A guide to recent progress in the profinite front

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1. **Preliminaries**
   - Profinite semigroups
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   - The monoid of continuous endomorphisms
2. **Tameness**
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- *Pseudovariety*: class of finite semigroups (or of finite monoids) closed under taking homomorphic images, subsemigroups (resp. submonoids) and finite direct products.

$\mathbf{S} = \{$all finite semigroups$\}$

$\mathbf{G} = \{$groups$\}$

$\mathbf{G}_p = \{p$-groups$\}$

$\mathbf{G}_{\mathrm{nil}} = \{$nilpotent groups$\}$

$\mathbf{G}_{\mathrm{sol}} = \{$solvable groups$\}$

$\mathbf{Ab} = \{$Abelian groups$\}$

$\mathbf{Com} = \{$commutative semigroups$\}$

$\mathbf{CR} = \{$unions of groups$\}$

$\mathbf{N} = \{$nilpotent semigroups$\}$

$\mathbf{D} = \{$idempotents are right zeros$\}$

$\mathbf{K} = \{$idempotents are left zeros$\}$

$\mathbf{Sl} = \{$semilattices$\}$

$\mathbf{J} = \{\mathcal{J}$-trivial$\}$

$\mathbf{R} = \{\mathcal{R}$-trivial$\}$

$\mathbf{L} = \{\mathcal{L}$-trivial$\}$

$\mathbf{A} = \{$aperiodic semigroups$\}$

$\mathbf{DV} = \{$regular $\mathcal{J}$-classes form subsemigroups in $\mathbf{V}\}$

$\mathbf{EV} = \{S \in \mathbf{S} : \langle E(S) \rangle \in \mathbf{V}\}$

$\mathbf{LV} = \{S \in \mathbf{S} : \forall e \in E(S),\ eSe \in \mathbf{V}\}$

$\mathbf{BV} = \{S \in \mathbf{S}$: every *block* of S belongs to $\mathbf{V}\}$

$\bar{\mathbf{H}} = \{S \in \mathbf{S} : \forall H \in \mathbf{G}\ (H \leq S \implies H \in \mathbf{V})\} \quad (\mathbf{H} \subseteq \mathbf{G})$

$\mathbf{V} * \mathbf{W} = \langle V * W : V \in \mathbf{V},\ W \in \mathbf{W} \rangle$

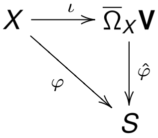
$\mathbf{V} \ⓜ \mathbf{W} = \langle S \in \mathbf{S} : \exists T \in \mathbf{W}\ \exists$ homo. $h : S \to T\ \forall e \in E(T),\ h^{-1}(t) \in \mathbf{V} \rangle$

- A *topological semigroup* is a semigroup S endowed with a topology such that the multiplication $S \times S \to S$ is continuous.
- If X is a topological space, then a continuous mapping $\varphi : X \to S$ is said to *generate* S if $S = \overline{\langle \varphi(X) \rangle}$.
  - We then also say that S is *X-generated (via φ)*.
- A *compact semigroup* is a topological semigroup whose topology is compact (and Hausdorff).
- Finite semigroups are viewed as topological semigroups under the discrete topology.
- A *residually* **V** semigroup is a topological semigroup S such that

$$\forall s_1, s_2 \in S \left(s_1 \neq s_2 \implies \exists T \in \mathbf{V}\, \exists \text{ cont. homo. } h : S \to T :\ h(s_1) \neq h(s_2)\right)$$

- A *pro-**V** semigroup* is a compact semigroup that is residually **V**.
- A *profinite semigroup* is a pro-**S** semigroup.
  - Equivalently, it is a compact totally disconnected semigroup.

- The *free pro-**V** semigroup on a topological space X* is defined by the following universal property: it is given by a continuous mapping $\iota : X \to \overline{\Omega}_X \mathbf{V}$ into a pro-**V** semigroup such that, for every continuous mapping $\varphi : X \to S$ into a pro-**V** semigroup there is a unique continuous homomorphism $\hat{\varphi}$ such that the following diagram commutes



- The obvious way to *construct* $\overline{\Omega}_X \mathbf{V}$ is to take the direct product of all X-generated pro-**V** semigroups S_i, via φ_i ($i \in I$), in the category of all X-generated topological semigroups, that is the closure of the subsemigroup generated by $\{(\varphi_i(x))_{i \in I} : \ x \in X\}$.
  - In fact, it suffices to take X-generated semigroups from **V**.
  - This is usually viewed as the projective limit of all X-generated semigroups from **V**.

- Another convenient way to *construct* $\overline{\Omega}_X\mathbf{V}$ is to consider the following pseudo-metric on the free semigroup X^+:

$$d(u,v) = 2^{-r(u,v)}$$

$$r(u,v) = \min\{|S| : S \in \mathbf{V}, \exists \text{ cont. mapping } h : X \to S;\ S \not\models u = v[h]\}$$

and to take the completion $\widehat{X^+}$, which is a metric space.

  - The multiplication in X^+ is uniformly continuous with respect the pseudo-metric d and, therefore, it extends to a continuous multiplication on $\widehat{X^+}$.
  - This only works well in case X is finite since otherwise $\overline{\Omega}_X\mathbf{V}$ is in general not metrizable.
  - The problem lies in that, for X infinite, X^+ is in general not covered by finitely many balls of radius 2^{-n} (for instance if $\mathbf{V}$ is nontrivial and X is totally disconnected), which entails that the completion $\widehat{X^+}$ is not compact, and therefore not a pro-$\mathbf{V}$ semigroup.
  - From hereon, unless something is explicitly stated to the contrary, X is taken to be a finite (discrete) set.

- A language $L \subseteq X^+$ is said to be ***V**-recognizable* if

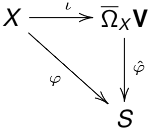
$$\exists S \in \mathbf{V}\ \exists \text{ homo. } \varphi : X^+ \to S : L = \varphi^{-1}\varphi(L)$$

- It is simple to show that $L \subseteq X^+$ is **V**-recognizable if and only if, denoting $\bar{\iota} : X^+ \to \overline{\Omega}_X\mathbf{V}$ the natural homomorphism,
  1. $\overline{\bar{\iota}(L)} \subseteq \overline{\Omega}_X\mathbf{V}$ is an open set
  2. $L = \bar{\iota}^{-1}(\overline{\bar{\iota}(L)})$.
- The second condition is superfluous if $\bar{\iota}$ is injective and the induced topology on X^+ (the *pro-**V** topology*) is discrete.
- This is the case whenever $\mathbf{V} \supseteq \mathbf{N}$. In fact

PROPOSITION (JA-JCCOSTA-ZEITOUN)

$\mathbf{V} \supseteq \mathbf{N}$ *if and only if for every finite alphabet X, the natural mapping $X^+ \to \overline{\Omega}_X\mathbf{V}$ is injective and its image is an open discrete subset of $\overline{\Omega}_X\mathbf{V}$.*

- The diagram describing the universal property of $X \to \overline{\Omega}_X\mathbf{V}$ suggests a natural way to interpret each $w \in \overline{\Omega}_X\mathbf{V}$ as an operation $w_S : S^X \to S$ on each pro-$\mathbf{V}$ semigroup S:



$$w_S(\varphi) = \hat{\varphi}(w)$$

- This interpretation commutes with all continuous homomorphisms between pro-$\mathbf{V}$ semigroups.
- Operations with this property with respect to a certain class of topological semigroups are called *($|X|$-ary) implicit operations* on the class.
- In particular, the homomorphisms between the members of $\mathbf{V}$ respect the enriched algebraic structure, obtained by adding all such operations.
- It is well-known that the correspondence $w \mapsto (w_S)_{S\in\mathbf{V}}$ is a bijection from $\overline{\Omega}_X\mathbf{V}$ to the set of all $|X|$-ary implicit operations on $\mathbf{V}$.

- Given $u, v \in \overline{\Omega}_X\mathbf{V}$ and a pro-$\mathbf{V}$ semigroup S, we write $S \models u = v$ if $u_S = v_S$.
- Such a formal equality $u = v$ is known as a *($\mathbf{V}$-)pseudoidentity*.
- Given a set Σ of pseudoidentities, the class

$$[\![\Sigma]\!]_{\mathbf{V}} = \{S \in \mathbf{V} : \ \forall \sigma \in \Sigma, \ S \models \sigma\}$$

Theorem (Reiterman)

Every subpseudovariety of $\mathbf{V}$ is of this form.

- In case $\mathbf{V} = \mathbf{S}$, we drop the index $\mathbf{V}$.

- Note that, every profinite semigroup S, given $s \in S$, the sequence $(s^{n!-1})_n$ converges to some element, denoted $s^{\omega-1}$.
- In fact, $s^{\omega-1}$ is the inverse of es in the group minimum ideal of $\overline{\langle s \rangle}$, with idempotent $e = s^{\omega-1}s$.
- We naturally also write s^{ω} for $ss^{\omega-1}$ and $s^{\omega+1}$ for ss^{ω}.

$$\begin{aligned}
\mathbf{S} &= [\![x = x]\!] \\
\mathbf{G} &= [\![x^{\omega} = 1]\!] \\
\mathbf{CR} &= [\![x^{\omega+1} = x]\!] \\
\mathbf{A} &= [\![x^{\omega+1} = x^{\omega}]\!] \\
\mathbf{N} &= [\![x^{\omega} = 0]\!]
\end{aligned}$$

$$\begin{aligned}
\mathbf{Com} &= [\![xy = yx]\!] \\
\mathbf{D} &= [\![xy^{\omega} = y^{\omega}]\!] \\
\mathbf{K} &= [\![x^{\omega}y = x^{\omega}]\!] \\
\mathbf{J} &= [\![(xy)^{\omega} = (yx)^{\omega}, x^{\omega+1} = x^{\omega}]\!] \\
\mathbf{R} &= [\![(xy)^{\omega}x = (xy)^{\omega}]\!] \\
\mathbf{L} &= [\![(xy)^{\omega}x = (yx)^{\omega}]\!]
\end{aligned}$$

- Let S be a compact semigroup. Then the continuous endomorphisms of S constitute a monoid *End*(S) under composition.
- As a function space, there are two classical candidates for topologies on End(S), namely:
  - the *pointwise convergence topology*, i.e., as a subspace of the product space S^S or in which a sequence converges if and only if it converges pointwise;
  - the *compact open topology*, which in our context can be described by stating that a sequence converges if and only if it converges it converges uniformly.
- The former topology is in a sense very familiar while the latter has the advantage that the *evaluation mapping*

$$\begin{aligned} \mathrm{End}(S) \times S &\longrightarrow S \\ (\varphi, s) &\longmapsto \varphi(s) \end{aligned}$$

is continuous.

THEOREM (JA (2003))

If S is a finitely generated profinite semigroup, then the two topologies coincide on $\mathrm{End}(S)$ and $\mathrm{End}(S)$ is a profinite semigroup with respect to them.

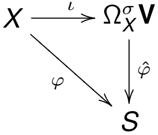
- In particular, if X is a finite set and $\mathbf{V}$ is any pseudovariety of semigroups, given $\varphi \in \mathrm{End}(\overline{\Omega}_X\mathbf{V})$, there is an idempotent $\varphi^\omega = \lim_{n\to\infty} \varphi^{n!}$, which we view as a canonical infinite iteration of the *substitution* φ.
- For example, the substitution $\varphi(x) = x^p$, where x is a variable, induces the idempotent $\varphi^\omega \in \mathrm{End}(\overline{\Omega}_{\{x\}}\mathbf{S})$, which satisfies

$$\varphi^\omega(x) = \lim_{n\to\infty} \varphi^{n!}(x) = \lim_{n\to\infty} x^{p^{n!}}.$$

We define x^{p^ω} to be $\varphi^\omega(x)$. It is easy to show that

$$\mathbf{G}_p = [\![x^{p^\omega} = 1]\!].$$

- Let σ be a set of implicit operations (on **S**), containing the binary multiplication. We call σ an *implicit signature*.
  - *Canonical* example: $\kappa = \{\_ \cdot \_,\ \_^{\omega-1}\}$.
- Recall that every profinite semigroup has a natural structure as a *σ-algebra*.
- Given a pseudovariety **V** and a finite set X, let $\iota : X \to \overline{\Omega}_X\mathbf{V}$ be the natural mapping.
- Denote by $\Omega_X^\sigma\mathbf{V}$ the σ-subalgebra generated by $\iota(X)$. The mapping $X \to \Omega_X^\sigma\mathbf{V}$ determined by ι has the suitable universal property of the *free σ-algebra on X* in the Birkhoff variety generated by **V**:



where now S is any member of the variety and $\hat{\varphi}$ is a homomorphism of σ-algebras.

- We say that a *system of equations* $(u_i = v_i)_{i \in I}$ over an alphabet X, with a set of clopen *constraints* $K_x \subseteq \overline{\Omega}_A\mathbf{S}$ *has a solution* γ *modulo* **V** if $\gamma : \overline{\Omega}_X\mathbf{S} \to \overline{\Omega}_A\mathbf{S}$ is a continuous homomorphism such that the following two conditions hold:

  1. $\forall x \in X,\ \gamma(x) \in K_x$;
  2. $\forall i \in I,\ \mathbf{V} \models \gamma(u_i) = \gamma(v_i)$.

- We say that **V** is *σ-reducible with respect to a class $\mathcal{C}$ of constrained systems of equations* if every constrained system in the class that has a solution $\gamma : \overline{\Omega}_X\mathbf{S} \to \overline{\Omega}_A\mathbf{S}$ modulo **V** admits such a solution satisfying $\gamma(X) \subseteq \Omega^{\sigma}_A\mathbf{S}$.

- Finally, we say that **V** is *σ-tame with respect to a class $\mathcal{C}$ of constrained systems of equations* if σ is an implicit signature such that:
  1. σ is recursively enumerable;
  2. the operations in σ are computable;
  3. the word problem for $\Omega_X^\sigma \mathbf{V}$ is decidable for every finite set X;
  4. **V** is σ-reducible with respect to $\mathcal{C}$.
- If this property holds, then
  - in case $\mathcal{C}$ consists of all constrained systems of equations associated with finite digraphs

    $$x \xrightarrow{y} z \quad \longmapsto \quad xy = z$$

    we simply say that **V** is *σ-tame*;
  - in case $\mathcal{C}$ consists of all constrained systems of equations of the form $u = v$ with $u, v \in \Omega_X^\sigma \mathbf{S}$, we say that **V** is *completely σ-tame*.

- Given a subset $L \subseteq S$ of a topological semigroup, let $\mathrm{cl}_S(L)$ denote its closure.
  - In the special case $S = \overline{\Omega}_X\mathbf{S}$, we let $\mathrm{cl}(L) = \mathrm{cl}_{\overline{\Omega}_X\mathbf{S}}(L)$.
  - If $S = \overline{\Omega}_X\mathbf{V}$, we let $\mathrm{cl}_{\sigma,\mathbf{V}}(L) = \mathrm{cl}_{\Omega_X^\sigma\mathbf{V}}(L)$.
- We say that the pseudovariety $\mathbf{V}$ is *σ-full* if it satisfies the following condition for every rational language $L \subseteq X^+$:

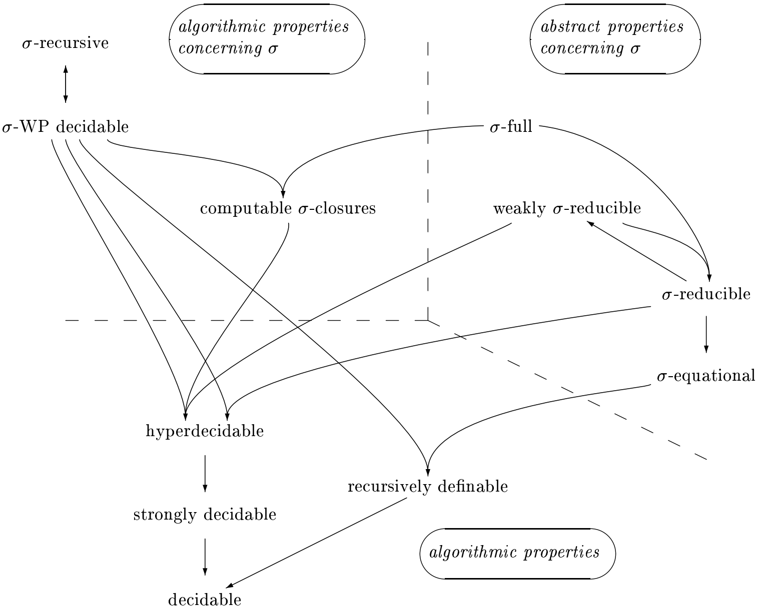
$$\mathrm{cl}_{\sigma,\mathbf{V}}(L) = p_{\mathbf{V}}\left(\mathrm{cl}(L) \cap \Omega_X^\sigma\mathbf{S}\right)$$

THEOREM (JA-STEINBERG (2000))

If $\mathbf{V}$ is a recursively enumerable pseudovariety of semigroups and σ is a recursively enumerable implicit signature consisting of computable operations such that the word problem for $\Omega_X^\sigma\mathbf{V}$ is decidable and $\mathbf{V}$ is σ-full, then there is an algorithm to compute the closure $\mathrm{cl}_{\sigma,\mathbf{V}}(L) \subseteq \Omega_X^\sigma\mathbf{V}$ for every rational language $L \subseteq X^+$.

- The proof of the preceding theorem is obtained by exhibiting two semi-algorithms, one for enumerating the elements in the closure, the other for its complement.
- Given a rational expression for the language, can one do better?
- For a subset L of a σ-algebra S, denote by $\langle L\rangle_\sigma$ the σ-subalgebra generated by L.
- We say that the *Pin-Reutenauer procedure holds for a pseudovariety* $\mathbf{V}$ *in the implicit signature* σ if, for all rational languages $K, L \subseteq X^+$, the following equations hold:

$$\mathrm{cl}_{\sigma,\mathbf{V}}(KL) = \mathrm{cl}_{\sigma,\mathbf{V}}(K) \cdot \mathrm{cl}_{\sigma,\mathbf{V}}(L),$$
$$\mathrm{cl}_{\sigma,\mathbf{V}}(L^+) = \langle \mathrm{cl}_{\sigma,\mathbf{V}}(L)\rangle_\sigma.$$



algorithmic properties concerning σ
abstract properties concerning σ
σ-recursive
σ-WP decidable
σ-full
computable σ-closures
weakly σ-reducible
σ-reducible
σ-equational
hyperdecidable
recursively definable
strongly decidable
algorithmic properties
decidable

- $\Omega_X^\kappa \mathbf{G}$ is the free group on X and so the solution of the word problem for $\Omega_X^\kappa \mathbf{G}$ is simple and well known.

Theorem (Ash (1991) + JA-Steinberg (2000))

G *is* κ*-tame.*

Theorem (Pin-Reutenauer (1991) + Ribes-Zalesskiĭ (1993) or Ash (1991))

The Pin-Reutenauer procedure holds for **G**.

Theorem (Delgado (2001))

G *is* κ*-full.*

Theorem (Coulbois-Khélif (1999))

G *is not completely* κ*-tame.*

- Since the free group is residually $\mathbf{G}_p$ [Baumslag (1965)], $\mathbf{G}_p$ cannot be κ-tame.
- $\mathbf{G}_p$ is not κ-full either [Steinberg (2001)].

THEOREM (JA (2002), BUILDING ON RESULTS OF RIBES-ZALESSKIĬ (1994), MARGOLIS-SAPIR-WEIL (2001), STEINBERG (2001))

Let $\mathbf{H}$ *be an extension-closed pseudovariety of finite groups. If there is an algorithm to decide whether a finite subset of the free group* $\Omega_X^\kappa \mathbf{G}$ *generates a dense subgroup in the pro-*$\mathbf{H}$ *topology (i.e., in the subspace topology* $\Omega_X^\kappa \mathbf{G} = \Omega_X^\kappa \mathbf{H} \subseteq \overline{\Omega}_X \mathbf{H}$*), then* $\mathbf{H}$ *is* σ*-tame, for a certain infinite implicit signature constructed by the infinite iteration of appropriate substitutions.*

- From the proof of this result, it follows that $\mathbf{H}$ is σ-full.

- Ribes-Zalesskiĭ (1994): an algorithm to compute the closure $\mathrm{cl}_{\kappa,\mathbf{G}_p}(L) \subseteq \Omega_X^\kappa \mathbf{G}_p$ for a rational language $L \subseteq X^+$.

COROLLARY

$\mathbf{G}_p$ *is tame.*

- Can one decide denseness of finitely generated subgroups of the free group for the pro-$\mathbf{G}_{\mathrm{sol}}$ topology?
- Likewise for the pro-$\mathbf{G}_{\mathrm{odd}}$ topology?

THEOREM

1. $\overline{\Omega}_X \mathbf{J} = \Omega_X^\kappa \mathbf{J}$.
2. *The variety of κ-algebras generated by* **J** *is defined by the identities*

$$x^{\omega-1}x = xx^{\omega-1} = x^{\omega-1},\ (x^\omega)^\omega = x^\omega$$
$$(xy)z = x(yz),\ (xy)^\omega = (yx)^\omega = (x^\omega y^\omega)^\omega.$$

3. *Solution of the word problem for* $\Omega_X^\kappa \mathbf{J}$.
4. **J** *is completely κ-tame.*
5. **J** *is κ-full.*
6. *The Pin-Reutenauer procedure holds for* **J**. *(Follows from the results presented by M. Zeitoun.)*

- JA-Weil (1997), JA-Zeitoun (2007): structural descriptions of $\overline{\Omega}_A$**R**.
- JA-Zeitoun (2007): efficient solution of the word problem for Ω^κ_X**R**.
- JA-JCCosta-Zeitoun (2007): **R** is completely κ-tame
- JA-JCCosta-Zeitoun: **R** is κ-full and the Pin-Reutenauer procedure holds for **R**. (See M. Zeitoun's talk.)
- Partial extension:
  - Moura: structural description of $\overline{\Omega}_X$**DA** and solution of the word problem for Ω^κ_X**DA**.

- McCammond (2001): an algorithm to decide the word problem for $\Omega_X^\kappa \mathbf{A}$.
- The algorithm consists in the rewriting of any given description of an element of $\Omega_X^\kappa \mathbf{A}$, as a κ-term, in a certain normalized form.
- The hard part of the proof of correctness of the algorithm consists in showing that distinct κ-terms in normal form represent distinct elements of $\Omega_X^\kappa \mathbf{A}$.
- McCammond's proof uses his own solution (1991) of the word problem for free Burnside semigroups

$$\langle X : \forall t,\ t^{n+1} = t^n \rangle$$

for sufficiently large n (which he had achieved for $n \geq 6$).

- JA-JCCosta-Zeitoun: a new proof of correctness of McCammond's algorithm is obtained as follows:
  - to each κ-term w, a descending sequence $L_n(w)$ of rational languages is associated;
  - $L_n(w)$ is shown to be star-free (i.e., **A**-recognizabe [Schützenberger (1965)]) provided w is in normal form and n is sufficiently large;
  - for distinct κ-terms in normal form v and w, it is shown that $L_n(v) \cap L_n(w) = \emptyset$ for sufficiently large n.

  See J. C. Costa's talk for details.

- So far, complete structural results for relatively free profinite semigroups $\overline{\Omega}_X\mathbf{V}$ concern relatively small pseudovarieties, like **J**, **R**, **DA**, **CR** [JA-Trotter (2001)] and some related pseudovarieties.
- These are all subpseudovarieties of **DS**, a pseudovariety for which there are plenty of idempotents: it is precisely characterized by the property that every regular $\mathcal{H}$-class contains an idempotent.
- Outside **DA**, there are not many cases for which the structure has been identified.
- Examples are the pseudovarieties $\mathbf{LSI} = \mathbf{SI} * \mathbf{D}$ and $\mathbf{ESI} = \mathbf{SI} * \mathbf{G}$, usually taking advantage of such decompositions to obtain the desired structural results.
  - JCCosta-Nogueira: **LSI** is completely κ-tame.

- For *large* pseudovarieties, say containing **A**, there are so far no such structural results but only partial information.
- There is nevertheless a free semigroup-like property that many large pseudovarieties share that may lead to complete structural results: *equidivisibility*.
- A *refinement* of a factorization $s_1 \cdots s_n$ in a semigroup is obtained by further factorizing some (possibly none) of the factors.
- A semigroup S is said to be *equidivisible* [McKnight-Storey (1969)] if any two factorizations of the same element admit a common refinement.

PROPOSITION (JA-ACOSTA (2009))

If $\mathbf{V}$ *is a pseudovariety closed under concatenation, then* $\overline{\Omega}_X\mathbf{V}$ *is equidivisible.*

THEOREM (STRAUBING (1979))

A pseudovariety $\mathbf{V}$ *is closed under concatenation if and only if it satisfies the equation* $\mathbf{V} = \mathbf{A} \circledcirc \mathbf{V}$.

- It is easy to deduce that every pseudovariety of the form $\bar{\mathbf{H}}$, where $\mathbf{H}$ is a pseudovariety of groups, is closed under concatenation.
- In particular, $\overline{\Omega}_X\mathbf{S}$ and $\overline{\Omega}_X\mathbf{A}$ are equidivisible.

Theorem (JA-ACosta-JCCosta-Zeitoun)

The following are equivalent for $\mathbf{V} \supseteq \mathbf{N} \cap \mathbf{Com}$*:*

1. $\mathbf{V}$ *is closed under concatenation;*
2. $\mathbf{A} ⓜ \mathbf{V} = \mathbf{V}$*;*
3. *the multiplication* $\overline{\Omega}_X\mathbf{V} \times \overline{\Omega}_X\mathbf{V} \to \overline{\Omega}_X\mathbf{V}$ *is an open mapping for every finite set* X*;*
4. *(lifting factorizations) for every finite set* X*, if* $u, v, w \in \overline{\Omega}_X\mathbf{S}$ *are such that* $\mathbf{V} \models uv = w$ *then there is a factorization* $w = u'v'$ *in* $\overline{\Omega}_X\mathbf{S}$ *such that* $\mathbf{V} \models u = u',\ v = v'$*.*

Corollary

If $\mathbf{V}$ *is closed under concatenation and the products* $u_1 \cdots u_m$ *and* $v_1 \cdots v_n$ *of elements of* $\overline{\Omega}_X\mathbf{S}$ *coincide in* $\mathbf{V}$*, then they admit refinements whose factors in corresponding positions are equal over* $\mathbf{V}$*.*

- By a *subshift* over a finite alphabet X we mean a closed subset of $X^{\mathbb{Z}}$ that is stable under arbitrary shifts of the origin (in the elements of $X^{\mathbb{Z}}$, which are viewed as bi-infinite words in the letters of X).
- It is well known that a subshift $\mathcal{X}$ is characterized by its *language of finite blocks* (or *finite factors*) $L(\mathcal{X})$, which is an arbitrary *factorial extendable* language.

THEOREM (JA (2005,2007))

For every $\mathbf{V} \supseteq \mathbf{LSI}$, *the correspondence*

$$\mathcal{S} \longmapsto J(\mathcal{X}) = \overline{L(\mathcal{X})} \setminus X^+ \subseteq \overline{\Omega}_X \mathbf{V}$$

defines a bijection between minimal subshifts and the $\mathcal{J}$*-maximal regular* $\mathcal{J}$*-classes of* $\overline{\Omega}_X \mathbf{V}$.

- We say that an endomorphism φ of X^+ is *primitive* if

$$\exists n\,\forall x, y \in X,\; x \text{ apears in } \varphi(y).$$

- The continuous extension $\hat{\varphi}$ to $\overline{\Omega}_X\mathbf{S}$ of an endomorphism φ of X^+ is also called a *finite substitution (of X)*.
- A finite substitution φ (of X) defines a subshift $\mathcal{X}_\varphi$ whose finite factors are the factors of some $\varphi^n(x)$ for arbitrarily large n.
- It is well known that, if φ is primitive, then $\mathcal{X}_\varphi$ is a minimal subshift.

THEOREM (JA (2005,2007))

Let $\varphi \in End(\overline{\Omega}_X\mathbf{S})$ be a finite primitive substitution. If φ induces an automorphism of the free group $\Omega_X^\kappa\mathbf{G}$, then the maximal subgroups of $J(\mathcal{X}_\varphi)$ are finitely generated free profinite groups.

- For further recent developments, se A. Costa's talk.

- So, just which profinite groups may appear as closed subgroups of free profinite semigroups?

THEOREM (RHODES-STEINBERG (2008))

Precisely the same that appear as closed subgroups of free profinite groups, namely the projective profinite groups.

- What about clopen subsemigroups?

THEOREM (JA-STEINBERG (2009))

Let $\mathbf{H}$ be an extension-closed pseudovariety of groups and X a finite set. Then the free clopen subsemigroups of $\overline{\Omega}_X\bar{\mathbf{H}}$ are the closures of $\bar{\mathbf{H}}$-recognizable free subsemigroups of X^+.

- See B. Steinberg's talk for more details and results on the minimum ideal.