Overview of Weyl–Titchmarsh theory for second order Sturm–Liouville equations on time scales

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This is joint work with prof. Roman Šimon Hilscher. It is well known that the second order linear differential equation can be divided into two cases depending on the count of its square-integrable solutions. Namely, in the *limit point case* there is exactly one (up to a multiplicative constant) square-integrable solution, and in the *limit circle case* there are two linearly independent square-integrable solutions. This dichotomy was initially investigated (by using a geometrical approach) by Weyl in his paper [6] from 1910. One of the most important contributions for the extension of this theory was made by Titchmarsh in the series of papers from 1939–1945 (especially in 1941), which were summarized in his book [5]. He re-proved Weyl's results by using an alternative method and established many properties of the fundamental function appearing in this theory, the so-called $m(\lambda)$-function. Hence in honour of the pioneers of this theory, it is called the *Weyl–Titchmarsh theory*.

The previous results were extended in many ways. First of all, there were weakened conditions put on the coefficients of the differential equation. Moreover, it was shown in Walker's paper from 1974, that every even order Sturm–Liouville differential equation can be written in the form of the linear Hamiltonian system. As a consequence, a huge attention is paid for studying the Weyl–Titchmarsh theory for continuous time linear Hamiltonian systems, especially by Clark, Everitt, Gesztesy, Hinton, Krall, and Shaw. An investigation of this theory for the second order difference equation was initiated by Hellinger and Nevanlinna in their papers from 1922. In the 20th century, the discrete theory was not investigated so intensively like the continuous theory. The discrete analogy of the results known for the linear Hamiltonian differential systems, i.e., the Weyl–Titchmarsh theory for discrete linear Hamiltonian systems, was given e.g. by Clark and Gesztesy (in 2004) and by Shi (in 2006).

It is known that discrete linear Hamiltonian systems and even order Sturm–Liouville difference equations are special cases of the discrete *symplectic* systems. Some of the results of the Weyl–Titchmarsh theory known for discrete Hamiltonian systems were given for discrete symplectic systems by Bohner and Sun in [1] and more results by the author and Clark in [2].

The last extensions of the Weyl–Titchmarsh theory are represented by dynamic equations. It was studied for linear Hamiltonian dynamic systems in [3] and, more generally, for symplectic dynamic systems in [4].

The previous overview showed that the Weyl–Titchmarsh theory is a very actual topic in the theory of differential, difference, and dynamic equations or systems. In this contribution, we present some fundamental elements of the Weyl–Titchmarsh theory for the second order dynamic equation

$$\big(r(t)x^{\Delta}\big)^{\Delta} + p(t)x^{\sigma} = \lambda w(t)\, x^{\sigma}, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{1}$$

where $a := \min \mathbb{T}$, the nonempty and closed subset of real numbers $\mathbb{T}$ is an unbounded time scale, $[a, \infty)_{\mathbb{T}} := [a, \infty) \cap \mathbb{T}$, $r(\cdot), p(\cdot), w(\cdot)$ are real functions such that $\inf_{t\in[a,b]_{\mathbb{T}}} |r(t)| > 0$ for all $b > a$, $r(\cdot), p(\cdot), w(\cdot) \in \mathrm{C}_{\mathrm{prd}}$, $w(t) > 0$ for all $t \in \mathbb{T}$, and $\lambda \in \mathbb{C}$ is a spectral parametr. An important role in this theory is played by the $m(\lambda)$-*function*, whose natural properties

$$m(\bar{\lambda}) = \overline{m(\lambda)} \quad \text{and} \quad \mathrm{Im}(\lambda) \cdot \mathrm{Im}\,(m(\lambda)) > 0$$

remain true on time scales. We construct the so-called *Weyl solution* and *Weyl disk*. We justify the terminology "disk" by its geometric properties, show explicitly the coordinates of the center of the disk, and calculate its radius. We show that the dichotomy mentioned above works in the same way (especially, that the Weyl solution is square-integrable) and present a necessary and sufficient criterion for the limit point case. Finally, we consider a nonhomogeneous problem associated with equation (1). We define the *Green function* and use it for expressing a solution of the nonhomogeneous problem.

All the presented results generalize and unify the continuous and discrete results, especially those in the original work by Titchmarsh in [5, identities (2.1.5), (2.1.6), (2.1.7), and (2.6.1)] and they are here derived as the special case of the results for symplectic dynamic system in [4].

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