $\bigcup_i (B_i \times B_i) \subseteq U$.

Proposition (Bourbaki)

Let $(X, \mathcal{U})$ be a quasi-uniform space. Then the completion of X is compact iff $\mathcal{U}$ is totally bounded.

An example

Let E be a set. Given a finite partition $X = \{X_1, \ldots, X_n\}$ of E, let

$$U_X = \bigcup_{1 \leqslant i \leqslant n} X_i \times X_i$$

The sets of the form U_X, where X runs over the class of finite partitions of E, form the basis of the profinite quasi-uniformity on E. The profinite completion of E is compact.

Part III

Lattices of languages

- J. Almeida, *Finite semigroups and universal algebra*, World Scientific Publishing Co. Inc., River Edge, NJ, 1994.
- M. Gehrke, S. Grigorieff et J.-E. Pin, Duality and equational theory of regular languages, in *ICALP 2008, Part II*, L. A. et al. (éd.), Berlin, 2008, pp. 246–257, *Lect. Notes Comp. Sci.* vol. 5126, Springer.

Lattices of languages

Let A be a finite alphabet. A lattice of languages is a set of regular languages of A^* containing $\emptyset$ and A^* and closed under finite intersection and finite union.

A lattice of languages is a quotienting algebra of languages if it is closed under the quotienting operations $L \to u^{-1}L$ and $L \to Lu^{-1}$, for each word $u \in A^*$.

The pro-$\mathcal{L}$ qu-structure

Let $\mathcal{L}$ be a lattice of languages. One defines a quasi-uniform structure on A^* generated by the finite intersections of sets of the form

$$U_L = (L \times A^*) \cup (A^* \times L^c) \qquad (L \in \mathcal{L})$$

and called the pro-$\mathcal{L}$ qu-structure.

The space A^* is quasi-metrizable iff $\mathcal{L}$ is countable. This is the case in particular if A is finite.

The pro-$\mathcal{L}$ completion

The completion of A^* for the pro-$\mathcal{L}$ (quasi)-uniform structure is called the pro-$\mathcal{L}$ completion of A^* and denoted $\widehat{A^*}^{\mathcal{L}}$.

The space $\widehat{A^*}^{\mathcal{L}}$ is an ordered topological space, compact and totally order disconnected: the points of the space are separated by the upwards saturated clopen sets.

Examples

If $\mathcal{L}$ finite or cofinite languages of A^*, then $\widehat{A^*}^{\mathcal{L}} = A^* \cup \{0\}$ and, for each word x, $x \leqslant 0$.

If $\mathcal{L}$ is the set of languages of the form FA^*, where F is finite, then $\widehat{A^*}^{\mathcal{L}} = A^* \cup A^\omega$ and for all words x, y, $xy \leqslant x$.

If $\mathcal{L}$ is the set of shuffle ideals, then $\widehat{A^*}^{\mathcal{L}}$ is countable and is structure is well understood. For each word u, x, v, $uxv \leqslant uv$.

The product

If $\mathcal{L}$ is a quotienting algebra of languages, then the product on A^* is uniformly continuous and $\widehat{A^*}^{\mathcal{L}}$ becomes a topological compact monoid.

Theorem (Almeida-A. Costa, GGP)

The product on $\widehat{A^*}^{\mathcal{L}}$ is an open map iff $\mathcal{L}$ is closed under product.

Properties of $\widehat{A^*}^{\mathcal{L}}$

Theorem (Almeida)

Let L be a language of A^*. Then TFCAE:

(1) $L \in \mathcal{L}$,

(2) $\overline{L}^{\mathcal{L}}$ *is clopen and* $\overline{L}^{\mathcal{L}} \cap A^* = L$,

(3) $L = K \cap A^*$ *for some clopen* K *of* $\widehat{A^*}^{\mathcal{L}}$.

In fact, the sets of the form $\overline{L}$, with $L \in \mathcal{L}$ form a basis for the pro-$\mathcal{L}$ topology. These sets are clopen.

$\mathcal{L}$-preserving functions

Definition. A function $f : A^* \to A^*$ is $\mathcal{L}$-preserving if, for each language $L \in \mathcal{L}$, $f^{-1}(L) \in \mathcal{L}$.

Theorem

A function from A^* to A^* is uniformly continuous for the pro-$\mathcal{L}$ uniform structure iff it is $\mathcal{L}$-preserving.

In particular, regular-preserving functions are exactly the uniformly continuous functions for the profinite metric d.

A well-known exercise...

If L is a language, its square root is $K = \{u \in A^* \mid u^2 \in L\}$.

Exercise. Show that the square root of a regular [star-free] language is regular [star-free].

Proof. Note that $K = f^{-1}(L)$, where $f(u) = u^2$. Let $\mathcal{L}$ be a quotienting algebra of languages. Since the product is uniformly continuous for $d_{\mathcal{L}}$, f is uniformly continuous. Thus f is $\mathcal{L}$-preserving.

Part IV

Equational theory

- M. Gehrke, S. Grigorieff et J.-E. Pin, Duality and equational theory of regular languages, *ICALP 2008*

Equations

Let u and v be words of A^*. A language L of A^* satisfies the equation $u \to v$ if

$$u \in L \Rightarrow v \in L$$

Let E be a set of equations of the form $u \to v$. Then the languages of A^* satisfying the equations of E form a lattice of languages.

Equational description of finite lattices

Proposition

A finite set of languages of A^* is a lattice of languages iff it can be defined by a set of equations of the form $u \to v$ with $u, v \in A^*$.

Therefore, there is an equational theory for finite lattices of languages. What about infinite lattices?

One needs the profinite world...

Profinite equations

Let (u, v) be a pair of profinite words of $\widehat{A^*}$. We say that a regular language L of A^* satisfies the profinite equation $u \to v$ if

$$u \in \overline{L} \Rightarrow v \in \overline{L}$$

Let $\eta : A^* \to M$ be the syntactic morphism of L. Then L satisfies the profinite equation $u \to v$ iff

$$\hat{\eta}(u) \in \eta(L) \Rightarrow \hat{\eta}(v) \in \eta(L)$$

Equational theory of lattices

Given a set E of equations of the form $u \to v$ (where u and v are profinite words), the set of all regular languages of A^* satisfying all the equations of E is called the set of languages defined by E.

Theorem (Gehrke, Grigorieff, Pin 2008)

A set of regular languages of A^* is a lattice of languages iff it can be defined by a set of equations of the form $u \to v$, where $u, v \in \widehat{A^*}$.

Equations of the form $u \leqslant v$

Let us say that a regular language satisfies the equation $u \leqslant v$ if, for all $x, y \in \widehat{A^*}$, it satisfies the equation $xvy \rightarrow xuy$.

Proposition

Let L be a regular language of A^*, let $(M, \leqslant_L)$ be its syntactic ordered monoid and let $\eta : A^* \rightarrow M$ be its syntactic morphism. Then L satisfies the equation $u \leqslant v$ iff $\hat{\eta}(u) \leqslant_L \hat{\eta}(v)$.

Quotienting algebras of languages

A lattice of languages is a quotienting algebra of languages if it is closed under the quotienting operations $L \to u^{-1}L$ and $L \to Lu^{-1}$, for each word $u \in A^*$.

Theorem

A set of regular languages of A^* is a quotienting algebra of languages iff it can be defined by a set of equations of the form $u \leqslant v$, where $u, v \in \widehat{A^*}$.

Boolean algebras

Let us write

$u \leftrightarrow v$ for $u \to v$ and $v \to u$,

$u = v$ for $u \leqslant v$ and $v \leqslant u$.

Theorem

(1) *A set of regular languages of A^* is a Boolean algebra iff it can be defined by a set of equations of the form $u \leftrightarrow v$.*

(2) *A set of regular languages of A^* is a Boolean quotienting algebra iff it can be defined by a set of equations of the form $u = v$.*

Interpreting equations

Let u and v be two profinite words.

Closed under		**Interpretation**
$\cup, \cap$	$u \to v$	$u \in \overline{L} \Rightarrow v \in \overline{L}$
$+$ quotient	$u \leqslant v$	$\forall x, y \quad xvy \to xuy$
$+$ complement (L^c)	$u \leftrightarrow v$	$u \to v$ and $v \to u$
$+$ quotient and L^c	$u = v$	$xuy \leftrightarrow xvy$

Identities

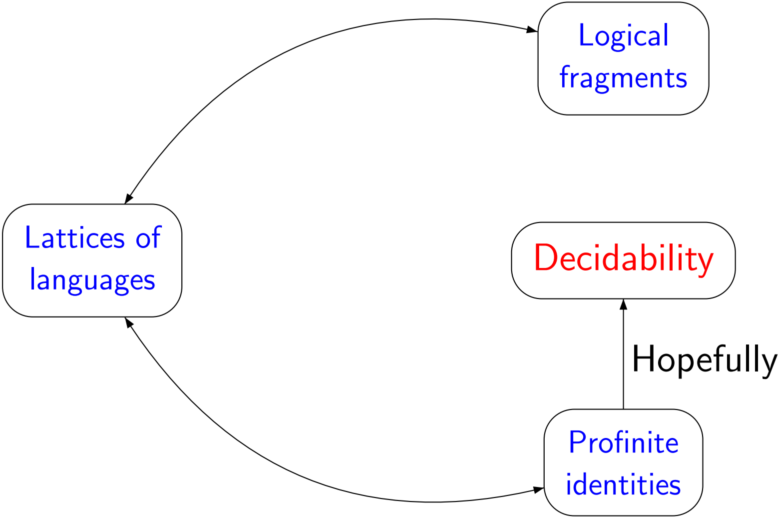
One can also recover Eilenberg's variety theorem and its variants by using identities. An identity is an equation in which letters are considered as variables.

Closed under inverse of $\cdots$ morphisms	**Interpretation of variables**
all	words
length increasing	nonempty words
length preserving	letters
length multiplying	words of equal length

Equational descriptions

- Every lattice of regular languages has an equational description.
- In particular, any class of regular languages defined by a fragment of logic closed under conjunctions and disjunctions (first order, monadic second order, temporal, etc.) admits an equational description.
- This result can also be adapted to languages of infinite words, words over ordinals or linear orders, and hopefully to tree languages.

The virtuous circle



Part V

Some examples

- Languages with zero
- Nondense languages
- Slender languages
- Sparse languages
- Examples from logic
- Examples of identities

A profinite word

Let us fix a total order on the alphabet A. Let $u_0, u_1, \ldots$ be the ordered sequence of all words of A^* in the induced shortlex order.

$1, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, \ldots$

Reilly and Zhang (see also Almeida-Volkov) proved that the sequence $(v_n)_{n \geqslant 0}$ defined by

$$v_0 = u_0, \; v_{n+1} = (v_n u_{n+1} v_n)^{(n+1)!}$$

is a Cauchy sequence, which converges to an idempotent ρ_A of the minimal ideal of $\widehat{A^*}$.

Languages with zero

A language with zero is a language whose syntactic monoid has a zero. The class of regular languages with zero is closed under Boolean operations and quotients, but not under inverse of morphisms.

Proposition

A regular language has a zero iff it satisfies the equation $x\rho_A = \rho_A = \rho_A x$ *for all* $x \in A^*$.

In the sequel, we simply write 0 for ρ_A to mean that L has a zero.

Nondense languages

A language L of A^* is dense if, for each word $u \in A^*$, $L \cap A^*uA^* \neq \emptyset$.

Regular non-dense or full languages form a lattice closed under quotients.

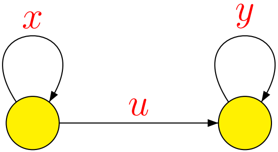
Theorem

A regular language of A^* is non-dense or full iff it satisfies the equations $x \leqslant 0$ for all $x \in A^*$.

Slender or full languages

A regular language is slender iff it is a finite union of languages of the form xu^*y, where $x, u, y \in A^*$.

Fact. A regular language is slender iff its minimal deterministic automaton does not contain any pair of connected cycles.



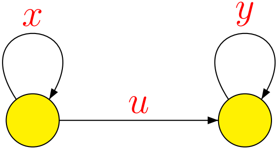
Two connected cycles, where $x, y \in A^+$ and $u \in A^*$.

Equations for slender languages

Denote by $i(x)$ the initial of a word x.

Theorem

Suppose that $|A| \geqslant 2$. A regular language of A^* is slender or full iff it satisfies the equations $x \leqslant 0$ for all $x \in A^*$ and the equation $x^\omega u y^\omega = 0$ for each $x, y \in A^+$, $u \in A^*$ such that $i(uy) \neq i(x)$.



Sparse languages

A regular language is sparse iff it is a finite union of languages of the form $u_0v_1^*u_1 \cdots v_n^*u_n$, where u_0, v_1, ..., v_n, u_n are words.

Theorem

Suppose that $|A| \geqslant 2$. A regular language of A^* is sparse or full iff it satisfies the equations $x \leqslant 0$ for all $x \in A^*$ and the equations $(x^\omega y^\omega)^\omega = 0$ for each $x, y \in A^+$ such that $i(x) \neq i(y)$.

Identities of well-known logical fragments

(1) Star-free languages: $x^{\omega+1} = x^{\omega}$. Captured by the logical fragment $FO[<]$.

(2) Finite unions of languages of the form $A^*a_1A^*a_2A^* \;\cdots\; a_kA^*$, where $a_1, \ldots, a_k$ are letters: $x \leqslant 1$. Captured by $\Sigma_1[<]$.

(3) Piecewise testable languages = Boolean closure of (3): $x^{\omega+1} = x^{\omega}$ and $(xy)^{\omega} = (yx)^{\omega}$. Captured by $\mathcal{B}\Sigma_1[<]$.

(4) Unambiguous star-free languages: $x^{\omega+1} = x^{\omega}$ and $(xy)^{\omega}(yx)^{\omega}(xy)^{\omega} = (xy)^{\omega}$. Captured by $FO_2[<]$ (first order with two variables) or by $\Sigma_2[<] \cap \Pi_2[<]$ or by unary temporal logic.

Another fragment of Büchi's sequential calculus

Denote by $\mathcal{B}\Sigma_1(S)$ the Boolean combinations of existential formulas in the signature $\{S, (\mathbf{a})_{a\in A}\}$. This logical fragment allows to specify properties like the factor aa occurs at least twice. Here is an equational description of the $\mathcal{B}\Sigma_1(S)$-definable languages, where $r, s, u, v, x, y \in A^*$:

$$ux^\omega v \leftrightarrow ux^{\omega+1}v$$
$$ux^\omega ry^\omega sx^\omega ty^\omega v \leftrightarrow ux^\omega ty^\omega sx^\omega ry^\omega v$$
$$x^\omega uy^\omega vx^\omega \leftrightarrow y^\omega vx^\omega uy^\omega$$
$$y(xy)^\omega \leftrightarrow (xy)^\omega \leftrightarrow (xy)^\omega x$$

Examples of length-multiplying identities

Length-multiplying identities: x and y represent words of the same length.

(1) Regular languages of AC^0: $(x^{\omega-1}y)^\omega = (x^{\omega-1}y)^{\omega+1}$. Captured by $FO[< +MOD]$.

(2) Finite union of languages of the form $(A^d)^*a_1(A^d)^*a_2(A^d)^* \cdots\ a_k(A^d)^*$, with $d > 0$: $x^{\omega-1}y \leqslant 1$ and $yx^{\omega-1} \leqslant 1$. Captured by $\Sigma_1[< +MOD]$.

Part VI

Duality

- J. Almeida, *Finite semigroups and universal algebra*, World Scientific Publishing Co. Inc., River Edge, NJ, 1994.
- M. Gehrke, S. Grigorieff et J.-E. Pin, Duality and equational theory of regular languages, *ICALP 2008*
- N. Pippenger, Regular languages and Stone duality, *Theory Comput. Syst.* **30**,2 (1997), 121–134.

Duality in a nutshell

The dual space of a distributive lattice is the set of its prime filters.

Elements	$\longleftrightarrow$	Prime filters
Boolean algebras	$\longleftrightarrow$	Topological spaces
Distributive lattices	$\longleftrightarrow$	Ordered topologial spaces
Sublattices	$\longleftrightarrow$	Quotient spaces
n-ary operations	$\longleftrightarrow$	$(n+1)$-ary relations

The dual space of $\mathrm{Reg}(A^*)$

Almeida (1989, implicitely) and Pippenger (1997, explicitely) proved:

Theorem

The dual space of the lattice of regular languages is the space of profinite words. Furthermore, the canonical embedding is given by the topologial closure: $e(L) = \overline{L}$.

Prime filters of $\mathrm{Reg}(A^*)$

- Given a prime filter p of $\mathrm{Reg}(A^*)$, there is a unique profinite word u such that, for every morphism from A^* onto a finite monoid M, $\hat\varphi(u)$ is the unique element $m \in M$ such that $\varphi^{-1}(m) \in p$.

- If u is a profinite word, the set

$$p_u = \{\ L \in \mathrm{Reg}(A^*) \mid \varphi^{-1}(\hat\varphi(u)) \subseteq L \text{ for some morphism } \varphi \text{ from } A^* \text{ onto a finite monoid } \}$$

is a prime filter of $\mathrm{Reg}(A^*)$.

A duality result

The right and left residuals of L by K are:

$$K\backslash L = \{u \in A^* \mid Ku \subseteq L\}$$
$$L/K = \{u \in A^* \mid uK \subseteq L\}$$

Theorem

The product on profinite words is the dual of the residuation operations on regular languages.

The identity $(H\backslash L)/K = H\backslash(L/K)$ in $\mathrm{Reg}(A^*)$ is equivalent to stating that the product is associative.

Back to the proof of the main result

Theorem

A set of regular languages of A^* is a lattice of languages iff it can be defined by a set of equations of the form $u \to v$, where $u, v \in \widehat{A^*}$.

The proof is an instantiation of the duality between sublattices of $\mathrm{Reg}(A^*)$ and preorders on its dual space $\widehat{A^*}$.

Let $\mathcal{L}$ be a lattice of languages. The preorder determining the quotient in the dual space is exactly the equational theory of $\mathcal{L}$.

Reductions

Given two sets X and Y, Y reduces to X if there exists a function f such that $X = f^{-1}(Y)$.

(1) In computability theory, f is Turing computable,

(2) In complexity theory, f is computable in polynomial time,

(3) In descriptive set theory, f is continuous.

Each of these reductions defines a partial preorder.

Proposal: Use $\mathcal{L}$-preserving functions as reductions and study the corresponding hierarchies.

Conclusion

Profinite topologies lead to an elegant theory and opens the door to more sophisticated topological tools: Stone-Priestley dualities, uniform spaces, spectral spaces, Wadge hierarchies.

Two difficult problems:
(1) Finding a set of equations defining a lattice can be difficult. In good cases, equations involve only words and simple profinite operators, like ω, but this is not the rule.
(2) Given a set of equations, one still needs to decide whether a given regular language satisfies these equations.