A NEW APPROACH TO MCCAMMOND'S SOLUTION OF THE ω-WORD PROBLEM FOR FINITE APERIODIC SEMIGROUPS

José Carlos Costa

with Jorge Almeida (U. Porto)
and Marc Zeitoun (U. Bordeaux)

Departamento de Matemática
Universidade do Minho
Braga, Portugal

Workshop on Equational Theory of Regular Languages

Brno, 5 March 2009

- An ω-term is a formal expression obtained from letters of an alphabet X using the operations of concatenation $(u, v) \mapsto uv$, and ω-power $u \mapsto u^\omega$.
- An ω-term has a natural *interpretation* on each finite semigroup S:

When do two ω-terms define the same operation over all finite aperiodic semigroups?

- An ω-term is a formal expression obtained from letters of an alphabet X using the operations of concatenation $(u, v) \mapsto uv$, and ω-power $u \mapsto u^\omega$.
- An ω-term has a natural *interpretation* on each finite semigroup S:
  - the concatenation is viewed as the semigroup multiplication
  - the ω-power is interpreted as the operation that sends each element $s \in S$ to the unique idempotent power s^ω of s.

When do two ω-terms define the same operation over all finite aperiodic semigroups?

- An ω-term is a formal expression obtained from letters of an alphabet X using the operations of concatenation $(u, v) \mapsto uv$, and ω-power $u \mapsto u^\omega$.
- An ω-term has a natural *interpretation* on each finite semigroup S:
  - the concatenation is viewed as the semigroup multiplication
  - the ω-power is interpreted as the operation that sends each element $s \in S$ to the unique idempotent power s^ω of s.

When do two ω-terms define the same operation over all finite aperiodic semigroups?

- An ω-term is a formal expression obtained from letters of an alphabet X using the operations of concatenation $(u, v) \mapsto uv$, and ω-power $u \mapsto u^\omega$.
- An ω-term has a natural *interpretation* on each finite semigroup S:
  - the concatenation is viewed as the semigroup multiplication
  - the ω-power is interpreted as the operation that sends each element $s \in S$ to the unique idempotent power s^ω of s.

When do two ω-terms define the same operation over all finite aperiodic semigroups?

- An ω-term is a formal expression obtained from letters of an alphabet X using the operations of concatenation $(u, v) \mapsto uv$, and ω-power $u \mapsto u^\omega$.
- An ω-term has a natural *interpretation* on each finite semigroup S:
    - the concatenation is viewed as the semigroup multiplication
    - the ω-power is interpreted as the operation that sends each element $s \in S$ to the unique idempotent power s^ω of s.

When do two ω-terms define the same operation over all finite aperiodic semigroups?

Rewriting rules for ω-terms

J. McCammond, *Normal forms for free aperiodic semigroups*, Int. J. Algebra and Computation **11** (2001), 581–625.

Theorem (McCammond's algorithm)

Using only rewriting rules resulting from reading the following identities in either direction, it is possible to transform any ω-term into a certain normal form, preserving its action on finite aperiodic semigroups:

1. $(x^\omega)^\omega = x^\omega$;
2. $(x^k)^\omega = x^\omega$ for $k \geq 2$;
3. $x^\omega x^\omega = x^\omega$;
4. $x^\omega x = x^\omega = x x^\omega$;
5. $(xy)^\omega x = x(yx)^\omega$.

Rewriting rules for ω-terms

J. McCammond, *Normal forms for free aperiodic semigroups*, Int. J. Algebra and Computation **11** (2001), 581–625.

Theorem (McCammond's algorithm)

Using only rewriting rules resulting from reading the following identities in either direction, it is possible to transform any ω-term into a certain normal form, preserving its action on finite aperiodic semigroups:

1. $(x^\omega)^\omega = x^\omega$;
2. $(x^k)^\omega = x^\omega$ for $k \geq 2$;
3. $x^\omega x^\omega = x^\omega$;
4. $x^\omega x = x^\omega = x x^\omega$;
5. $(xy)^\omega x = x(yx)^\omega$.

McCammond's normal form

- McCammond represents ω-terms over an alphabet X as well-parenthesized words in the alphabet

$$Y = X \uplus \{(,)\},$$

for which the parentheses are thus viewed as letters. The set of such words is then identified with the set U_X of ω-terms over X.

- Fix a total ordering of the alphabet X, and extend it to Y by letting $(< x <)$ for all $x \in X$.
- A *Lyndon word* is a primitive word that is minimal, with respect to the lexicographic ordering, in its conjugacy class.
- Alternatively, a word $w \in X^+$ is a *Lyndon word* if and only if $w < v$ for any proper non-empty suffix v of w.

McCammond's normal form

- McCammond represents ω-terms over an alphabet X as well-parenthesized words in the alphabet

$$Y = X \uplus \{(,)\},$$

for which the parentheses are thus viewed as letters. The set of such words is then identified with the set U_X of ω-terms over X.

- Fix a total ordering of the alphabet X, and extend it to Y by letting $(< x <)$ for all $x \in X$.

- A *Lyndon word* is a primitive word that is minimal, with respect to the lexicographic ordering, in its conjugacy class.

- Alternatively, a word $w \in X^+$ is a *Lyndon word* if and only if $w < v$ for any proper non-empty suffix v of w.

McCammond's normal form

- McCammond represents ω-terms over an alphabet X as well-parenthesized words in the alphabet

$$Y = X \uplus \{(,)\},$$

  for which the parentheses are thus viewed as letters. The set of such words is then identified with the set U_X of ω-terms over X.
- Fix a total ordering of the alphabet X, and extend it to Y by letting $(< x <)$ for all $x \in X$.
- A *Lyndon word* is a primitive word that is minimal, with respect to the lexicographic ordering, in its conjugacy class.
- Alternatively, a word $w \in X^+$ is a *Lyndon word* if and only if $w < v$ for any proper non-empty suffix v of w.

McCammond's normal form

- McCammond represents ω-terms over an alphabet X as well-parenthesized words in the alphabet

$$Y = X \uplus \{(,)\},$$

  for which the parentheses are thus viewed as letters. The set of such words is then identified with the set U_X of ω-terms over X.
- Fix a total ordering of the alphabet X, and extend it to Y by letting $(< x <)$ for all $x \in X$.
- A *Lyndon word* is a primitive word that is minimal, with respect to the lexicographic ordering, in its conjugacy class.
- Alternatively, a word $w \in X^+$ is a *Lyndon word* if and only if $w < v$ for any proper non-empty suffix v of w. So, every Lyndon word is an unbordered word.

McCammond's normal form

- McCammond represents ω-terms over an alphabet X as well-parenthesized words in the alphabet

$$Y = X \uplus \{(,)\},$$

  for which the parentheses are thus viewed as letters. The set of such words is then identified with the set U_X of ω-terms over X.
- Fix a total ordering of the alphabet X, and extend it to Y by letting $(< x <)$ for all $x \in X$.
- A *Lyndon word* is a primitive word that is minimal, with respect to the lexicographic ordering, in its conjugacy class.
- Alternatively, a word $w \in X^+$ is a *Lyndon word* if and only if $w < v$ for any proper non-empty suffix v of w. So, every Lyndon word is an unbordered word.

McCammond's normal form

- McCammond represents ω-terms over an alphabet X as well-parenthesized words in the alphabet

$$Y = X \uplus \{(,)\},$$

  for which the parentheses are thus viewed as letters. The set of such words is then identified with the set U_X of ω-terms over X.
- Fix a total ordering of the alphabet X, and extend it to Y by letting $(< x <)$ for all $x \in X$.
- A *Lyndon word* is a primitive word that is minimal, with respect to the lexicographic ordering, in its conjugacy class.
- Alternatively, a word $w \in X^+$ is a *Lyndon word* if and only if $w < v$ for any proper non-empty suffix v of w. So, every Lyndon word is an unbordered word.

McCammond's normal form

McCammond's *normal form* is defined recursively:

- The *rank 0 normal forms* are the words from X^+.
- Assuming that the *rank i normal forms* have been defined, a *rank $i+1$ normal form* is a word from Y^+ of the form

$$\alpha_0(\beta_1)\alpha_1(\beta_2)\cdots\alpha_{n-1}(\beta_n)\alpha_n,$$

where the α_j and β_k are ω-terms such that:

McCammond's normal form

McCammond's *normal form* is defined recursively:

- The *rank 0 normal forms* are the words from X^+.
- Assuming that the *rank i normal forms* have been defined, a *rank $i+1$ normal form* is a word from Y^+ of the form

$$\alpha_0(\beta_1)\alpha_1(\beta_2)\cdots\alpha_{n-1}(\beta_n)\alpha_n,$$

where the α_j and β_k are ω-terms such that:

- each β_k is a Lyndon word of rank i;
- each intermediate α_j is not a prefix of a power of β_j nor a suffix of a power of β_{j+1};
- replacing each factor (β_k) by $\beta_k\beta_k$, we obtain a normal form of rank i;
- at least one of the preceding properties fails if we remove from α_j a prefix β_j or a suffix β_{j+1} for $0 < j < n$;
- β_1 is not a suffix of α_0 and β_n is not a prefix of α_n.

McCammond's normal form

McCammond's *normal form* is defined recursively:

- The *rank 0 normal forms* are the words from X^+.
- Assuming that the *rank i normal forms* have been defined, a *rank $i+1$ normal form* is a word from Y^+ of the form

$$\alpha_0(\beta_1)\alpha_1(\beta_2)\cdots\alpha_{n-1}(\beta_n)\alpha_n,$$

where the α_j and β_k are ω-terms such that:

  - each β_k is a Lyndon word of rank i;
  - each intermediate α_j is not a prefix of a power of β_j nor a suffix of a power of β_{j+1};
  - replacing each factor (β_k) by $\beta_k\beta_k$, we obtain a normal form of rank i;
  - at least one of the preceding properties fails if we remove from α_j a prefix β_j or a suffix β_{j+1} for $0 < j < n$;
  - β_1 is not a suffix of α_0 and β_n is not a prefix of α_n.

McCammond's normal form

McCammond's *normal form* is defined recursively:

- The *rank 0 normal forms* are the words from X^+.
- Assuming that the *rank i normal forms* have been defined, a *rank $i+1$ normal form* is a word from Y^+ of the form

$$\alpha_0(\beta_1)\alpha_1(\beta_2)\cdots\alpha_{n-1}(\beta_n)\alpha_n,$$

where the α_j and β_k are ω-terms such that:

  - each β_k is a Lyndon word of rank i;
  - each intermediate α_j is not a prefix of a power of β_j nor a suffix of a power of β_{j+1};
  - replacing each factor (β_k) by $\beta_k\beta_k$, we obtain a normal form of rank i;
  - at least one of the preceding properties fails if we remove from α_j a prefix β_j or a suffix β_{j+1} for $0 < j < n$;
  - β_1 is not a suffix of α_0 and β_n is not a prefix of α_n.

McCammond's normal form

McCammond's *normal form* is defined recursively:

- The *rank 0 normal forms* are the words from X^+.
- Assuming that the *rank i normal forms* have been defined, a *rank $i+1$ normal form* is a word from Y^+ of the form

$$\alpha_0(\beta_1)\alpha_1(\beta_2)\cdots\alpha_{n-1}(\beta_n)\alpha_n,$$

where the α_j and β_k are ω-terms such that:

  - each β_k is a Lyndon word of rank i;
  - each intermediate α_j is not a prefix of a power of β_j nor a suffix of a power of β_{j+1};
  - replacing each factor (β_k) by $\beta_k\beta_k$, we obtain a normal form of rank i;
  - at least one of the preceding properties fails if we remove from α_j a prefix β_j or a suffix β_{j+1} for $0 < j < n$;
  - β_1 is not a suffix of α_0 and β_n is not a prefix of α_n.

McCammond's normal form

McCammond's *normal form* is defined recursively:

- The *rank 0 normal forms* are the words from X^+.
- Assuming that the *rank i normal forms* have been defined, a *rank $i+1$ normal form* is a word from Y^+ of the form

$$\alpha_0(\beta_1)\alpha_1(\beta_2)\cdots\alpha_{n-1}(\beta_n)\alpha_n,$$

where the α_j and β_k are ω-terms such that:

  - each β_k is a Lyndon word of rank i;
  - each intermediate α_j is not a prefix of a power of β_j nor a suffix of a power of β_{j+1};
  - replacing each factor (β_k) by $\beta_k\beta_k$, we obtain a normal form of rank i;
  - at least one of the preceding properties fails if we remove from α_j a prefix β_j or a suffix β_{j+1} for $0 < j < n$;
  - β_1 is not a suffix of α_0 and β_n is not a prefix of α_n.

McCammond's normal form

McCammond's *normal form* is defined recursively:

- The *rank 0 normal forms* are the words from X^+.
- Assuming that the *rank i normal forms* have been defined, a *rank $i+1$ normal form* is a word from Y^+ of the form

$$\alpha_0(\beta_1)\alpha_1(\beta_2)\cdots\alpha_{n-1}(\beta_n)\alpha_n,$$

where the α_j and β_k are ω-terms such that:

  - each β_k is a Lyndon word of rank i;
  - each intermediate α_j is not a prefix of a power of β_j nor a suffix of a power of β_{j+1};
  - replacing each factor (β_k) by $\beta_k\beta_k$, we obtain a normal form of rank i;
  - at least one of the preceding properties fails if we remove from α_j a prefix β_j or a suffix β_{j+1} for $0 < j < n$;
  - β_1 is not a suffix of α_0 and β_n is not a prefix of α_n.

McCammond's solution of the ω-word problem over A

The following ω-terms are in McCammond's *normal form*, where a and b are letters with $a < b$,

- $(a)ab(b)$ is the normal form, for instance, of $(a)(b)$ and of $a(a^5)a(a^2)b^8(b)$.
- $b(ab)abaa(a)b(aab)$ is the normal form of $(ba)(a)ba(aba)ab$.
- $((a)ab(b)ba)(a)ab(b)$ is the normal form of $((a)(b))$.

Theorem (McCammond'2001)

Two ω-terms coincide in all finite aperiodic semigroups if and only if they have the same normal form.

J. McCammond, *The solution to the word problem for the relatively free semigroups satisfying $T^a = T^{a+b}$ with $a \geq 6$*, Int. J. Algebra and Computation **11** (1991), 1-32.

McCammond's solution of the ω-word problem over **A**

The following ω-terms are in McCammond's *normal form*, where a and b are letters with $a < b$,

- $(a)ab(b)$ is the normal form, for instance, of $(a)(b)$ and of $a(a^5)a(a^2)b^8(b)$.
- $b(ab)abaa(a)b(aab)$ is the normal form of $(ba)(a)ba(aba)ab$.
- $((a)ab(b)ba)(a)ab(b)$ is the normal form of $((a)(b))$.

Theorem (McCammond'2001)

Two ω-terms coincide in all finite aperiodic semigroups if and only if they have the same normal form.

J. McCammond, *The solution to the word problem for the relatively free semigroups satisfying $T^a = T^{a+b}$ with $a \geq 6$*, Int. J. Algebra and Computation **11** (1991), 1-32.

McCammond's solution of the ω-word problem over **A**

The following ω-terms are in McCammond's *normal form*, where a and b are letters with $a < b$,

- $(a)ab(b)$ is the normal form, for instance, of $(a)(b)$ and of $a(a^5)a(a^2)b^8(b)$.
- $b(ab)abaa(a)b(aab)$ is the normal form of $(ba)(a)ba(aba)ab$.
- $\big((a)ab(b)ba\big)(a)ab(b)$ is the normal form of $\big((a)(b)\big)$.

Theorem (McCammond'2001)

Two ω-terms coincide in all finite aperiodic semigroups if and only if they have the same normal form.

J. McCammond, *The solution to the word problem for the relatively free semigroups satisfying $T^a = T^{a+b}$ with $a \geq 6$*, Int. J. Algebra and Computation **11** (1991), 1-32.

McCammond's solution of the ω-word problem over A

The following ω-terms are in McCammond's *normal form*, where a and b are letters with $a < b$,

- $(a)ab(b)$ is the normal form, for instance, of $(a)(b)$ and of $a(a^5)a(a^2)b^8(b)$.
- $b(ab)abaa(a)b(aab)$ is the normal form of $(ba)(a)ba(aba)ab$.
- $\big((a)ab(b)ba\big)(a)ab(b)$ is the normal form of $\big((a)(b)\big)$.

Theorem (McCammond'2001)

Two ω-terms coincide in all finite aperiodic semigroups if and only if they have the same normal form.

J. McCammond, *The solution to the word problem for the relatively free semigroups satisfying $T^a = T^{a+b}$ with $a \geq 6$*, Int. J. Algebra and Computation **11** (1991), 1-32.

McCammond's solution of the ω-word problem over **A**

The following ω-terms are in McCammond's *normal form*, where a and b are letters with $a < b$,

- $(a)ab(b)$ is the normal form, for instance, of $(a)(b)$ and of $a(a^5)a(a^2)b^8(b)$.
- $b(ab)abaa(a)b(aab)$ is the normal form of $(ba)(a)ba(aba)ab$.
- $((a)ab(b)ba)(a)ab(b)$ is the normal form of $((a)(b))$.

Theorem (McCammond'2001)

Two ω-terms coincide in all finite aperiodic semigroups if and only if they have the same normal form.

J. McCammond, *The solution to the word problem for the relatively free semigroups satisfying $T^a = T^{a+b}$ with $a \geq 6$*, Int. J. Algebra and Computation **11** (1991), 1-32.

McCammond's solution of the ω-word problem over **A**

The following ω-terms are in McCammond's *normal form*, where a and b are letters with $a < b$,

- $(a)ab(b)$ is the normal form, for instance, of $(a)(b)$ and of $a(a^5)a(a^2)b^8(b)$.
- $b(ab)abaa(a)b(aab)$ is the normal form of $(ba)(a)ba(aba)ab$.
- $\big((a)ab(b)ba\big)(a)ab(b)$ is the normal form of $\big((a)(b)\big)$.

Theorem (McCammond'2001)

Two ω-terms coincide in all finite aperiodic semigroups if and only if they have the same normal form.

J. McCammond, *The solution to the word problem for the relatively free semigroups satisfying $T^a = T^{a+b}$ with $a \geq 6$*, Int. J. Algebra and Computation **11** (1991), 1-32.

Languages associated with ω-terms

- Let n be a positive integer.
- For a word $\alpha \in X^+$, we let $E_n(\alpha) = \{\alpha\}$.
- For an ω-term $\alpha = \alpha_0(\beta_1)\alpha_1 \cdots (\beta_r)\alpha_r$ where all the β_j have the same rank i and all the α_j have rank at most i, we let

$$E_n(\alpha) = \{\alpha_0\beta_1^{n_1}\alpha_1 \cdots \beta_r^{n_r}\alpha_r : n_1, \ldots, n_r \geq n\}.$$

- For a set W of ω-terms, we let

$$E_n(W) = \bigcup_{\alpha \in W} E_n(\alpha).$$

- For an ω-term α of rank k, we let

$$L_n(\alpha) = (E_n)^k(\alpha).$$

- For a set W of ω-terms, we let

$$L_n(W) = \bigcup_{\alpha \in W} L_n(\alpha).$$

Languages associated with ω-terms

- Let n be a positive integer.
- For a word $\alpha \in X^+$, we let $E_n(\alpha) = \{\alpha\}$.
- For an ω-term $\alpha = \alpha_0(\beta_1)\alpha_1 \cdots (\beta_r)\alpha_r$ where all the β_j have the same rank i and all the α_j have rank at most i, we let

$$E_n(\alpha) = \{\alpha_0\beta_1^{n_1}\alpha_1 \cdots \beta_r^{n_r}\alpha_r : n_1, \dots, n_r \geq n\}.$$

- For a set W of ω-terms, we let

$$E_n(W) = \bigcup_{\alpha \in W} E_n(\alpha).$$

- For an ω-term α of rank k, we let

$$L_n(\alpha) = (E_n)^k(\alpha).$$

- For a set W of ω-terms, we let

$$L_n(W) = \bigcup_{\alpha \in W} L_n(\alpha).$$

Languages associated with ω-terms

- Let n be a positive integer.
- For a word $\alpha \in X^+$, we let $E_n(\alpha) = \{\alpha\}$.
- For an ω-term $\alpha = \alpha_0(\beta_1)\alpha_1\cdots(\beta_r)\alpha_r$ where all the β_j have the same rank i and all the α_j have rank at most i, we let

$$E_n(\alpha) = \{\alpha_0\beta_1^{n_1}\alpha_1\cdots\beta_r^{n_r}\alpha_r : n_1, \dots, n_r \geq n\}.$$

- For a set W of ω-terms, we let

$$E_n(W) = \bigcup_{\alpha \in W} E_n(\alpha).$$

- For an ω-term α of rank k, we let

$$L_n(\alpha) = (E_n)^k(\alpha).$$

- For a set W of ω-terms, we let

$$L_n(W) = \bigcup_{\alpha \in W} L_n(\alpha).$$

Languages associated with ω-terms

- Let n be a positive integer.
- For a word $\alpha \in X^+$, we let $E_n(\alpha) = \{\alpha\}$.
- For an ω-term $\alpha = \alpha_0(\beta_1)\alpha_1 \cdots (\beta_r)\alpha_r$ where all the β_j have the same rank i and all the α_j have rank at most i, we let

$$E_n(\alpha) = \{\alpha_0 \beta_1^{n_1} \alpha_1 \cdots \beta_r^{n_r} \alpha_r : n_1, \ldots, n_r \geq n\}.$$

- For a set W of ω-terms, we let

$$E_n(W) = \bigcup_{\alpha \in W} E_n(\alpha).$$

- For an ω-term α of rank k, we let

$$L_n(\alpha) = (E_n)^k(\alpha).$$

- For a set W of ω-terms, we let

$$L_n(W) = \bigcup_{\alpha \in W} L_n(\alpha).$$

Languages associated with ω-terms

- Let n be a positive integer.
- For a word $\alpha \in X^+$, we let $E_n(\alpha) = \{\alpha\}$.
- For an ω-term $\alpha = \alpha_0(\beta_1)\alpha_1 \cdots (\beta_r)\alpha_r$ where all the β_j have the same rank i and all the α_j have rank at most i, we let

$$E_n(\alpha) = \{\alpha_0 \beta_1^{n_1} \alpha_1 \cdots \beta_r^{n_r} \alpha_r : n_1, \dots, n_r \geq n\}.$$

- For a set W of ω-terms, we let

$$E_n(W) = \bigcup_{\alpha \in W} E_n(\alpha).$$

- For an ω-term α of rank k, we let

$$L_n(\alpha) = (E_n)^k(\alpha).$$

- For a set W of ω-terms, we let

$$L_n(W) = \bigcup_{\alpha \in W} L_n(\alpha).$$

Languages associated with ω-terms

- Let n be a positive integer.
- For a word $\alpha \in X^+$, we let $E_n(\alpha) = \{\alpha\}$.
- For an ω-term $\alpha = \alpha_0(\beta_1)\alpha_1 \cdots (\beta_r)\alpha_r$ where all the β_j have the same rank i and all the α_j have rank at most i, we let

$$E_n(\alpha) = \{\alpha_0\beta_1^{n_1}\alpha_1 \cdots \beta_r^{n_r}\alpha_r : n_1, \ldots, n_r \geq n\}.$$

- For a set W of ω-terms, we let

$$E_n(W) = \bigcup_{\alpha \in W} E_n(\alpha).$$

- For an ω-term α of rank k, we let

$$L_n(\alpha) = (E_n)^k(\alpha).$$

- For a set W of ω-terms, we let

$$L_n(W) = \bigcup_{\alpha \in W} L_n(\alpha).$$

An important parameter

- Let
$$\alpha = \alpha_0(\beta_1)\alpha_1\cdots(\beta_r)\alpha_r$$
be an ω-term of rank $i \geq 1$ where each β_j has rank $i-1$ and each α_j has rank at most $i-1$.

Let $\mu(\alpha)$ denote the integer
$$2(2+\max\{|\beta_j\alpha_j\beta_{j+1}|, |\beta_r\alpha_r|, |\alpha_0\beta_1| : j = 1, \ldots, r-1\}).$$

An important parameter

- Let

$$\alpha = \alpha_0(\beta_1)\alpha_1 \cdots (\beta_r)\alpha_r$$

be an ω-term of rank $i \geq 1$ where each β_j has rank $i - 1$ and each α_j has rank at most $i - 1$.

Let $\mu(\alpha)$ denote the integer

$$2(2 + \max\{|\beta_j \alpha_j \beta_{j+1}|, |\beta_r \alpha_r|, |\alpha_0 \beta_1| : j = 1, \ldots, r-1\}).$$

An important parameter

- Let

$$\alpha = \alpha_0(\beta_1)\alpha_1\cdots(\beta_r)\alpha_r$$

be an ω-term of rank $i \geq 1$ where each β_j has rank $i-1$ and each α_j has rank at most $i-1$.

Let $\mu(\alpha)$ denote the integer

$$2(2+\max\{|\beta_j\alpha_j\beta_{j+1}|, |\beta_r\alpha_r|, |\alpha_0\beta_1| : j = 1, \ldots, r-1\}).$$

- In case α is a word, we let

$$\mu(\alpha) = |\alpha|.$$

Key results

Theorem (Star freeness)

Whenever α is an ω-term in normal form and $n \geq \mu(\alpha)$, the language $L_n(\alpha)$ is star free.

Theorem (Separation)

Let α and β be two ω-terms in normal form and let $n > \max\{|\alpha|, |\beta|, \mu(\alpha), \mu(\beta)\}$. Then

$$L_n(\alpha) \cap L_n(\beta) \neq \varnothing \implies \alpha = \beta.$$

Key results

Theorem (Star freeness)

Whenever α is an ω-term in normal form and $n \geq \mu(\alpha)$, the language $L_n(\alpha)$ is star free.

Theorem (Separation)

Let α and β be two ω-terms in normal form and let $n > \max\{|\alpha|, |\beta|, \mu(\alpha), \mu(\beta)\}$. Then

$$L_n(\alpha) \cap L_n(\beta) \neq \varnothing \implies \alpha = \beta.$$

Application: McCammond's Theorem

Corollary (McCammond's Theorem reproved)

If α and β are ω-words in normal form such that $p_{\mathbf{A}}(\alpha) = p_{\mathbf{A}}(\beta)$, then $\alpha = \beta$.

Proof.

Let n be any integer greater than $\max\{|\alpha|, |\beta|, \mu(\alpha), \mu(\beta)\}$.

By the Separation Theorem, it suffices to show that $L_n(\alpha) \cap L_n(\beta) \neq \emptyset$.

Suppose to the contrary that $L_n(\alpha) \cap L_n(\beta) = \emptyset$.

Since $L_n(\alpha)$ and $L_n(\beta)$ are star-free languages by the Star-freeness Theorem, their closures in $\overline{\Omega}_X \mathbf{A}$, which are respectively $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha)))$ and $p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta)))$, are clopen subsets whose intersection with X^+ give, respectively, $L_n(\alpha)$ and $L_n(\beta)$.

Hence $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha))) \cap p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta))) = \emptyset$.

Since $\alpha \in \mathrm{cl}(L_n(\alpha))$ and $\beta \in \mathrm{cl}(L_n(\beta))$, it follows that $p_{\mathbf{A}}(\alpha) \neq p_{\mathbf{A}}(\beta)$, in contradiction with the hypothesis. $\square$

Application: McCammond's Theorem

Corollary (McCammond's Theorem reproved)

If α and β are ω-words in normal form such that $p_{\mathbf{A}}(\alpha) = p_{\mathbf{A}}(\beta)$, then $\alpha = \beta$.

Proof.

Let n be any integer greater than $\max\{|\alpha|, |\beta|, \mu(\alpha), \mu(\beta)\}$.

By the Separation Theorem, it suffices to show that $L_n(\alpha) \cap L_n(\beta) \neq \emptyset$.

Suppose to the contrary that $L_n(\alpha) \cap L_n(\beta) = \emptyset$.

Since $L_n(\alpha)$ and $L_n(\beta)$ are star-free languages by the Star-freeness Theorem, their closures in $\overline{\Omega}_X\mathbf{A}$, which are respectively $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha)))$ and $p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta)))$, are clopen subsets whose intersection with X^+ give, respectively, $L_n(\alpha)$ and $L_n(\beta)$.

Hence $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha))) \cap p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta))) = \emptyset$.

Since $\alpha \in \mathrm{cl}(L_n(\alpha))$ and $\beta \in \mathrm{cl}(L_n(\beta))$, it follows that $p_{\mathbf{A}}(\alpha) \neq p_{\mathbf{A}}(\beta)$, in contradiction with the hypothesis. ∎

Application: McCammond's Theorem

Corollary (McCammond's Theorem reproved)

If α and β are ω-words in normal form such that $p_{\mathbf{A}}(\alpha) = p_{\mathbf{A}}(\beta)$, then $\alpha = \beta$.

Proof.

Let n be any integer greater than $\max\{|\alpha|, |\beta|, \mu(\alpha), \mu(\beta)\}$.

By the Separation Theorem, it suffices to show that $L_n(\alpha) \cap L_n(\beta) \neq \varnothing$.

Suppose to the contrary that $L_n(\alpha) \cap L_n(\beta) = \varnothing$.

Since $L_n(\alpha)$ and $L_n(\beta)$ are star-free languages by the Star-freeness Theorem, their closures in $\overline{\Omega}_X \mathbf{A}$, which are respectively $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha)))$ and $p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta)))$, are clopen subsets whose intersection with X^+ give, respectively, $L_n(\alpha)$ and $L_n(\beta)$.

Hence $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha))) \cap p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta))) = \varnothing$.

Since $\alpha \in \mathrm{cl}(L_n(\alpha))$ and $\beta \in \mathrm{cl}(L_n(\beta))$, it follows that $p_{\mathbf{A}}(\alpha) \neq p_{\mathbf{A}}(\beta)$, in contradiction with the hypothesis. □

Application: McCammond's Theorem

Corollary (McCammond's Theorem reproved)

If α and β are ω-words in normal form such that $p_{\mathbf{A}}(\alpha) = p_{\mathbf{A}}(\beta)$, then $\alpha = \beta$.

Proof.

Let n be any integer greater than $\max\{|\alpha|, |\beta|, \mu(\alpha), \mu(\beta)\}$.

By the Separation Theorem, it suffices to show that $L_n(\alpha) \cap L_n(\beta) \neq \emptyset$.

Suppose to the contrary that $L_n(\alpha) \cap L_n(\beta) = \emptyset$.

Since $L_n(\alpha)$ and $L_n(\beta)$ are star-free languages by the Star-freeness Theorem, their closures in $\overline{\Omega}_X \mathbf{A}$, which are respectively $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha)))$ and $p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta)))$, are clopen subsets whose intersection with X^+ give, respectively, $L_n(\alpha)$ and $L_n(\beta)$.

Hence $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha))) \cap p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta))) = \emptyset$.

Since $\alpha \in \mathrm{cl}(L_n(\alpha))$ and $\beta \in \mathrm{cl}(L_n(\beta))$, it follows that $p_{\mathbf{A}}(\alpha) \neq p_{\mathbf{A}}(\beta)$, in contradiction with the hypothesis. □

Application: McCammond's Theorem

Corollary (McCammond's Theorem reproved)

If α and β are ω-words in normal form such that $p_{\mathbf{A}}(\alpha) = p_{\mathbf{A}}(\beta)$, then $\alpha = \beta$.

Proof.

Let n be any integer greater than $\max\{|\alpha|, |\beta|, \mu(\alpha), \mu(\beta)\}$.

By the Separation Theorem, it suffices to show that $L_n(\alpha) \cap L_n(\beta) \neq \varnothing$.

Suppose to the contrary that $L_n(\alpha) \cap L_n(\beta) = \varnothing$.

Since $L_n(\alpha)$ and $L_n(\beta)$ are star-free languages by the Star-freeness Theorem, their closures in $\overline{\Omega}_X \mathbf{A}$, which are respectively $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha)))$ and $p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta)))$, are clopen subsets whose intersection with X^+ give, respectively, $L_n(\alpha)$ and $L_n(\beta)$.

Hence $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha))) \cap p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta))) = \varnothing$.

Since $\alpha \in \mathrm{cl}(L_n(\alpha))$ and $\beta \in \mathrm{cl}(L_n(\beta))$, it follows that $p_{\mathbf{A}}(\alpha) \neq p_{\mathbf{A}}(\beta)$, in contradiction with the hypothesis. □

Application: McCammond's Theorem

Corollary (McCammond's Theorem reproved)

If α and β are ω-words in normal form such that $p_{\mathbf{A}}(\alpha) = p_{\mathbf{A}}(\beta)$, then $\alpha = \beta$.

Proof.

Let n be any integer greater than $\max\{|\alpha|, |\beta|, \mu(\alpha), \mu(\beta)\}$.

By the Separation Theorem, it suffices to show that $L_n(\alpha) \cap L_n(\beta) \neq \varnothing$.

Suppose to the contrary that $L_n(\alpha) \cap L_n(\beta) = \varnothing$.

Since $L_n(\alpha)$ and $L_n(\beta)$ are star-free languages by the Star-freeness Theorem, their closures in $\overline{\Omega}_X \mathbf{A}$, which are respectively $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha)))$ and $p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta)))$, are clopen subsets whose intersection with X^+ give, respectively, $L_n(\alpha)$ and $L_n(\beta)$.

Hence $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha))) \cap p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta))) = \varnothing$.

Since $\alpha \in \mathrm{cl}(L_n(\alpha))$ and $\beta \in \mathrm{cl}(L_n(\beta))$, it follows that $p_{\mathbf{A}}(\alpha) \neq p_{\mathbf{A}}(\beta)$, in contradiction with the hypothesis. $\square$

Application: McCammond's Theorem

Corollary (McCammond's Theorem reproved)

If α and β are ω-words in normal form such that $p_{\mathbf{A}}(\alpha) = p_{\mathbf{A}}(\beta)$, then $\alpha = \beta$.

Proof.

Let n be any integer greater than $\max\{|\alpha|, |\beta|, \mu(\alpha), \mu(\beta)\}$.

By the Separation Theorem, it suffices to show that $L_n(\alpha) \cap L_n(\beta) \neq \varnothing$.

Suppose to the contrary that $L_n(\alpha) \cap L_n(\beta) = \varnothing$.

Since $L_n(\alpha)$ and $L_n(\beta)$ are star-free languages by the Star-freeness Theorem, their closures in $\overline{\Omega}_X \mathbf{A}$, which are respectively $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha)))$ and $p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta)))$, are clopen subsets whose intersection with X^+ give, respectively, $L_n(\alpha)$ and $L_n(\beta)$.

Hence $p_{\mathbf{A}}(\mathrm{cl}(L_n(\alpha))) \cap p_{\mathbf{A}}(\mathrm{cl}(L_n(\beta))) = \varnothing$.

Since $\alpha \in \mathrm{cl}(L_n(\alpha))$ and $\beta \in \mathrm{cl}(L_n(\beta))$, it follows that $p_{\mathbf{A}}(\alpha) \neq p_{\mathbf{A}}(\beta)$, in contradiction with the hypothesis. □

Example of a language $L_n(\alpha)$ not star-free

We do not know whether the bound $n \geq \mu(\alpha)$ is optimal but we do know that some bound is required, that is that $L_n(\alpha)$ *may not be star-free* for α in normal form.

Example

Let

$$\alpha = (a^\omega abb^\omega a^2 b^2)^\omega.$$

Then α is in normal form and

$$L_1(\alpha) \cap (a^2b^2)^* = ((a^2b^2)^2)^+$$

so that $L_1(\alpha)$ is not star-free since

- *$(a^2b^2)^*$ is star-free and*
- *$((a^2b^2)^2)^+$ is not.*

Example of a language $L_n(\alpha)$ not star-free

We do not know whether the bound $n \geq \mu(\alpha)$ is optimal but we do know that some bound is required, that is that $L_n(\alpha)$ *may not be star-free* for α in normal form.

Example

Let

$$\alpha = (a^{\omega} abb^{\omega} a^2 b^2)^{\omega}.$$

Then α is in normal form and

$$L_1(\alpha) \cap (a^2b^2)^* = ((a^2b^2)^2)^+$$

so that $L_1(\alpha)$ is not star-free since

- $(a^2b^2)^*$ *is star-free and*
- $((a^2b^2)^2)^+$ *is not.*

Example of a language $L_n(\alpha)$ not star-free

We do not know whether the bound $n \geq \mu(\alpha)$ is optimal but we do know that some bound is required, that is that $L_n(\alpha)$ *may not be star-free* for α in normal form.

Example

Let

$$\alpha = (a^\omega abb^\omega a^2b^2)^\omega .$$

Then α is in normal form and

$$L_1(\alpha) \cap (a^2b^2)^* = \left((a^2b^2)^2\right)^+$$

so that $L_1(\alpha)$ is not star-free since

- $(a^2b^2)^*$ *is star-free and*
- $\left((a^2b^2)^2\right)^+$ *is not.*

Example of a language $L_n(\alpha)$ not star-free

We do not know whether the bound $n \geq \mu(\alpha)$ is optimal but we do know that some bound is required, that is that $L_n(\alpha)$ *may not be star-free* for α in normal form.

Example

Let

$$\alpha = (a^\omega abb^\omega a^2b^2)^\omega.$$

Then α is in normal form and

$$L_1(\alpha) \cap (a^2b^2)^* = \left((a^2b^2)^2\right)^+$$

so that $L_1(\alpha)$ is not star-free since

- $(a^2b^2)^*$ *is star-free and*
- $\left((a^2b^2)^2\right)^+$ *is not.*

Example of a language $L_n(\alpha)$ not star-free

We do not know whether the bound $n \geq \mu(\alpha)$ is optimal but we do know that some bound is required, that is that $L_n(\alpha)$ *may not be star-free* for α in normal form.

Example

Let

$$\alpha = (a^\omega abb^\omega a^2b^2)^\omega.$$

Then α is in normal form and

$$L_1(\alpha) \cap (a^2b^2)^* = \left((a^2b^2)^2\right)^+$$

so that $L_1(\alpha)$ is not star-free since

- *$(a^2b^2)^*$ is star-free and*
- *$\left((a^2b^2)^2\right)^+$ is not.*

Circular normal form

We say that an ω-term α is in *circular normal form* if either:

- α is a word;
- α is a term of rank $i \geq 1$ of the form $\alpha = (\delta_1)\gamma_1 \cdots (\delta_r)\gamma_r$ such that each term $(\delta_k)\gamma_k(\delta_{k+1})$ with $k \in \{1, \ldots, r\}$ is in normal form, where we consider $\gamma_{r+1} = \gamma_1$.

Lemma

Let α be a primitive ω-term of rank $i \geq 0$ in circular normal form and let $n \geq \mu(\alpha)$. If $L_n(\alpha)$ is a star-free language, then so is $L_n(\alpha)^*$.

Circular normal form

We say that an ω-term α is in *circular normal form* if either:

- α is a word;
- α is a term of rank $i \geq 1$ of the form $\alpha = (\delta_1)\gamma_1 \cdots (\delta_r)\gamma_r$ such that each term $(\delta_k)\gamma_k(\delta_{k+1})$ with $k \in \{1, \ldots, r\}$ is in normal form, where we consider $\gamma_{r+1} = \gamma_1$.

Lemma

Let α be a primitive ω-term of rank $i \geq 0$ in circular normal form and let $n \geq \mu(\alpha)$. If $L_n(\alpha)$ is a star-free language, then so is $L_n(\alpha)^*$.

Circular normal form

We say that an ω-term α is in *circular normal form* if either:

- α is a word;
- α is a term of rank $i \geq 1$ of the form $\alpha = (\delta_1)\gamma_1 \cdots (\delta_r)\gamma_r$ such that each term $(\delta_k)\gamma_k(\delta_{k+1})$ with $k \in \{1, \ldots, r\}$ is in normal form, where we consider $\gamma_{r+1} = \gamma_1$.

Lemma

Let α be a primitive ω-term of rank $i \geq 0$ in circular normal form and let $n \geq \mu(\alpha)$. If $L_n(\alpha)$ is a star-free language, then so is $L_n(\alpha)^*$.

Circular normal form

We say that an ω-term α is in *circular normal form* if either:

- α is a word;
- α is a term of rank $i \geq 1$ of the form $\alpha = (\delta_1)\gamma_1 \cdots (\delta_r)\gamma_r$ such that each term $(\delta_k)\gamma_k(\delta_{k+1})$ with $k \in \{1, \ldots, r\}$ is in normal form, where we consider $\gamma_{r+1} = \gamma_1$.

Lemma

Let α be a primitive ω-term of rank $i \geq 0$ in circular normal form and let $n \geq \mu(\alpha)$. If $L_n(\alpha)$ is a star-free language, then so is $L_n(\alpha)^*$.

Proof of star-freeness of $L_n(\alpha)$

Theorem (Star freeness)

Whenever α is an ω-term in normal form and $n \geq \mu(\alpha)$, the language $L_n(\alpha)$ is star free.

Proof.

Let $i = \operatorname{rank} \alpha$. If $i = 0$, then $L_n(\alpha) = \{\alpha\}$ is a star-free language.
Assume that $i \geq 1$, and let $\alpha = \gamma_0(\delta_1)\gamma_1 \cdots (\delta_r)\gamma_r$.
Claim: $L_n(\gamma_0)$, $L_n(\delta_j)$ and $L_n(\delta_j\gamma_j)$ are star-free languages.
As a consequence, each language

$$L_n((\delta_j)\gamma_j) = L_n(\delta_j)^* L_n(\delta_j)^{n-1} L_n(\delta_j\gamma_j)$$

is also star-free. Hence

$$L_n(\alpha) = L_n(\gamma_0) L_n((\delta_1)\gamma_1) \cdots L_n((\delta_r)\gamma_r)$$

is star-free, as stated in the theorem. □

Proof of star-freeness of $L_n(\alpha)$

Theorem (Star freeness)

Whenever α is an ω-term in normal form and $n \geq \mu(\alpha)$, the language $L_n(\alpha)$ is star free.

Proof.

Let $i = \text{rank}\ \alpha$. If $i = 0$, then $L_n(\alpha) = \{\alpha\}$ is a star-free language.
Assume that $i \geq 1$, and let $\alpha = \gamma_0(\delta_1)\gamma_1 \cdots (\delta_r)\gamma_r$.
Claim: $L_n(\gamma_0)$, $L_n(\delta_j)$ and $L_n(\delta_j\gamma_j)$ are star-free languages.
As a consequence, each language

$$L_n((\delta_j)\gamma_j) = L_n(\delta_j)^* L_n(\delta_j)^{n-1} L_n(\delta_j\gamma_j)$$

is also star-free. Hence

$$L_n(\alpha) = L_n(\gamma_0) L_n((\delta_1)\gamma_1) \cdots L_n((\delta_r)\gamma_r)$$

is star-free, as stated in the theorem. □

Proof of star-freeness of $L_n(\alpha)$

Theorem (Star freeness)

Whenever α is an ω-term in normal form and $n \geq \mu(\alpha)$, the language $L_n(\alpha)$ is star free.

Proof.

Let $i = \text{rank}\ \alpha$. If $i = 0$, then $L_n(\alpha) = \{\alpha\}$ is a star-free language.
Assume that $i \geq 1$, and let $\alpha = \gamma_0(\delta_1)\gamma_1 \cdots (\delta_r)\gamma_r$.
Claim: $L_n(\gamma_0)$, $L_n(\delta_j)$ and $L_n(\delta_j\gamma_j)$ are star-free languages.
As a consequence, each language

$$L_n((\delta_j)\gamma_j) = L_n(\delta_j)^* L_n(\delta_j)^{n-1} L_n(\delta_j\gamma_j)$$

is also star-free. Hence

$$L_n(\alpha) = L_n(\gamma_0) L_n((\delta_1)\gamma_1) \cdots L_n((\delta_r)\gamma_r)$$

is star-free, as stated in the theorem. □

Proof of star-freeness of $L_n(\alpha)$

Theorem (Star freeness)

Whenever α is an ω-term in normal form and $n \geq \mu(\alpha)$, the language $L_n(\alpha)$ is star free.

Proof.

Let $i = \operatorname{rank} \alpha$. If $i = 0$, then $L_n(\alpha) = \{\alpha\}$ is a star-free language.
Assume that $i \geq 1$, and let $\alpha = \gamma_0(\delta_1)\gamma_1 \cdots (\delta_r)\gamma_r$.
Claim: $L_n(\gamma_0)$, $L_n(\delta_j)$ and $L_n(\delta_j\gamma_j)$ are star-free languages.
As a consequence, each language

$$L_n((\delta_j)\gamma_j) = L_n(\delta_j)^* L_n(\delta_j)^{n-1} L_n(\delta_j\gamma_j)$$

is also star-free. Hence

$$L_n(\alpha) = L_n(\gamma_0) L_n((\delta_1)\gamma_1) \cdots L_n((\delta_r)\gamma_r)$$

is star-free, as stated in the theorem. ∎

PROOF OF STAR-FREENESS OF $L_n(\alpha)$

THEOREM (STAR FREENESS)

Whenever α is an ω-term in normal form and $n \geq \mu(\alpha)$, the language $L_n(\alpha)$ is star free.

PROOF.

Let $i = \operatorname{rank} \alpha$. If $i = 0$, then $L_n(\alpha) = \{\alpha\}$ is a star-free language. Assume that $i \geq 1$, and let $\alpha = \gamma_0(\delta_1)\gamma_1 \cdots (\delta_r)\gamma_r$.
Claim: $L_n(\gamma_0)$, $L_n(\delta_j)$ and $L_n(\delta_j\gamma_j)$ are star-free languages.
As a consequence, each language

$$L_n\big((\delta_j)\gamma_j\big) = L_n(\delta_j)^* L_n(\delta_j)^{n-1} L_n(\delta_j\gamma_j)$$

is also star-free. Hence

$$L_n(\alpha) = L_n(\gamma_0) L_n\big((\delta_1)\gamma_1\big) \cdots L_n\big((\delta_r)\gamma_r\big)$$

is star-free, as stated in the theorem. □

Claim: $L_n(\gamma_0)$, $L_n(\delta_j)$ and $L_n(\delta_j\gamma_j)$ are star-free languages, where $\alpha = \gamma_0(\delta_1)\gamma_1\cdots(\delta_r)\gamma_r$.

Proof of the Claim.

By induction on $i \geq 1$, the case $i = 1$ being clear since $\gamma_j, \delta_j \in X^*$. Suppose that $i \geq 2$ and assume inductively that the claim holds for terms of rank smaller than i. Consider the term

$$\alpha' = \gamma_0\delta_1\delta_1\gamma_1\cdots\delta_r\delta_r\gamma_r \in E_2(\alpha).$$

Then α' is a rank $i-1 \geq 1$ term in normal form and we may apply to it the induction hypothesis. Hence, if $\alpha' = u_0(v_1)u_1\cdots(v_s)u_s$ is the normal form expression of α', then $L_n(u_0)$, $L_n(v_k)$, and $L_n(v_k u_k)$ are star-free, whence so are $L_n(u_0)$, $L_n((v_k))$, *and* $L_n((v_k)u_k)$. Since each factor $\gamma_0, \delta_j, \delta_j\gamma_j$ must be a product of some of the factors $u_0, (v_k), (v_k)u_k$ it follows that the languages $L_n(\gamma_0), L_n(\delta_j), L_n(\delta_j\gamma_j)$ are star-free, thus proving the induction step. □

Claim: $L_n(\gamma_0)$, $L_n(\delta_j)$ and $L_n(\delta_j\gamma_j)$ are star-free languages, where $\alpha = \gamma_0(\delta_1)\gamma_1 \cdots (\delta_r)\gamma_r$.

Proof of the Claim.

By induction on $i \geq 1$, the case $i = 1$ being clear since $\gamma_j, \delta_j \in X^*$. Suppose that $i \geq 2$ and assume inductively that the claim holds for terms of rank smaller than i. Consider the term

$$\alpha' = \gamma_0\delta_1\delta_1\gamma_1 \cdots \delta_r\delta_r\gamma_r \in E_2(\alpha).$$

Then α' is a rank $i - 1 \geq 1$ term in normal form and we may apply to it the induction hypothesis. Hence, if $\alpha' = u_0(v_1)u_1 \cdots (v_s)u_s$ is the normal form expression of α', then $L_n(u_0)$, $L_n(v_k)$, and $L_n(v_k u_k)$ are star-free, whence so are $L_n(u_0)$, $L_n((v_k))$, *and* $L_n((v_k)u_k)$. Since each factor $\gamma_0, \delta_j, \delta_j\gamma_j$ must be a product of some of the factors $u_0, (v_k), (v_k)u_k$ it follows that the languages $L_n(\gamma_0), L_n(\delta_j), L_n(\delta_j\gamma_j)$ are star-free, thus proving the induction step. □

Claim: $L_n(\gamma_0)$, $L_n(\delta_j)$ and $L_n(\delta_j\gamma_j)$ are star-free languages, where $\alpha = \gamma_0(\delta_1)\gamma_1\cdots(\delta_r)\gamma_r$.

Proof of the Claim.

By induction on $i \geq 1$, the case $i = 1$ being clear since $\gamma_j, \delta_j \in X^*$. Suppose that $i \geq 2$ and assume inductively that the claim holds for terms of rank smaller than i. Consider the term

$$\alpha' = \gamma_0\delta_1\delta_1\gamma_1\cdots\delta_r\delta_r\gamma_r \in E_2(\alpha).$$

Then α' is a rank $i - 1 \geq 1$ term in normal form and we may apply to it the induction hypothesis. Hence, if $\alpha' = u_0(v_1)u_1\cdots(v_s)u_s$ is the normal form expression of α', then $L_n(u_0)$, $L_n(v_k)$, and $L_n(v_k u_k)$ are star-free, whence so are $L_n(u_0)$, $L_n((v_k))$, *and* $L_n((v_k)u_k)$. Since each factor $\gamma_0, \delta_j, \delta_j\gamma_j$ must be a product of some of the factors $u_0, (v_k), (v_k)u_k$ it follows that the languages $L_n(\gamma_0), L_n(\delta_j), L_n(\delta_j\gamma_j)$ are star-free, thus proving the induction step. □

Claim: $L_n(\gamma_0)$, $L_n(\delta_j)$ and $L_n(\delta_j\gamma_j)$ are star-free languages, where $\alpha = \gamma_0(\delta_1)\gamma_1\cdots(\delta_r)\gamma_r$.

Proof of the Claim.

By induction on $i \geq 1$, the case $i = 1$ being clear since $\gamma_j, \delta_j \in X^*$. Suppose that $i \geq 2$ and assume inductively that the claim holds for terms of rank smaller than i. Consider the term

$$\alpha' = \gamma_0\delta_1\delta_1\gamma_1\cdots\delta_r\delta_r\gamma_r \in E_2(\alpha).$$

Then α' is a rank $i - 1 \geq 1$ term in normal form and we may apply to it the induction hypothesis. Hence, if $\alpha' = u_0(v_1)u_1\cdots(v_s)u_s$ is the normal form expression of α', then $L_n(u_0)$, $L_n(v_k)$, and $L_n(v_ku_k)$ are star-free, whence so are $L_n(u_0)$, $L_n((v_k))$, *and* $L_n((v_k)u_k)$. Since each factor $\gamma_0, \delta_j, \delta_j\gamma_j$ must be a product of some of the factors $u_0, (v_k), (v_k)u_k$ it follows that the languages $L_n(\gamma_0), L_n(\delta_j), L_n(\delta_j\gamma_j)$ are star-free, thus proving the induction step. □

Claim: $L_n(\gamma_0)$, $L_n(\delta_j)$ and $L_n(\delta_j\gamma_j)$ are star-free languages, where $\alpha = \gamma_0(\delta_1)\gamma_1\cdots(\delta_r)\gamma_r$.

Proof of the Claim.

By induction on $i \geq 1$, the case $i = 1$ being clear since $\gamma_j, \delta_j \in X^*$. Suppose that $i \geq 2$ and assume inductively that the claim holds for terms of rank smaller than i. Consider the term

$$\alpha' = \gamma_0\delta_1\delta_1\gamma_1\cdots\delta_r\delta_r\gamma_r \in E_2(\alpha).$$

Then α' is a rank $i - 1 \geq 1$ term in normal form and we may apply to it the induction hypothesis. Hence, if $\alpha' = u_0(v_1)u_1\cdots(v_s)u_s$ is the normal form expression of α', then $L_n(u_0)$, $L_n(v_k)$, and $L_n(v_ku_k)$ are star-free, whence so are $L_n(u_0)$, $L_n((v_k))$, *and* $L_n((v_k)u_k)$. Since each factor $\gamma_0, \delta_j, \delta_j\gamma_j$ must be a product of some of the factors $u_0, (v_k), (v_k)u_k$ it follows that the languages $L_n(\gamma_0), L_n(\delta_j), L_n(\delta_j\gamma_j)$ are star-free, thus proving the induction step. □

Claim: $L_n(\gamma_0)$, $L_n(\delta_j)$ and $L_n(\delta_j\gamma_j)$ are star-free languages, where $\alpha = \gamma_0(\delta_1)\gamma_1\cdots(\delta_r)\gamma_r$.

Proof of the Claim.

By induction on $i \geq 1$, the case $i = 1$ being clear since $\gamma_j, \delta_j \in X^*$. Suppose that $i \geq 2$ and assume inductively that the claim holds for terms of rank smaller than i. Consider the term

$$\alpha' = \gamma_0\delta_1\delta_1\gamma_1\cdots\delta_r\delta_r\gamma_r \in E_2(\alpha).$$

Then α' is a rank $i-1 \geq 1$ term in normal form and we may apply to it the induction hypothesis. Hence, if $\alpha' = u_0(v_1)u_1\cdots(v_s)u_s$ is the normal form expression of α', then $L_n(u_0)$, $L_n(v_k)$, and $L_n(v_ku_k)$ are star-free, whence so are $L_n(u_0)$, $L_n((v_k))$, and $L_n((v_k)u_k)$. Since each factor $\gamma_0, \delta_j, \delta_j\gamma_j$ must be a product of some of the factors $u_0, (v_k), (v_k)u_k$ it follows that the languages $L_n(\gamma_0), L_n(\delta_j), L_n(\delta_j\gamma_j)$ are star-free, thus proving the induction step. □

When is $L_n(\alpha)^*$ star-free?

Notice that, for an ω-term α and a fixed positive integer n:

$L_n(\alpha)^*$ is star-free $\Leftrightarrow$

$\Leftrightarrow$ the syntactic semigroup $S(L_n(\alpha)^*)$ is aperiodic

$\Leftrightarrow$ $S(L_n(\alpha)^*)$ verifies the pseudoidentity $x^\omega = x^{\omega+1}$

$\Leftrightarrow$ $S(L_n(\alpha)^*)$ ultimately verifies the identity $x^N = x^{N+1}$

$\Leftrightarrow$ $\forall N > K(\alpha, n) \forall u, v, z \in X^* \left(uz^N v \in L_n(\alpha)^* \Leftrightarrow uz^{N+1} v \in L_n(\alpha)^* \right)$

$\Leftrightarrow$ $\forall N > K(\alpha, n) \forall u, v, z \in X^* \left(uz^N v \in L_n(\alpha)^* \Rightarrow uz^{N+1} v \in L_n(\alpha)^* \right)$

We could further assume z to be a Lyndon word.

Lemma

Let z_1 and z_2 be Lyndon words and suppose that w is a word such that $|w| \geq |z_1| + |z_2|$ and w is a factor of both a power of z_1 and a power of z_2. Then $z_1 = z_2$.

When is $L_n(\alpha)^*$ star-free?

Notice that, for an ω-term α and a fixed positive integer n:

$L_n(\alpha)^*$ is star-free $\Leftrightarrow$

$\Leftrightarrow$ the syntactic semigroup $S\big(L_n(\alpha)^*\big)$ is aperiodic

$\Leftrightarrow$ $S\big(L_n(\alpha)^*\big)$ verifies the pseudoidentity $x^\omega = x^{\omega+1}$

$\Leftrightarrow$ $S\big(L_n(\alpha)^*\big)$ ultimately verifies the identity $x^N = x^{N+1}$

$\Leftrightarrow$ $\forall N > K(\alpha, n) \forall u, v, z \in X^* \left(uz^N v \in L_n(\alpha)^* \Leftrightarrow uz^{N+1} v \in L_n(\alpha)^* \right)$

$\Leftrightarrow$ $\forall N > K(\alpha, n) \forall u, v, z \in X^* \left(uz^N v \in L_n(\alpha)^* \Rightarrow uz^{N+1} v \in L_n(\alpha)^* \right)$

We could further assume z to be a Lyndon word.

Lemma

Let z_1 and z_2 be Lyndon words and suppose that w is a word such that $|w| \geq |z_1| + |z_2|$ and w is a factor of both a power of z_1 and a power of z_2. Then $z_1 = z_2$.

When is $L_n(\alpha)^*$ star-free?

Notice that, for an ω-term α and a fixed positive integer n:

$L_n(\alpha)^*$ is star-free $\Leftrightarrow$

$\Leftrightarrow$ the syntactic semigroup $S\big(L_n(\alpha)^*\big)$ is aperiodic

$\Leftrightarrow$ $S\big(L_n(\alpha)^*\big)$ verifies the pseudoidentity $x^{\omega} = x^{\omega+1}$

$\Leftrightarrow$ $S\big(L_n(\alpha)^*\big)$ ultimately verifies the identity $x^N = x^{N+1}$

$\Leftrightarrow$ $\forall N > K(\alpha, n) \forall u, v, z \in X^* \left(uz^N v \in L_n(\alpha)^* \Leftrightarrow uz^{N+1} v \in L_n(\alpha)^* \right)$

$\Leftrightarrow$ $\forall N > K(\alpha, n) \forall u, v, z \in X^* \left(uz^N v \in L_n(\alpha)^* \Rightarrow uz^{N+1} v \in L_n(\alpha)^* \right)$

We could further assume z to be a Lyndon word.

Lemma

Let z_1 and z_2 be Lyndon words and suppose that w is a word such that $|w| \geq |z_1| + |z_2|$ and w is a factor of both a power of z_1 and a power of z_2. Then $z_1 = z_2$.

When is $L_n(\alpha)^*$ star-free?

Notice that, for an ω-term α and a fixed positive integer n:

$L_n(\alpha)^*$ is star-free $\Leftrightarrow$

$\Leftrightarrow$ the syntactic semigroup $S\big(L_n(\alpha)^*\big)$ is aperiodic

$\Leftrightarrow$ $S\big(L_n(\alpha)^*\big)$ verifies the pseudoidentity $x^{\omega} = x^{\omega+1}$

$\Leftrightarrow$ $S\big(L_n(\alpha)^*\big)$ ultimately verifies the identity $x^N = x^{N+1}$

$\Leftrightarrow$ $\forall N > K(\alpha, n) \forall u, v, z \in X^* \big(uz^N v \in L_n(\alpha)^* \Leftrightarrow uz^{N+1} v \in L_n(\alpha)^*\big)$

$\Leftrightarrow$ $\forall N > K(\alpha, n) \forall u, v, z \in X^* \big(uz^N v \in L_n(\alpha)^* \Rightarrow uz^{N+1} v \in L_n(\alpha)^*\big)$

We could further assume z to be a Lyndon word.

Lemma

Let z_1 and z_2 be Lyndon words and suppose that w is a word such that $|w| \geq |z_1| + |z_2|$ and w is a factor of both a power of z_1 and a power of z_2. Then $z_1 = z_2$.

When is $L_n(\alpha)^*$ star-free?

Notice that, for an ω-term α and a fixed positive integer n:

$L_n(\alpha)^*$ is star-free $\Leftrightarrow$

$\Leftrightarrow$ the syntactic semigroup $S(L_n(\alpha)^*)$ is aperiodic

$\Leftrightarrow$ $S(L_n(\alpha)^*)$ verifies the pseudoidentity $x^\omega = x^{\omega+1}$

$\Leftrightarrow$ $S(L_n(\alpha)^*)$ ultimately verifies the identity $x^N = x^{N+1}$

$$\Leftrightarrow \quad \forall N > K(\alpha,n) \forall u,v,z \in X^* \left(uz^N v \in L_n(\alpha)^* \Leftrightarrow uz^{N+1} v \in L_n(\alpha)^* \right)$$

$$\Leftrightarrow \quad \forall N > K(\alpha,n) \forall u,v,z \in X^* \left(uz^N v \in L_n(\alpha)^* \Rightarrow uz^{N+1} v \in L_n(\alpha)^* \right)$$

We could further assume z to be a Lyndon word.

Lemma

Let z_1 and z_2 be Lyndon words and suppose that w is a word such that $|w| \geq |z_1| + |z_2|$ and w is a factor of both a power of z_1 and a power of z_2. Then $z_1 = z_2$.

When is $L_n(\alpha)^*$ star-free?

Notice that, for an ω-term α and a fixed positive integer n:

$L_n(\alpha)^*$ is star-free $\Leftrightarrow$

$\Leftrightarrow$ the syntactic semigroup $S\big(L_n(\alpha)^*\big)$ is aperiodic

$\Leftrightarrow$ $S\big(L_n(\alpha)^*\big)$ verifies the pseudoidentity $x^\omega = x^{\omega+1}$

$\Leftrightarrow$ $S\big(L_n(\alpha)^*\big)$ ultimately verifies the identity $x^N = x^{N+1}$

$$\Leftrightarrow \quad \forall N > K(\alpha,n)\,\forall u,v,z \in X^* \left(uz^N v \in L_n(\alpha)^* \Leftrightarrow uz^{N+1}v \in L_n(\alpha)^*\right)$$

$$\Leftrightarrow \quad \forall N > K(\alpha,n)\,\forall u,v,z \in X^* \left(uz^N v \in L_n(\alpha)^* \Rightarrow uz^{N+1}v \in L_n(\alpha)^*\right)$$

We could further assume z to be a Lyndon word.

Lemma

Let z_1 and z_2 be Lyndon words and suppose that w is a word such that $|w| \geq |z_1| + |z_2|$ and w is a factor of both a power of z_1 and a power of z_2. Then $z_1 = z_2$.

When is $L_n(\alpha)^*$ star-free?

Notice that, for an ω-term α and a fixed positive integer n:

$$
\begin{aligned}
& L_n(\alpha)^* \text{ is star-free} \Leftrightarrow \\
\Leftrightarrow\quad & \text{the syntactic semigroup } S\left(L_n(\alpha)^*\right) \text{ is aperiodic} \\
\Leftrightarrow\quad & S\left(L_n(\alpha)^*\right) \text{ verifies the pseudoidentity } x^\omega = x^{\omega+1} \\
\Leftrightarrow\quad & S\left(L_n(\alpha)^*\right) \text{ ultimately verifies the identity } x^N = x^{N+1} \\
\Leftrightarrow\quad & \forall N > K(\alpha, n) \forall u, v, z \in X^* \left(uz^N v \in L_n(\alpha)^* \Leftrightarrow uz^{N+1} v \in L_n(\alpha)^*\right) \\
\Leftrightarrow\quad & \forall N > K(\alpha, n) \forall u, v, z \in X^* \left(uz^N v \in L_n(\alpha)^* \Rightarrow uz^{N+1} v \in L_n(\alpha)^*\right)
\end{aligned}
$$

We could further assume z to be a Lyndon word.

Lemma

Let z_1 and z_2 be Lyndon words and suppose that w is a word such that $|w| \geq |z_1| + |z_2|$ and w is a factor of both a power of z_1 and a power of z_2. Then $z_1 = z_2$.

When is $L_n(\alpha)^*$ star-free?

Notice that, for an ω-term α and a fixed positive integer n:

$$
\begin{aligned}
& L_n(\alpha)^* \text{ is star-free} \Leftrightarrow \\
\Leftrightarrow\ & \text{the syntactic semigroup } S\left(L_n(\alpha)^*\right) \text{ is aperiodic} \\
\Leftrightarrow\ & S\left(L_n(\alpha)^*\right) \text{ verifies the pseudoidentity } x^\omega = x^{\omega+1} \\
\Leftrightarrow\ & S\left(L_n(\alpha)^*\right) \text{ ultimately verifies the identity } x^N = x^{N+1} \\
\Leftrightarrow\ & \forall N > K(\alpha, n) \forall u, v, z \in X^* \left(uz^N v \in L_n(\alpha)^* \Leftrightarrow uz^{N+1} v \in L_n(\alpha)^*\right) \\
\Leftrightarrow\ & \forall N > K(\alpha, n) \forall u, v, z \in X^* \left(uz^N v \in L_n(\alpha)^* \Rightarrow uz^{N+1} v \in L_n(\alpha)^*\right)
\end{aligned}
$$

We could further assume z to be a Lyndon word.

Lemma

Let z_1 and z_2 be Lyndon words and suppose that w is a word such that $|w| \geq |z_1| + |z_2|$ and w is a factor of both a power of z_1 and a power of z_2. Then $z_1 = z_2$.

Lemma

Let α be a term of rank 1 in circular normal form and let $n \geq \mu(\alpha)$. If $z^\ell \in L_n(\alpha)$ then there exists a term of rank 1 in circular normal form $\bar{\alpha}$ such that $\alpha = \bar{\alpha}^\ell$ and $z \in L_n(\bar{\alpha})$.

Proof

Let $\alpha = (v_1)u_1 \cdots (v_r)u_r$ and $w = z^\ell$. Since $z^\ell \in L_n(\alpha)$, it follows that $w = v_1^{n_1}u_1 \cdots v_r^{n_r}u_r$, with $n_1, \ldots, n_r \geq n$.

If $\ell = 1$ then we can just take $\bar{\alpha} = \alpha$. If $\ell > 1$,

***Case 1:** $z < v_1$. If $v_1 = z^k$, then we set $t = z$. Otherwise, let t be as in*

Then t is both a proper suffix and a proper prefix of v_1, which contradicts the hypothesis that v_1 is a Lyndon word.

(continues)

Lemma

Let α be a term of rank 1 in circular normal form and let $n \geq \mu(\alpha)$. If $z^\ell \in L_n(\alpha)$ then there exists a term of rank 1 in circular normal form $\bar{\alpha}$ such that $\alpha = \bar{\alpha}^\ell$ and $z \in L_n(\bar{\alpha})$.

Proof

Let $\alpha = (v_1)u_1 \cdots (v_r)u_r$ and $w = z^\ell$. Since $z^\ell \in L_n(\alpha)$, it follows that $w = v_1^{n_1}u_1 \cdots v_r^{n_r}u_r$, with $n_1, \ldots, n_r \geq n$.

If $\ell = 1$ then we can just take $\bar{\alpha} = \alpha$. If $\ell > 1$,

Case 1: $z < v_1$. If $v_1 = z^k$, then we set $t = z$. Otherwise, let t be as in

Then t is both a proper suffix and a proper prefix of v_1, which contradicts the hypothesis that v_1 is a Lyndon word.

(continues)

Lemma

Let α be a term of rank 1 in circular normal form and let $n \geq \mu(\alpha)$. If $z^\ell \in L_n(\alpha)$ then there exists a term of rank 1 in circular normal form $\bar{\alpha}$ such that $\alpha = \bar{\alpha}^\ell$ and $z \in L_n(\bar{\alpha})$.

Proof

Let $\alpha = (v_1)u_1 \cdots (v_r)u_r$ and $w = z^\ell$. Since $z^\ell \in L_n(\alpha)$, it follows that $w = v_1^{n_1}u_1 \cdots v_r^{n_r}u_r$, with $n_1, \ldots, n_r \geq n$.
If $\ell = 1$ then we can just take $\bar{\alpha} = \alpha$. If $\ell > 1$,

Case 1: $z < v_1$. If $v_1 = z^k$, then we set $t = z$. Otherwise, let t be as in

Then t is both a proper suffix and a proper prefix of v_1, which contradicts the hypothesis that v_1 is a Lyndon word.

(continues)

Lemma

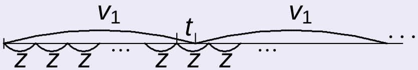
Let α be a term of rank 1 in circular normal form and let $n \geq \mu(\alpha)$. If $z^\ell \in L_n(\alpha)$ then there exists a term of rank 1 in circular normal form $\bar{\alpha}$ such that $\alpha = \bar{\alpha}^\ell$ and $z \in L_n(\bar{\alpha})$.

Proof

Let $\alpha = (v_1)u_1 \cdots (v_r)u_r$ and $w = z^\ell$. Since $z^\ell \in L_n(\alpha)$, it follows that $w = v_1^{n_1}u_1 \cdots v_r^{n_r}u_r$, with $n_1, \ldots, n_r \geq n$.

If $\ell = 1$ then we can just take $\bar{\alpha} = \alpha$. If $\ell > 1$,

***Case 1**: $z < v_1$. If $v_1 = z^k$, then we set $t = z$. Otherwise, let t be as in*



Then t is both a proper suffix and a proper prefix of v_1, which contradicts the hypothesis that v_1 is a Lyndon word.

(continues)

Proof (continuation).

Case 2: $z = v_1^k$ is also impossible.
Case 3: $v_1^{k-1} < z < v_1^k$ for some $k \geq 2$, is also impossible.

Therefore $v_1^{n_1} < z$ and z is not a prefix of any power of v_1. In particular, $|z| > n \geq \mu[\alpha] > |u_r|$. Hence for every $i \in \{1, \dots, \ell - 1\}$ there exists $m_i \in \{2, \dots, r\}$ such that

$$|v_1^{n_1} u_1 \cdots v_{m_i-1}^{n_{m_i-1}}| \leq |z^i| < |v_1^{n_1} u_1 \cdots v_{m_i-1}^{n_{m_i-1}} u_{m_i-1} v_{m_i}^{n_{m_i}}|.$$

Taking into account that $v_1^n < z$ it follows that $v_1 = v_{m_i}$, $u_1 = u_{m_i}$ and $v_2 = v_{m_i+1}$. Inductively, one shows that $v_j = v_{m_i+j-1}$ and $u_j = u_{m_i+j-1}$ for all j, which proves that the word $\bar{\alpha} = (v_1)u_1 \cdots (v_{m_1-1})u_{m_1-1}$ is such that $\alpha = \bar{\alpha}^\ell$ and $z \in L_n(\bar{\alpha})$. □

Proof (continuation).

Case 2: $z = v_1^k$ is also impossible.
Case 3: $v_1^{k-1} < z < v_1^k$ for some $k \geq 2$, is also impossible.

Therefore $v_1^{n_1} < z$ and z is not a prefix of any power of v_1. In particular, $|z| > n \geq \mu[\alpha] > |u_r|$. Hence for every $i \in \{1, \ldots, \ell - 1\}$ there exists $m_i \in \{2, \ldots, r\}$ such that

$$|v_1^{n_1} u_1 \cdots v_{m_i-1}^{n_{m_i-1}}| \leq |z^i| < |v_1^{n_1} u_1 \cdots v_{m_i-1}^{n_{m_i-1}} u_{m_i-1} v_{m_i}^{n_{m_i}}|.$$

Taking into account that $v_1^n < z$ it follows that $v_1 = v_{m_i}$, $u_1 = u_{m_i}$ and $v_2 = v_{m_i+1}$. Inductively, one shows that $v_j = v_{m_i+j-1}$ and $u_j = u_{m_i+j-1}$ for all j, which proves that the word $\bar{\alpha} = (v_1)u_1 \cdots (v_{m_1-1})u_{m_1-1}$ is such that $\alpha = \bar{\alpha}^\ell$ and $z \in L_n(\bar{\alpha})$. □

Proof (continuation).

Case 2: $z = v_1^k$ is also impossible.
Case 3: $v_1^{k-1} < z < v_1^k$ for some $k \geq 2$, is also impossible.

Therefore $v_1^{n_1} < z$ and z is not a prefix of any power of v_1. In particular, $|z| > n \geq \mu[\alpha] > |u_r|$. Hence for every $i \in \{1, \dots, \ell - 1\}$ there exists $m_i \in \{2, \dots, r\}$ such that

$$|v_1^{n_1} u_1 \cdots v_{m_i-1}^{n_{m_i-1}}| \leq |z^i| < |v_1^{n_1} u_1 \cdots v_{m_i-1}^{n_{m_i-1}} u_{m_i-1} v_{m_i}^{n_{m_i}}|.$$

Taking into account that $v_1^n < z$ it follows that $v_1 = v_{m_i}$, $u_1 = u_{m_i}$ and $v_2 = v_{m_i+1}$. Inductively, one shows that $v_j = v_{m_i+j-1}$ and $u_j = u_{m_i+j-1}$ for all j, which proves that the word $\bar{\alpha} = (v_1)u_1 \cdots (v_{m_1-1})u_{m_1-1}$ is such that $\alpha = \bar{\alpha}^\ell$ and $z \in L_n(\bar{\alpha})$. □

Proof (continuation).

Case 2: $z = v_1^k$ is also impossible.
Case 3: $v_1^{k-1} < z < v_1^k$ for some $k \geq 2$, is also impossible.

Therefore $v_1^{n_1} < z$ and z is not a prefix of any power of v_1. In particular, $|z| > n \geq \mu[\alpha] > |u_r|$. Hence for every $i \in \{1, \ldots, \ell - 1\}$ there exists $m_i \in \{2, \ldots, r\}$ such that

$$|v_1^{n_1} u_1 \cdots v_{m_i-1}^{n_{m_i-1}}| \leq |z^i| < |v_1^{n_1} u_1 \cdots v_{m_i-1}^{n_{m_i-1}} u_{m_i-1} v_{m_i}^{n_{m_i}}|.$$

Taking into account that $v_1^n < z$ it follows that $v_1 = v_{m_i}$, $u_1 = u_{m_i}$ and $v_2 = v_{m_i+1}$. Inductively, one shows that $v_j = v_{m_i+j-1}$ and $u_j = u_{m_i+j-1}$ for all j, which proves that the word $\bar{\alpha} = (v_1)u_1 \cdots (v_{m_1-1})u_{m_1-1}$ is such that $\alpha = \bar{\alpha}^\ell$ and $z \in L_n(\bar{\alpha})$. □

Proof (continuation).

Case 2: $z = v_1^k$ is also impossible.
Case 3: $v_1^{k-1} \prec z \prec v_1^k$ for some $k \geq 2$, is also impossible.

Therefore $v_1^{n_1} \prec z$ and z is not a prefix of any power of v_1. In particular, $|z| > n \geq \mu[\alpha] > |u_r|$. Hence for every $i \in \{1, \ldots, \ell - 1\}$ there exists $m_i \in \{2, \ldots, r\}$ such that

$$|v_1^{n_1} u_1 \cdots v_{m_i-1}^{n_{m_i-1}}| \leq |z^i| < |v_1^{n_1} u_1 \cdots v_{m_i-1}^{n_{m_i-1}} u_{m_i-1} v_{m_i}^{n_{m_i}}|.$$

Taking into account that $v_1^n \prec z$ it follows that $v_1 = v_{m_i}$, $u_1 = u_{m_i}$ and $v_2 = v_{m_i+1}$. Inductively, one shows that $v_j = v_{m_i+j-1}$ and $u_j = u_{m_i+j-1}$ for all j, which proves that the word $\bar{\alpha} = (v_1)u_1 \cdots (v_{m_1-1})u_{m_1-1}$ is such that $\alpha = \bar{\alpha}^\ell$ and $z \in L_n(\bar{\alpha})$. □

Proof (continuation).

Case 2: $z = v_1^k$ is also impossible.
Case 3: $v_1^{k-1} < z < v_1^k$ for some $k \geq 2$, is also impossible.

Therefore $v_1^{n_1} < z$ and z is not a prefix of any power of v_1. In particular, $|z| > n \geq \mu[\alpha] > |u_r|$. Hence for every $i \in \{1, \ldots, \ell - 1\}$ there exists $m_i \in \{2, \ldots, r\}$ such that

$$|v_1^{n_1} u_1 \cdots v_{m_i-1}^{n_{m_i-1}}| \leq |z^i| < |v_1^{n_1} u_1 \cdots v_{m_i-1}^{n_{m_i-1}} u_{m_i-1} v_{m_i}^{n_{m_i}}|.$$

Taking into account that $v_1^n < z$ it follows that $v_1 = v_{m_i}$, $u_1 = u_{m_i}$ and $v_2 = v_{m_i+1}$. Inductively, one shows that $v_j = v_{m_i+j-1}$ and $u_j = u_{m_i+j-1}$ for all j, which proves that the word $\bar{\alpha} = (v_1)u_1 \cdots (v_{m_1-1})u_{m_1-1}$ is such that $\alpha = \bar{\alpha}^\ell$ and $z \in L_n(\bar{\alpha})$. □

In case α is a primitive term we get the result.

Lemma

Let α be a primitive term of rank $i \geq 0$ in circular normal form and let $n \geq \mu(\alpha)$. If $z^\ell \in L_n(\alpha)^k$ then $z \in L_n(\alpha)^m$ for some m such that $1 \leq m \leq k$.

Theorem

If $v \in \overline{\Omega}_X\mathbf{A}$ is a factor of $u \in \Omega_X^\omega\mathbf{A}$, then $v \in \Omega_X^\omega\mathbf{A}$.

- For $w \in \overline{\Omega}_X\mathbf{A}$, let
  - $\mathcal{F}(w)$ be the set of all finite factors $u \in X^+$ of w;
  - $P(w)$ be the language of all ω-terms in normal form which define factors of w.

Theorem

Let $w \in \overline{\Omega}_X\mathbf{A}$. Then $w \in \Omega_X^\omega\mathbf{A}$ if and only if w satisfies the following finiteness conditions:

- *$\mathcal{F}(w)$ has no infinite factor-anti-chains;*
- *$P(w)$ is a rational language.*

THEOREM

If $v \in \overline{\Omega}_X\mathbf{A}$ is a factor of $u \in \Omega_X^\omega\mathbf{A}$, then $v \in \Omega_X^\omega\mathbf{A}$.

- For $w \in \overline{\Omega}_X\mathbf{A}$, let
  - $\mathcal{F}(w)$ be the set of all finite factors $u \in X^+$ of w;
  - $P(w)$ be the language of all ω-terms in normal form which define factors of w.

THEOREM

Let $w \in \overline{\Omega}_X\mathbf{A}$. Then $w \in \Omega_X^\omega\mathbf{A}$ if and only if w satisfies the following finiteness conditions:

- *$\mathcal{F}(w)$ has no infinite factor-anti-chains;*
- *$P(w)$ is a rational language.*

Theorem

If $v \in \overline{\Omega}_X \mathbf{A}$ is a factor of $u \in \Omega_X^\omega \mathbf{A}$, then $v \in \Omega_X^\omega \mathbf{A}$.

- For $w \in \overline{\Omega}_X \mathbf{A}$, let
  - $\mathcal{F}(w)$ be the set of all finite factors $u \in X^+$ of w;
  - $P(w)$ be the language of all ω-terms in normal form which define factors of w.

Theorem

Let $w \in \overline{\Omega}_X \mathbf{A}$. Then $w \in \Omega_X^\omega \mathbf{A}$ if and only if w satisfies the following finiteness conditions:

- *$\mathcal{F}(w)$ has no infinite factor-anti-chains;*
- *$P(w)$ is a rational language.*

Theorem

If $v \in \overline{\Omega}_X \mathbf{A}$ is a factor of $u \in \Omega_X^\omega \mathbf{A}$, then $v \in \Omega_X^\omega \mathbf{A}$.

- For $w \in \overline{\Omega}_X \mathbf{A}$, let
  - $\mathcal{F}(w)$ be the set of all finite factors $u \in X^+$ of w;
  - $P(w)$ be the language of all ω-terms in normal form which define factors of w.

Theorem

Let $w \in \overline{\Omega}_X \mathbf{A}$. Then $w \in \Omega_X^\omega \mathbf{A}$ if and only if w satisfies the following finiteness conditions:

- *$\mathcal{F}(w)$ has no infinite factor-anti-chains;*
- *$P(w)$ is a rational language.*