ARCHIVUM MATHEMATICUM (BRNO)
Tomus 32 (1996), 000 – 000

RELATIONS BETWEEN LINEAR CONNECTIONS ON THE TANGENT BUNDLE AND CONNECTIONS ON THE JET BUNDLE OF A FIBRED MANIFOLD

JOSEF JANYŠKA AND MARCO MODUGNO

For Ivan Kolář, on the occasion of his 60th birthday.

Abstract. All natural operations transforming linear connections on the tangent bundle of a fibred manifold to connections on the 1-jet bundle are classified. It is proved that such operators form a 2-parameter family (with real coefficients).

Introduction

This paper is motivated by the bijective relation between time-preserving linear connections on space-time with absolute time and affine connections on 1-jet bundle of space-time, [1], [2], [3]. We would like to know if similar relation holds also for a general fibred manifold and so we study all natural operations transforming linear connections on the tangent bundle of a fibred manifold to connections on the 1-jet bundle. We prove that such operators form a 2-parameter family (with real coefficients) and we give its coordinate and geometric expressions.

Our operator are natural in the sense of [4] and [5].

All manifolds and mappings are assumed to be smooth.

1. Linear connections

Let $p : Y \to X$ be a fibred manifold with a local fibred coordinate chart $(x^\lambda, x^i) = (x^A)$, $\lambda = 1, \dots, \dim X = n$, $i = 1, \dots, \dim Y - \dim X = m$, $A = 1, \dots, \dim Y = n + m$.

A linear connection Λ on the bundle $\pi_X : TX \to X$ and a linear connection K on the bundle $\pi_Y : TY \to Y$ can be expressed, respectively, by tangent valued

1991 *Mathematics Subject Classification.* 53C05, 53C15, 58A20.
Key words and phrases. Tangent bundle, jet bundle, connection, natural operator.
This research has been supported by grant No. 201/96/0079 of GA CR, GNFM of CNR, MURST, University of Florence and Contract ERB CHRXCT 930096 of EEC.

forms

$$\Lambda : TX \to T^*X \underset{TX}{\otimes} TTX \,,$$
$$K : TY \to T^*Y \underset{TY}{\otimes} TTY$$

with coordinate expressions, respectively,

$$\Lambda = d^\varphi \otimes (\partial_\varphi + \Lambda_\varphi{}^\lambda{}_\psi \dot{x}^\psi \dot{\partial}_\lambda) \,, \qquad \Lambda_\varphi{}^\lambda{}_\psi \in C^\infty(X) \,, \tag{1.1}$$
$$K = d^A \otimes (\partial_A + K_A{}^C{}_B \dot{x}^B \dot{\partial}_C) \,, \qquad K_A{}^C{}_B \in C^\infty(Y) \,, \tag{1.2}$$

where $(x^\varphi, \dot{x}^\varphi)$ and $(x^A, \dot{x}^A)$ are the induced coordinate charts on TX and TY, respectively. The connections Λ and K are also characterised by the vertical projections $\nu_\Lambda : TTX \to TX$ and $\nu_K : TTY \to TY$, respectively, or equivalently by the forms $\nu_\Lambda : TX \to T^*TX \otimes_{TX} TX$ and $\nu_K : TY \to T^*TY \otimes_{TY} TY$ with coordinate expressions, respectively,

$$\nu_\Lambda = (\dot{d}^\lambda - \Lambda_\varphi{}^\lambda{}_\psi \dot{x}^\psi d^\varphi) \otimes \partial_\lambda \,, \tag{1.3}$$
$$\nu_K = (\dot{d}^A - K_B{}^A{}_C \dot{x}^C d^B) \otimes \partial_A \,. \tag{1.4}$$

Let us denote by $K \otimes_Y \Lambda^*$ the tensor product of the connection K and the pullback of the dual connection Λ^* with respect to p, i.e.

$$K \otimes_Y \Lambda^* : T^*X \underset{Y}{\otimes} TY \to T^*Y \underset{T^*X \underset{Y}{\otimes} TY}{\otimes} T(T^*X \underset{Y}{\otimes} TY)$$

with coordinate expression, in the induced fibred coordinate chart (x^A, x^A_λ) on $q : T^*X \otimes_Y TY \to Y$,

$$K \otimes_Y \Lambda^* = d^A \otimes \big(\partial_A + (K_A{}^C{}_B \delta^\mu_\lambda - \delta^C_B \Lambda_A{}^\mu{}_\lambda) x^B_\mu \partial^\lambda_C\big) \,, \tag{1.5}$$

where we put $\Lambda_i{}^\mu{}_\lambda = 0$. The connection $K \otimes_Y \Lambda^*$ can be defined by the vertical projection $\nu_{K \otimes_Y \Lambda^*} : T(T^*X \otimes_Y TY) \to T^*X \otimes_Y TY$. We have the coordinate expression

$$\nu_{K \otimes_Y \Lambda^*} = (d^A_\mu - (K_B{}^A{}_C \delta^\kappa_\mu - \delta^A_C \Lambda_B{}^\kappa{}_\mu) x^C_\kappa d^B) \otimes \partial^\mu_A \,. \tag{1.6}$$

A linear connection K on TY is said to be *projectable* on a linear connection Λ on TX if the following diagram commutes

$$\begin{array}{ccc} TTY & \xrightarrow{\ \nu_K\ } & TY \\ {\scriptstyle TTp}\downarrow & & \downarrow{\scriptstyle Tp} \\ TTX & \xrightarrow{\ \nu_\Lambda\ } & TX \end{array}$$

A pair of linear connections (K, Λ) is said to be *fibre preserving* if the covariant derivative of dp with respect to $K \otimes_Y \Lambda^*$ vanishes, i.e. $\nabla_{K \otimes_Y \Lambda^*}(dp) = 0$.

Lemma 1.1. Let K be a linear connection on TY and Λ a linear connection on TX. The following three conditions are equivalent

i) K is projectable on Λ.

ii) The pair (K, Λ) is fibre preserving.

iii) In a fibred coordinate chart $K_A{}^\lambda{}_i = K_j{}^\lambda{}_B = 0$ and $K_\mu{}^\lambda{}_\nu = \Lambda_\mu{}^\lambda{}_\nu$.

PROOF. It can be proved by using (1.3), (1.4) and (1.6). □

2. Contact mappings

We deal with the natural complementary contact maps

$$д : J_1Y \underset{Y}{\times} TX \to TY\,, \quad \vartheta : J_1Y \underset{Y}{\times} TY \to VY\,,$$

or equivalently

$$д : J_1Y \to T^*X \underset{Y}{\otimes} TY\,, \quad \vartheta : J_1Y \to T^*Y \underset{Y}{\otimes} VY\,,$$

which split the natural exact sequence

$$0 \to VY \to TY \xrightarrow{dp} TX \to 0\,, \tag{2.1}$$

through the exact sequence over J_1Y

$$0 \to J_1Y \underset{X}{\times} TX \xrightarrow{д} J_1Y \underset{Y}{\times} TY \xrightarrow{\vartheta} J_1Y \underset{Y}{\times} VY \to 0\,. \tag{2.2}$$

We have the coordinate expressions

$$д = d^\lambda \otimes д_\lambda = d^\lambda \otimes (\partial_\lambda + x^i_\lambda \partial_i)\,, \quad \vartheta = \vartheta^i \otimes \partial_i = (d^i - x^i_\lambda d^\lambda) \otimes \partial_i\,, \tag{2.3}$$

where $(x^\lambda, x^i; x^i_\lambda)$ is the induced coordinate chart on J_1Y.

We recall the canonical isomorphism

$$VJ_1Y \simeq J_1Y \underset{Y}{\times} (T^*X \underset{Y}{\otimes} VY)$$

given by

$$\partial^\lambda_i \mapsto d^\lambda \otimes \partial_i\,.$$

3. Induced connection

A connection Γ on the affine bundle $\pi^1_0 : J_1Y \to Y$ can be expressed by a tangent valued form

$$\Gamma : J_1Y \to T^*Y \underset{J_1Y}{\otimes} TJ_1Y$$

with coordinate expression

$$\Gamma = d^A \otimes (\partial_A + \Gamma_A{}^i_\lambda \partial^\lambda_i)\,, \qquad \Gamma_A{}^i_\lambda \in C^\infty(J_1Y)\,. \tag{3.1}$$

Using the identification of VJ_1Y and $T^*X \otimes_Y VY$, the connection Γ can be characterised by the vertical projection $\nu_\Gamma : TJ_1Y \to T^*X \otimes_{J_1Y} VY$, or equivalently by the form $\nu_\Gamma : J_1Y \to T^*X \otimes_Y T^*J_1Y \otimes_{J_1Y} VY$. In coordinates we have

$$\nu_\Gamma = d^\lambda \otimes (d^i_\lambda - \Gamma_A{}^i_\lambda d^A) \otimes \partial_i \, . \tag{3.2}$$

The connection Γ is affine if and only if its coordinate expression is of the type

$$\Gamma_A{}^i_\lambda = \Gamma_A{}^i{}_\lambda{}^\mu_j x^j_\mu + \Gamma_A{}^i_\lambda{}_{\cdot}^{\cdot} \, , \qquad \Gamma_A{}^i{}_\lambda{}^\mu_j \, , \, \Gamma_A{}^i_\lambda{}_{\cdot}^{\cdot} \in C^\infty(Y) \, .$$

Theorem 3.1. Let Λ be a linear connection on TX and K a linear connection on TY. The map

$$\nu_\Gamma = \vartheta \circ \nu_{(K \otimes \Lambda^*)} \circ T Д$$

given by the following diagram

$$\begin{array}{ccccc}
TJ_1Y & \xrightarrow{\nu_\Gamma} & VJ_1Y & \xrightarrow{\simeq} & J_1Y \underset{Y}{\times} (T^*X \otimes VY) \\
\Big\downarrow {\scriptstyle (\pi_{J_1Y}, TД)} & & & & \Big\uparrow {\scriptstyle (\mathrm{id}_{J_1Y} \times (\mathrm{id}_{T^*X} \otimes \vartheta))} \\
J_1Y \underset{Y}{\times} T(T^*X \otimes TY) & & \xrightarrow{(\mathrm{id}_{J_1Y} \times \nu_{K \otimes \Lambda^*})} & & J_1Y \underset{Y}{\times} (T^*X \otimes TY)
\end{array}$$

turns out to be a connection on the bundle $\pi^1_0 : J_1Y \to Y$. Moreover, we have the coordinate expression

$$\Gamma_A{}^i_\lambda = K_A{}^i{}_j x^j_\lambda + K_A{}^i{}_\lambda - x^i_\mu (K_A{}^\mu{}_j x^j_\lambda + K_A{}^\mu{}_\lambda) \, , \tag{3.3}$$

i.e. the connection Γ is independent of Λ.

Thus, we have obtained a natural operator

$$\chi : K \mapsto \Gamma$$

transforming linear connections on TY into connections on J_1Y.

Proof. It can be proved in coordinates by using (2.3), (1.6) and (3.2). □

Lemma 3.1. If (K, Λ) are fibre preserving, then the induced connection $\chi(K)$ on J_1Y is affine.

Proof. From the coordinate expression (3.3), for a pair of fibre preserving connections K and Λ, we get

$$\Gamma_A{}^i_\lambda = (\delta^\mu_\lambda K_A{}^i{}_j - \delta^i_j K_A{}^\mu{}_\lambda) x^j_\mu + K_A{}^i{}_\lambda \, , \tag{3.4}$$

where we put $K_k{}^\mu{}_\lambda = 0$ and $K_\nu{}^\mu{}_\lambda = \Lambda_\nu{}^\mu{}_\lambda$. □

Remark 3.1. *In Galilei relativistic theory* [1], [2], [3], *the base manifold (time) is assumed to be 1-dimensional and affine. A linear connection on space–time is said to be time–preserving if it is projectable on the canonical flat connection on the base.* (3.4) *then implies that the relation between time–preserving linear connections on space–time and affine connections on its 1-jet bundle is bijective. But for* $\dim X > 1$ *and the flat connection on an affine base manifold this relation is not one-to-one.*

4. Curvature

The curvatures of a linear connection K on TY and of a connection Γ on J_1Y are, respectively, the 2–forms

$$R_K = \frac{1}{2}[K,K] : TY \to \wedge^2 T^*Y \underset{Y}{\otimes} TY\,,$$
$$R_\Gamma = \frac{1}{2}[\Gamma,\Gamma] : J_1Y \to \wedge^2 T^*Y \underset{Y}{\otimes} (T^*X \underset{Y}{\otimes} VY)\,,$$

with coordinate expressions

$$R_K = (R_K)_{AB}{}^C{}_D \dot{x}^D d^A \wedge d^B \otimes \partial_C = = (\frac{\partial K_B{}^C{}_D}{\partial x^A} + K_A{}^E{}_D K_B{}^C{}_E)\dot{x}^D d^A \wedge d^B \otimes \partial_C \tag{4.1}$$

and

$$R_\Gamma = (R_\Gamma)_{AB}{}^i{}_\lambda d^A \wedge d^B \otimes d^\lambda \otimes \partial_i = = (\frac{\partial \Gamma_B{}^i{}_\lambda}{\partial x^A} + \Gamma_A{}^j{}_\mu \frac{\partial \Gamma_B{}^i{}_\lambda}{\partial x^j_\mu}) d^A \wedge d^B \otimes d^\lambda \otimes \partial_i\,, \tag{4.2}$$

respectively.

Theorem 4.1. If Γ is the connection on J_1Y induced by a linear connection K on TY, then we have

$$R_\Gamma = \vartheta \circ R_K \circ \text{д}\,,$$

according to the following commutative diagram

$$\begin{array}{ccc} J_1Y \underset{Y}{\times} T^*X \underset{Y}{\otimes} TY & \xrightarrow{\ \mathrm{id}_{J_1Y} \times (\mathrm{id}_{T^*X} \otimes R_K)\ } & J_1Y \underset{Y}{\times} (T^*X \underset{Y}{\otimes} \wedge^2 T^*Y \underset{Y}{\otimes} TY) \\ \big\uparrow{\scriptstyle (\mathrm{id}_{J_1Y},\,\text{д})} & & \big\downarrow{\scriptstyle \vartheta} \\ J_1Y & \xrightarrow{\ R_\Gamma\ } & \wedge^2 T^*Y \underset{Y}{\otimes} (T^*X \underset{Y}{\otimes} VY) \end{array}$$

i.e. in coordinates

$$(R_\Gamma)_{AB}{}^i{}_\lambda = (R_K)_{AB}{}^i{}_j x^j_\lambda + (R_K)_{AB}{}^i{}_\lambda - x^i_\mu\big((R_K)_{AB}{}^\mu{}_j x^j_\lambda + (R_K)_{AB}{}^\mu{}_\lambda\big)\,.$$

Proof. It can be proved by using (3.3), (4.1) and (4.2). □

5. Main theorem

Let us denote by $G^k_{(n,m)}$, $k \geq 0$, the group of k-order jets of diffeomorphisms of $\mathbb{R}^{n+m}$ which preserve the origin and the fibration $\mathbb{R}^{n+m} \to \mathbb{R}^n$, i.e. $G^k_{(n,m)}$ is the subgroup in G^k_{n+m} given by $a^\lambda_{iA_1\dots A_r} = 0$, $r = 0, \dots, k-1$. We have the canonical group homomorphism $\pi^l_k : G^l_{(n,m)} \to G^k_{(n,m)}$, $l > k$, and we denote by $K^{(l,k)}_{(n,m)}$ its kernel.

Let us denote by $Q = \mathbb{R}^m \otimes \mathbb{R}^{n*} \times \mathbb{R}^{n+m} \otimes \otimes^2 \mathbb{R}^{(n+m)*}$ the $G^2_{(n,m)}$-space with coordinates $(x^i_\lambda, K_B{}^A{}_C)$ and the left action of the group $G^2_{(n,m)}$ given by

$$\bar{x}^i_\lambda = a^i_j x^j_\mu \tilde{a}^\mu_\lambda + a^i_\mu \tilde{a}^\mu_\lambda \,,$$
$$\bar{K}_B{}^A{}_C = a^A_L K_M{}^L{}_N \tilde{a}^M_B \tilde{a}^N_C + a^A_L \tilde{a}^L_{BC} \,.$$

Let us denote by $\tilde{Q} = \mathbb{R}^m \otimes \mathbb{R}^{n*} \times \mathbb{R}^{n+m} \otimes \wedge^2 \mathbb{R}^{(n+m)*}$ the $G^1_{(n,m)}$-space with coordinates $(x^i_\lambda, T_B{}^A{}_C)$ and the tensor action of the group $G^1_{(n,m)}$. We denote by $tor : Q \to \tilde{Q}$ the $G^2_{(n,m)}$-equivariant mapping given by the antisymmetrisation of subindices $K_B{}^A{}_C$, i.e.

$$x^i_\lambda = x^i_\lambda \,, \quad T_B{}^A{}_C = 1/2(K_B{}^A{}_C - K_C{}^A{}_B) \,.$$

Let us consider the space $S = \mathbb{R}^{(n+m)*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^m$ with coordinates $(u_A{}^i{}_\lambda)$ and the action of the group $G^1_{(n,m)}$ given by

$$\bar{u}_A{}^i{}_\lambda = a^i_j u_B{}^j{}_\mu \tilde{a}^B_A \tilde{a}^\mu_\lambda \,.$$

Lemma 5.1. All $G^2_{(n,m)}$-equivariant mappings from Q to S are of the form

$$\begin{aligned} u_A{}^i{}_\lambda = {} & k_1 (T_A{}^i{}_j x^j_\lambda + T_A{}^i{}_\lambda - x^i_\mu T_A{}^\mu{}_j x^j_\lambda - x^i_\mu T_A{}^\mu{}_\lambda) \\ & + k_2 (\delta^i_A T_D{}^D{}_j x^j_\lambda + \delta^i_A T_D{}^D{}_\lambda - x^i_\mu \delta^\mu_A T_D{}^D{}_j x^j_\lambda - x^i_\mu \delta^\mu_A T_D{}^D{}_\lambda) \,, \end{aligned} \tag{5.1}$$

where $T_B{}^A{}_C = 1/2(K_B{}^A{}_C - K_C{}^A{}_B)$.

Proof. The proof uses the standard techniques of computation of $G^2_{(n,m)}$-equivariant mappings, [4], and we can divide it into three steps. We omit technical computations.

Step 1. Let $f : Q \to S$ be a $G^2_{(n,m)}$-equivariant mapping. From the equivariancy of f with respect to $K^{(2,1)}_{(n,m)}$ we get that f is of the form $f = \tilde{f} \circ tor$, where $\tilde{f} : \tilde{Q} \to S$ is a $G^1_{(n,m)}$-equivariant mapping, so it is sufficient to classify all mappings $\tilde{f}$.

Step 2. Let us denote by h_m the homotheties of $\mathbb{R}^m$. From the equivariancy of $\tilde{f}$ with respect to $(h_n \times \mathrm{id}_{\mathbb{R}^m})$ and $(\mathrm{id}_{\mathbb{R}^n} \times h_m)$ we get that $\tilde{f}$ is polynomial and any monomial is linear in $T_B{}^A{}_C$ and of maximum degree 3 in x^i_λ. Coefficients are absolute invariant tensors and we have a polynomial with 33 coefficients.

Step 3. Finally, using equivariancy with respect to diffeomorphisms $(x^\lambda, x^i) \mapsto (x^\lambda, x^i + a^i_\mu x^\mu)$, we find relations between coefficients of $\tilde{f}$ and we get (5.1). □

Theorem 5.1. All natural operations transforming a linear connection K on TY into connections on J_1Y form the following 2–parameter family

$$\chi(K) + (\mathrm{id}\otimes д^*\otimes\vartheta)(k_1 T_K + k_2 I\otimes\hat{T}_K)\,, \tag{5.2}$$

where $k_1, k_2 \in \mathbb{R}$, T_K is the torsion tensor of K, $\hat{}$ denotes the contraction and I is the identity tensor on TY.

Proof. Any natural connection on J_1Y is of the form $\chi(K) + \Phi(K)$, where Φ is an operator (over J_1Y) transforming K into a section of $T^*Y\otimes_Y T^*X\otimes_Y VY$. So it is sufficient to classify all operators Φ. The generalized Peetre theorem implies that any operator Φ is of finite order, [4], [8].

Using homogeneity conditions, [4, Proposition 25.2], we get that all finite order operators Φ are of order 0 ($\Phi(K)$ depends only on coefficients of K and not on their derivatives).

All 0-order operators Φ are in a bijective correspondence with $G^2_{(n,m)}$-equivariant mappings from Q to S and it is easy to see that the operator corresponding to the mapping of Lemma 5.1 is $(\mathrm{id}\otimes д^*\otimes\vartheta)(k_1 T_K + k_2 I\otimes\hat{T}_K)$. □

Corollary 5.1. For a torsion free connection K the connection $\chi(K)$ is the unique natural connection on J_1Y given by K. □

Another geometrical description of Theorem 5.1 is based on the following theorem, [4, Proposition 25.2].

Theorem 5.2. All natural operations transforming a linear connection K on TY into linear connections on TY form the following 3–parameter family

$$K + k_1 T_K + k_2 I\otimes\hat{T}_K + k_3 \hat{T}_K\otimes I\,,$$

where $k_1, k_2, k_3 \in \mathbb{R}$. □

Theorem 5.1 now can be interpreted by applying the operator χ on the family of connections from Theorem 5.2. Then the resulting connection on J_1Y does not depend on k_3 and it is easy to see that

$$\chi(K + k_1 T_K + k_2 I\otimes\hat{T}_K + k_3\hat{T}_K\otimes I) = \chi(K) + (\mathrm{id}\otimes д^*\otimes\vartheta)(k_1 T_K + k_2 I\otimes\hat{T}_K)\,.$$

References

1. D. Canarutto, A. Jadczyk, M. Modugno: *Quantum mechanics of a spin particle in a curved spacetime with absolute time*, Rep. on Math. Phys., **36**, 1 (1995), 95–140.
2. A. Jadczyk, M. Modugno: *An outline of a new geometric approach to Galilei general relativistic quantum mechanics*, in C.N. Yang, M.L. Ge and X. W. Zhou editors, Proc. XXI Int. Conf. on Differential geometric methods in theoretical physics, Tianjin 5-9 June 1992, 543-556, Singapore, 1992, World Scientific.
3. A. Jadczyk, M. Modugno: Galilei general relativistic quantum mechanics, book manuscript, 1994.
4. I. Kolář, P. W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer–Verlag, 1993.

5. D. Krupka, J. Janyška, *Lectures on Differential Invariants*, Folia Fac. Sci. Nat. Univ. Purkynianae Brunensis, Brno, 1990.
6. L. Mangiarotti, M. Modugno: *Connections and differential calculus on fibred manifolds*, Istituto di Matematica Applicata "G. Sansone", Università di Firenze, 1989.
7. A. Nijenhuis, *Natural bundles and their general properties*, Diff. Geom., in honour of K. Yano, Kinokuniya, Tokyo 1972, pp. 317–334.
8. J. Slovák, *On the finite order of some operators*, Proc. Conf. Diff. Geom. and Its Appl. (communications), Brno 1986, Published by J. E. Purkyně University 1987, 283-294.

Josef Janyška
Department of Mathematics, Masaryk University
Janáčkovo nám 2a, 662 95 Brno, Czech Republic
E-mail: janyska@math.muni.cz

Marco Modugno
Department of Applied Mathematics "G. Sansone"
Via S. Marta 3, 50139 Florence, Italy
E-mail: modugno@ingfi1.ing.unifi.it