Two contributions from symbolic dynamics to the comprehension of Green's relations in relatively free profinite semigroups

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Joint work with Jorge Almeida, José Carlos Costa and Marc Zeitoun

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Subshifts

A symbolic dynamical system of $A^{\mathbb{Z}}$, also called subshift, is a nonempty subset of $\mathcal{X}$ such that

- $\mathcal{X}$ is topologically closed,
- $\sigma(\mathcal{X}) \subseteq \mathcal{X}$,
- $\sigma^{-1}(\mathcal{X}) \subseteq \mathcal{X}$.

$$\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}, \qquad x_i \in A$$

$L(\mathcal{X}) = \{u \in A^+ : u = x_i x_{i+1} \dots x_{i+n}$ for some $x \in \mathcal{X}$, $i \in \mathbb{Z}$, $n \geq 0\}$.

Factorial, prolongable and irreducible sets

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- ... factorial if

$$u \in K \text{ and } v \text{ is a factor of } u \Longrightarrow v \in K;$$

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The mapping $\mathcal{X} \mapsto L(\mathcal{X})$ is a bijection between the set of subshifts of $A^{\mathbb{Z}}$ and the nonempty factorial prolongable languages of A^+.

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A subshift is irreducible if $L(\mathcal{X})$ is irreducible.

Free profinite semigroups and symbolic dynamics

From hereon V contains $\mathscr{L}$Sl.

- $\overline{L(\mathcal{X})}$: the topological closure of $L(\mathcal{X})$ in $\overline{\Omega}_A\mathsf{V}$
- $\mathcal{M}(\mathcal{X})$: the set of elements of $\overline{\Omega}_A\mathsf{V}$ whose finite factors belong to $L(\mathcal{X})$.

- One has $\overline{L(\mathcal{X})} \subseteq \mathcal{M}(\mathcal{X})$.
- In general, the equality does not hold.

The $\mathcal{J}$-classes $\mathcal{J}(\mathcal{X})$ and $\mathcal{JM}(\mathcal{X})$

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This $\mathcal{J}$-class is regular.

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- $\mathcal{J}(\mathcal{X})$: the minimal $\mathcal{J}$-class of $\overline{L(\mathcal{X})}$.
- $\mathcal{JM}(\mathcal{X})$: the minimal $\mathcal{J}$-class of $\mathcal{M}(\mathcal{X})$.

The minimal case

A subshift is minimal if it does not contain proper subshifts.

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Example

For the Fibonacci substitution

$$\varphi(a) = ab, \quad \varphi(b) = a,$$

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Theorem (Almeida, 2003)

If $\mathcal{X}$ is a minimal subshift, then $\mathcal{J}(\mathcal{X}) = \mathcal{JM}(\mathcal{X})$ is a maximal regular $\mathcal{J}$-class.
All maximal regular $\mathcal{J}$-classes are of this form.

Chains of regular $\mathcal{R}$-classes

Suppose $|A| > 1$.

Theorem (J. C. Costa, 2001)

There is a $<_{\mathcal{R}}$-chain of $2^{\aleph_0}$ elements of $\overline{\Omega}_A \mathsf{V}$.

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Theorem (First contribution. In a work with J. Almeida, J. C. Costa and M. Zeitoun)

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Let f be a function $\mathrm{Dom} f \subseteq \mathscr{C} \to \overline{\Omega}_A \mathsf{V}$ such that

- $f(\mathcal{X}) \in \mathcal{J}\mathcal{M}(\mathcal{X})$
- $\mathcal{X} \supsetneq \mathcal{Y} \Leftrightarrow f(\mathcal{X}) <_{\mathscr{R}} f(\mathcal{Y}) \qquad (\mathcal{X}, \mathcal{Y} \in \mathrm{Dom} f)$

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For every $u, v \in \mathcal{J}\mathcal{M}(\mathcal{X})$ there is $w \in \overline{\Omega}_A \mathsf{V}$,
depending only on the finite suffixes of u and on the finite prefixes of v,
such that $uwv \in \mathcal{J}\mathcal{M}(\mathcal{X})$.

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Suppose $\mathrm{Dom} f \neq \mathscr{C}$. Let $\mathcal{Z} \in \mathscr{C} \setminus \mathrm{Dom} f$ and $v \in \mathcal{JM}(\mathcal{Z})$.
Let u be an accumulation point of $(f(\mathcal{X}))_{\mathcal{X} \subsetneq \mathcal{Z}}$.

$$f' : \mathcal{X} \in \mathrm{Dom} f \cup \{\mathcal{Z}\} \mapsto \begin{cases} f(\mathcal{X}) & \text{if } \mathcal{X} \subsetneq \mathcal{Z}, \\ uwv & \text{if } \mathcal{X} = \mathcal{Z}, \\ uwvw' f(\mathcal{X}) & \text{if } \mathcal{Z} \subsetneq \mathcal{X}, \end{cases}$$

Chains of regular $\mathcal{R}$-classes

*Theorem (First contribution.
In a work with J. Almeida, J. C. Costa and M. Zeitoun)*

*Suppose $|A| > 1$.
There is a $<_{\mathcal{R}}$-chain of $2^{\aleph_0}$ regular elements of $\overline{\Omega}_A V$,
with a minimum at the minimal ideal, and with a subsequence converging to this minimum.*

The proof uses the upper semi-continuity of the *entropy of pseudowords*, another concept borrowed from symbolic dynamics by Almeida and Volkov (2006).

Rees matrix representation

A regular $\mathcal{J}$-class of a compact semigroup is isomorphic to a *Rees matrix partial compact semigroup* $\mathscr{M}(I, G, \Lambda; P) = I \times G \times \Lambda$, where

- I and Λ are compact spaces;
- G is a compact group;
- P is a continuous partial function $\Lambda \times I \to G$;
- $(i_1, g_1, \lambda_1)(i_2, g_2, \lambda_2) = (i_1, g_1 P(\lambda_1, i_2) g_2, \lambda_2)$.

Right and left rays

We say that a right infinite sequence

$$x_0 x_1 x_2 x_3 \cdots$$

of elements of A is a right ray.
If

$$x = \cdots x_{-3} x_{-2} x_{-1} . x_0 x_1 x_2 x_3 \cdots$$

then $x_0 x_1 x_2 \cdots$ is a right ray of x and we use the notation

$$\overrightarrow{x} = x_0 x_1 x_2 x_3 \cdots$$

Dually, one defines *left ray* and $\overleftarrow{x}$.

Right and left rays of a pseudoword

Let u be an infinite pseudoword.
Let

$$u_0 u_1 \cdots u_{n-2} u_{n-1}$$

be the prefix of length n of u.

Definition

Right ray defined by u:

$$\overrightarrow{u} = u_0 u_1 u_2 \cdots u_{n-2} u_{n-1} u_n u_{n+1} \cdots$$

Right and left rays of $\mathcal{X}$

Definition

$$\overrightarrow{\mathcal{X}} = \{\overrightarrow{x} : x \in \mathcal{X}\}$$

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$$z \in \overrightarrow{\mathcal{X}} \Leftrightarrow \exists y \in A^{\mathbb{Z}^-} : y.z \in \mathcal{X}$$

A parametrization of $\mathcal{R}$-classes and $\mathcal{L}$-classes

Let $\mathcal{X}$ be a **minimal subshift** and $u, v \in \mathcal{J}(\mathcal{X})$.

Lemma

- $u \mathrel{\mathcal{R}} v$ if and only if $\overrightarrow{u} = \overrightarrow{v}$
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Corollary

$u \mathrel{\mathcal{H}} v$ if and only if $\overleftarrow{u}.\overrightarrow{u} = \overleftarrow{v}.\overrightarrow{v}$.

A closed coordinate system

Let $\mathcal{X}$ be a minimal subshift.

Let G be a maximal subgroup of $\mathcal{J}(\mathcal{X})$ with idempotent e.

There are families

$$(l_y)_{y\in\overleftarrow{\mathcal{X}}} \qquad (r_z)_{z\in\overrightarrow{\mathcal{X}}}$$

such that

- $l_y \mathrel{\mathcal{R}} e$ and l_y is in the $\mathcal{L}$-class determined by y;
- $r_z \mathrel{\mathcal{R}} e$ and r_z is in the $\mathcal{R}$-class determined by z;
- $l_y \in G \Rightarrow l_y = e, \quad r_z \in G \Rightarrow r_z = e$;

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- the sets $\{l_y : y \in \overleftarrow{\mathcal{X}}\}$ and $\{r_z : z \in \overrightarrow{\mathcal{X}}\}$ are closed;

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- the sets $\{l_y : y \in \overleftarrow{\mathcal{X}}\}$ and $\{r_z : z \in \overrightarrow{\mathcal{X}}\}$ are closed;
- the maps $y \mapsto l_y$ and $z \mapsto r_z$ are continuous.

Rees matrix representation of $\mathcal{J}(\mathcal{X})$

Let $\mathcal{X}$ be a minimal subshift.

$$P : \overleftarrow{\mathcal{X}} \times \overrightarrow{\mathcal{X}} \rightarrow G$$

$$(y, z) \mapsto \begin{cases} l_y r_z & \text{if } y.z \in \mathcal{X} \\ \text{not defined} & \text{if } y.z \notin \mathcal{X} \end{cases}$$

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Second contribution. With J. Almeida.

$$\mathscr{M}(\overrightarrow{\mathcal{X}}, G, \overleftarrow{\mathcal{X}}; P) \rightarrow \mathcal{J}(\mathcal{X})$$

$$(z, g, y) \mapsto r_z g l_y.$$

The Sturmian case

If $\mathcal{X}$ is a Sturmian subshift of $\{a,b\}^{\mathbb{Z}}$, then there are $x, y \in \mathcal{X}$ such that

- $x = \cdots x_{-4}x_{-3}x_{-2}a.bx_1x_2x_3\cdots$;
- $y = \cdots x_{-4}x_{-3}x_{-2}b.ax_1x_2x_3\cdots$;
- if $z, w \in \mathcal{X}$ have a common (right or left) ray, then $z = w$ or $\{z, w\} = \{\sigma^n(x), \sigma^n(y)\}$ for some $n \in \mathbb{Z}$.

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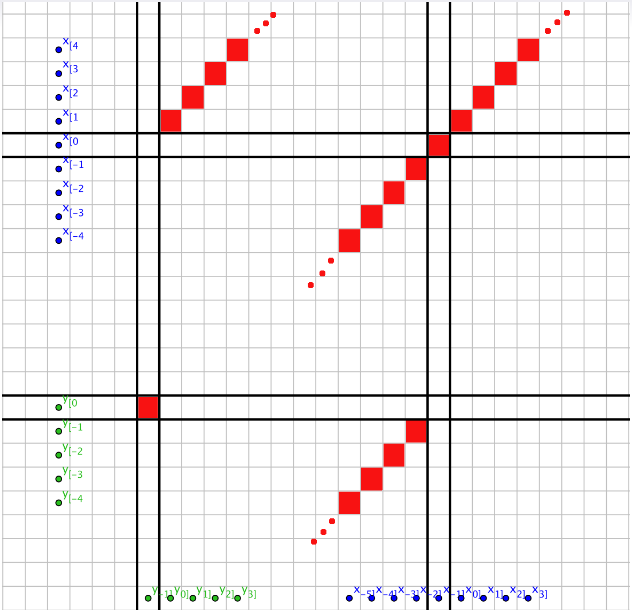
Given $z \in \mathcal{X}$, the right ray

$$z_n z_{n+1} z_{n+2} z_{n+3} \cdots$$

and the left ray

$$\cdots z_{m-3} z_{m-2} z_{m-1} z_m$$

are respectively denoted by $z_{[n}$ and $z_{m]}$.


x[4
x[3
x[2
x[1
x[0
x[-1
x[-2
x[-3
x[-4
y[0
y[-1
y[-2
y[-3
y[-4
y-1]
y0]
y1]
y2]
y3]
x-5]
x-4]
x-3]
x-2]
x-1]
x0]
x1]
x2]
x3]

The Rauzy graph $\Sigma_n(\mathcal{X})$

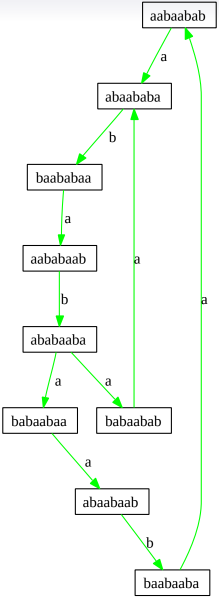
- The edges are the words in $L(\mathcal{X})$ with length $n+1$.
- The vertices are the words in $L(\mathcal{X})$ with length n.
- The edge $a_1 a_2 \dots a_{n-1} a_n$ has origin in $a_1 a_2 \dots a_{n-1}$ and terminus in $a_2 \dots a_{n-1} a_n$.

The centrally labeled Rauzy graph $\Sigma_{2n}(\mathcal{X})$

We assign to each edge of $\Sigma_{2n}(\mathcal{X})$ its middle letter.
This defines a nondeterministic automaton over the alphabet A with transitions

$$a_1 a_2 \dots a_{2n} \xrightarrow{a_{n+1}} a_2 \dots a_{2n} a_{2n+1}$$

defined precisely when $a_1 a_2 \dots a_{2n} a_{2n+1}$ belongs to $L(\mathcal{X})$.


aabaabab
a
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aababaab
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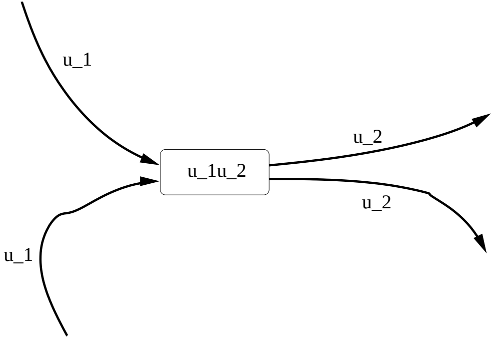
A crucial property

Two paths labeled $u = u_1 u_2$, with $|u_1| = |u_2| = n$.



The transition semigroup of $\Sigma_{2n}(\mathcal{X})$.

- The transition homomorphism of $\Sigma_{2n}(\mathcal{X})$ is denoted by η_n.
- The transition semigroup of $\Sigma_{2n}(\mathcal{X})$ is denoted by $T_n(\mathcal{X})$.

If $\mathcal{X}$ is irreducible then $\Sigma_{2n}(\mathcal{X})$ is strongly connected and $T_n(\mathcal{X})$ has a 0-minimum $\mathcal{J}$-class, denoted $\mathcal{J}_n(\mathcal{X})$.

A homomorphism of partial semigroups

Let $m \geq 2n$.

$$\begin{aligned} \psi_{m,n} : T_m(\mathcal{X}) \setminus \{0\} &\rightarrow T_n(\mathcal{X}) \setminus \{0\} \\ \eta_m(u) &\mapsto \eta_n(u) \end{aligned}$$

Let $s_1, s_2 \in T_m(\mathcal{X})$.
If $s_1 s_2 \neq 0$ then $\psi_{m,n}(s_1)\psi_{m,n}(s_2) \neq 0$ and
$\psi_{m,n}(s_1 s_2) = \psi_{m,n}(s_1)\psi_{m,n}(s_2)$.

Lemma

If $m \geq 2n$ then $\psi_{m,n}(\mathcal{J}_m(\mathcal{X})) = \mathcal{J}_n(\mathcal{X})$.

A projective limit of partial semigroups

- $n \preceq m \Leftrightarrow 2n \leq m$.
- A directed system:

$$\mathscr{J}(\mathcal{X}) = \{\psi_{m,n} : \mathcal{J}_m(\mathcal{X}) \to \mathcal{J}_n(\mathcal{X}) \mid n, m \in \mathbb{Z}^+, n \preceq m\}$$

- For $u \in \mathcal{M}(\mathcal{X})$, let $\theta_n(u)$ be an element of A^* such that $\mathrm{i}_{2n}(u) \cdot w \cdot \mathrm{t}_{2n}(u) \in L(\mathcal{X})$.

A well-defined continuous function:

$$\begin{aligned} \psi : \mathcal{J}\mathcal{M}(\mathcal{X}) &\to \varprojlim \mathscr{J}(\mathcal{X}) \\ u &\mapsto \Big(\eta_n\big(\mathrm{i}_{2n}(u) \cdot \theta_n(u) \cdot \mathrm{t}_{2n}(u)\big)\Big)_n \end{aligned}$$

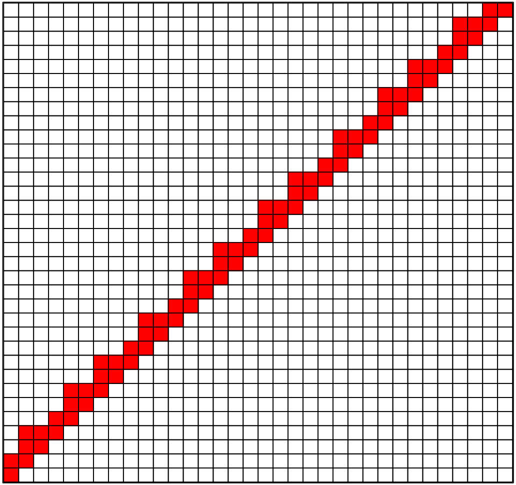
Second contribution revisited

Theorem

- *The mapping* $\psi : \mathcal{JM}(\mathcal{X}) \to \varprojlim \mathscr{J}(\mathcal{X})$ *is an onto homomorphism of partial semigroups.*
- *A pair* (u, v) *of elements of* $\mathcal{JM}(\mathcal{X})$ *belongs to the kernel of* ψ *if and only if* $\overleftarrow{u}\,.\,\overrightarrow{u} = \overleftarrow{v}\,.\,\overrightarrow{v}$.

Corollary

Suppose $\mathsf{V} \subseteq \mathsf{A}$.
If $\mathcal{X}$ *is a minimal subshift then* $\psi : \mathcal{JM}(\mathcal{X}) \to \varprojlim \mathscr{J}(\mathcal{X})$ *is a continuous isomorphism of compact partial semigroups.*



Which idempotents are not lost?

Lemma

Let s be an element of $\mathcal{J}_n(\mathcal{X})$. Let $m \geq 2n$.

Then $s = \psi_n(e)$ for some idempotent e of $\mathcal{J}\mathcal{M}(\mathcal{X})$ if and only if $s = \psi_{m,n}(t)$ for some idempotent t of $\mathcal{J}_m(\mathcal{X})$.