On almost periodic sequences and functions in metric spaces

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We will study almost periodic sequences and functions which attain values in a metric space $\mathcal{M}$ with a metric ϱ. We introduce the almost periodicity in $\mathcal{M}$ using the Bohr concept; i.e. we will consider the following definition:

Definition 1. A sequence $\{\psi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{M}$ is *almost periodic* if for every $\varepsilon > 0$, there exists a positive integer $p(\varepsilon)$ such that any set consisting of $p(\varepsilon)$ consecutive integers contains at least one integer l for which

$$\varrho\left(\psi_{k+l}, \psi_k\right) < \varepsilon, \quad k \in \mathbb{Z}.$$

A *continuous* function $\psi : \mathbb{R} \to \mathcal{M}$ is *almost periodic* if for every $\varepsilon > 0$, there exists a number $p(\varepsilon) > 0$ with the property that any interval of length $p(\varepsilon)$ of the real line contains at least one point s for which

$$\varrho\left(\psi(t+s), \psi(t)\right) < \varepsilon, \quad t \in \mathbb{R}.$$

We will mention some basic properties of almost periodic sequences and functions where our process will be analogous to the theory of almost periodic functions of real variable with fuzzy real numbers as values which is developed in [1]. Then we will present modifiable methods for constructing almost periodic functions and sequences in $\mathcal{M}$ which can be found in [3], [4], respectively.

These methods facilitate to construct almost periodic sequences and functions having specific properties. Our aim is to construct almost periodic sequences and functions whose ranges contain or consist of arbitrarily given subsets of $\mathcal{M}$ satisfying certain conditions. It is known that, for any bounded countable set of real numbers, there exists an almost periodic sequence whose range is the set and which attains each value in the set periodically. We extend this result into:

Theorem 1. *Let any countable and totally bounded set $M \subseteq \mathcal{M}$ be given. There exists an almost periodic sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ satisfying*

$$\{\psi_k;\ k \in \mathbb{Z}\} = M$$

with the property that, for any $l \in \mathbb{Z}$, there exists $q(l) \in \mathbb{N}$ such that

$$\psi_l = \psi_{l+jq(l)}, \quad j \in \mathbb{Z}.$$

We add that the range of any almost periodic sequence or function in $\mathcal{M}$ is totally bounded and that any almost periodic function is uniformly continuous. Considering these facts, we formulate the below given corresponding theorems for almost periodic functions.

Theorem 2. *Let $\varphi : \mathbb{R} \to \mathcal{M}$ be any uniformly continuous function such that the set $\{\varphi(k);\ k \in \mathbb{Z}\}$ is finite and the set $\{\varphi(t);\ t \in \mathbb{R}\}$ is totally bounded. There exists an almost periodic function ψ with the property that*

$$\{\psi(k);\ k \in \mathbb{Z}\} = \{\varphi(k);\ k \in \mathbb{Z}\}, \quad \{\psi(t);\ t \in \mathbb{R}\} = \{\varphi(t);\ t \in \mathbb{R}\}$$

and that, for any $l \in \mathbb{Z}$, there exists $q(l) \in \mathbb{N}$ for which

$$\psi(l + s) = \psi(l + s + jq(l)), \quad j \in \mathbb{Z},\ s \in [0, 1).$$

As an example which illustrates the previous theorem, we mention the next statement.

Corollary 1. *For any continuous function $F : [0, 1] \to \mathcal{M}$, there exists an almost periodic function ψ with the property that*

$$\{\psi(t);\ t \in \mathbb{R}\} = \{F(t);\ t \in (0, 1)\}.$$

Theorem 3. *Let any countable and totally bounded set $M \subseteq \mathcal{M}$ be given. If all $x, y \in M$ can be connected in $\mathcal{M}$ by continuous curves which depend uniformly continuously on x and y, then there exists an almost periodic function $\psi : \mathbb{R} \to \mathcal{M}$ such that*

$$\{\psi(k);\ k \in \mathbb{Z}\} = M$$

and that, for any $l \in \mathbb{Z}$, there exists $q(l) \in \mathbb{N}$ for which

$$\psi(l + s) = \psi(l + s + jq(l)), \quad j \in \mathbb{Z},\ s \in [0, 1).$$

We remark that the connection by arcs of given values in Theorem 3 is motivated by the well-known result, called the general approximation theorem, published in [2].

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References

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