Distributional integrals

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One way to define an integral is through properties of its primitive. This is a function whose derivative is in some sense equal to the integrand. For example, if $f:\mathbb{R}\to\mathbb{R}$ then $f\in L^1$ if and only if there is an absolutely continuous primitive $F:\mathbb{R}\to\mathbb{R}$ of bounded variation such that $F'=f$ almost everywhere. For the Denjoy or Henstock–Kurzweil integral, the primitives are ACG^* functions. For the wide Denjoy integral the primitives are ACG functions and the approximate derivative is used. Each of these spaces of primitives is contained in the continuous functions on the extended real line. By using $C(\overline{\mathbb{R}})$ as the set of primitives and the distributional derivative, we obtain an integral that contains the Lebesgue, Denjoy and wide Denjoy integrals.

Denote the Schwartz distributions on the real line by $\mathcal{D}'(\mathbb{R})$. This is the topological dual of the test function space $\mathcal{D}(\mathbb{R})=C_c^\infty(\mathbb{R})$. The distributional derivative is denoted F'.

Let $\mathcal{B}_c=\{F:\mathbb{R}\to\mathbb{R}\mid F\in C^0(\overline{\mathbb{R}}), F(-\infty)=0, F(\infty)\in\mathbb{R}\}$. Then $\mathcal{B}_c$ is a Banach space under the uniform norm $\|\cdot\|_\infty$. The space of integrable distributions is then $\mathcal{A}_c=\{f\in\mathcal{D}'(\mathbb{R})\mid f=F' \text{ for } F\in\mathcal{B}_c\}$. If $f\in\mathcal{A}_c$ then the *continuous primitive integral* of f is $\int_a^b f=F(b)-F(a)$ where $F\in\mathcal{B}_c$ is the unique primitive of f and $-\infty\le a<b\le\infty$. The Alexiewicz norm of f is $\sup_{x\in\mathbb{R}}\left|\int_{-\infty}^x f\right|=\|F\|_\infty$. Under the Alexiewicz norm, $\mathcal{A}_c$ is a separable Banach space isometrically isomorphic to $\mathcal{B}_c$. It follows that $\mathcal{A}_c$ contains all functions integrable in the Lebesgue, Denjoy or wide Denjoy sense. And, $\mathcal{A}_c$ is the completion of these spaces in the Alexiewicz norm. Many of the usual properties of integrals hold. The fundamental theorem of calculus is built into the definition. The dual space and multipliers are functions of bounded variation. There is an integration by parts formula, change of variables, second mean value theorem, convergence theorems. $\mathcal{A}_c$ is a Banach lattice under the ordering $f\preceq g$ in $\mathcal{A}_c$ if and only if $F(x)\le G(x)$ for all $x\in\mathbb{R}$, where $F,G\in\mathcal{B}_c$ are the respective primitives of f,g. The space $\mathcal{A}_c$ is also a Banach algebra: $fg=(FG)'$. Convolutions and Fourier series have also been studied for this integral. See [1], [2] and [4].

A function on the real line is called *regulated* if it has a left limit and a right limit at each point and real limits at infinity. By taking the primitives to be regulated functions we obtain a Banach space of integrable distributions that contains all of those above and also all signed Radon measures. The Dirac measure is $\delta=H'$ where $H=\chi_{(0,\infty)}$ is the Heaviside step function. Since H is regulated, δ is integrable in this sense. See [3].

Other generalisations are to consider higher derivatives within the class of continuous or regulated primitives [5] and to use L^p functions as the space of primitives [6].

Acknowledgement

Supported by the Natural Sciences and Engineering Research Council of Canada.

References

[1] E. Talvila, *The distributional Denjoy integral*, Real Anal. Exchange **33**(2008), 51–82.

[2] E. Talvila, *Convolutions with the continuous primitive integral*, Abstr. Appl. Anal. (2009) Art. ID 307404, 18 pp.

[3] E. Talvila, *The regulated primitive integral*, Illinois J. Math. (in press).

[4] E. Talvila, *Fourier series with the continuous primitive integral*, (to appear).

[5] E. Talvila, *Integrals and Banach spaces for finite order distributions*, (to appear).

[6] E. Talvila, *The L^p primitive integral*, (preprint).