On periodic solution of second-order differential equation with singularities

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Our objective will be to find sufficient conditions to ensure the existence of positive periodic solutions for a family of second order non autonomous differential equations with singular nonlinearities

$$u''(t) + f(u(t))u'(t) + g(u(t)) = h(t, u(t)) \qquad \text{for a. e. } t \in [0, \omega], \tag{1}$$

where $f, g \in C\big((0, +\infty); \mathbb{R}\big)$ and $h \in \mathrm{Kar}\left([0, \omega] \times \mathbb{R}; \mathbb{R}\right)$.

This problem has relevance from a mathematical and physical viewpoint. There are countless works on the mathematical equation (1), for example [3, 4, 5, 6]; but always under conditions of regularity of the function f. Among our contributions to a mathematical level it is included the consideration of singularities in f, g and the production of new conditions for the existence of a positive periodic solution, ever in the classical case, when the function f is continuous. In this work we present a review of the literature to this kind of equations and compare with our results.

From the physical point of view, also equation (1) becomes relevant. In 1907 Rayleigh proposed to explain the physical phenomenon that caused the deterioration of the propellers of war ships by an equation which modelates the dynamics of a bubble. Motivated by this problem, the scientist proposed new equations to give an explanation to the phenomenon. But it wasn't until 1949 when Plesset proposes his equation

$$\rho\left[R\ddot{R} + \frac{3}{2}\dot{R}^2\right] = [P_v - P_\infty(t)] + P_{g_0}\left(\frac{R_0}{R}\right)^{3k} - \frac{2S}{R} - \frac{4\mu\dot{R}}{R}. \tag{2}$$

Although it has seemingly intractable, it was a quantitative progress in the topic. The advent in 1946 of the first computer and the development of numerical methods would allow to study solutions of the equation and thus provide more or less precise explanation of this phenomenon. Since then, the use of numerical methods has been used in important works about cavitation, because this phenomenon appears ubiquitous in Nature. For the first time, we propose to do a variable change $R(t) = u^{\frac{2}{5}}(t)$ in the equation (2). The resulting equation is

$$\ddot{u} = \frac{5\left[P_v - P_\infty(t)\right]}{2\rho}u^{\frac{1}{5}} + \left(\frac{5P_{g_0}R_0^{3k}}{2\rho}\right)\frac{1}{u^{\frac{6k-1}{5}}} - \frac{5S}{u^{\frac{1}{5}}} - 4\mu\frac{\dot{u}}{u^{\frac{4}{5}}}.$$

This equation corresponds to a particular case of the equation (1) when $h(t, u) = \frac{5[p_v - p_\infty(t)]}{2\rho}u^{\frac{1}{5}}$, $g(u) = \frac{5S}{u^{\frac{1}{5}}} - \frac{\frac{5p_{g_0}R_0^{3k}}{2\rho}}{u^{\frac{6k-1}{5}}}$ and $f(u) = \frac{4\mu}{u^{\frac{1}{5}}}$. It shows the practical need to assume that the function f may have a singularity in 0.

As an application, we formulate analytical results that consolidate the physical and numerical works by some authors, see [1, 2].

The mathematical tools used in this work essencially are: *Lower and upper solutions methods and Schaeffer's fixed point theorem*.

Acknowledgement

The research was supported by projects MTM2008-02502.

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