System of linear differential equations of second order with constant coefficients and constant delay

Zdeněk Svoboda

Brno, Czech Republic

We discuss the asymptotic properties of the solutions of the system of linear differential equations of second order with constant coefficients and constant delay in the form:

$$\ddot{x}(t) + \Omega^2 x(t-\tau) = 0, \tag{1}$$

where $\tau > 0$ is a constant, $t \geq 0$, $x \in \mathbb{R}^n$ $x(t)$ is twice continuously differentiable vector function a Ω is a nonsingular constant matrix.

For the investigation of the structure of solution of these systems are important the concepts of delayed matrix cosine of a polynomial of of the degree $2k$ which is defined as:

$$\operatorname{Cos}_\tau \Omega t := \begin{cases} \Theta, & -\infty < t < -\tau; \\ I, & -\tau \leq t < 0; \\ \vdots & \vdots \\ I - \Omega^2 \frac{t^2}{2!} + \dots (-1)^k \Omega^{2k} \frac{[t-(k-1)\tau]^{2k}}{(2k)!}, & (k-1)\tau \leq t < k\tau. \end{cases}$$

and delayed matrix sine of a polynomial of of the degree $2k+1$ for more details see [4]. Here is derived the theorem which describes the solution of (1) satisfying the initial conditions $x(t) = \varphi(t)$, $\dot{x}(t) = \dot{\varphi}(t)$ for $-\tau \leq t < 0$ in the form:

$$x(t) = (\operatorname{Cos}_\tau \Omega t)\varphi(-\tau) + \Omega^{-1}\left[(\operatorname{Sin}_\tau \Omega t)\dot{\varphi}(-\tau) + \int_{-\tau}^{0} \operatorname{Sin}_\tau \Omega(t-\tau-\xi)\ddot{\varphi}(\xi)d\xi\right].$$

Analogous theorem for nonhomogenous equation (1) is derived in this paper, too. These results are obtained by step by step method and due to it the definition of delayed matrix cosine resp. sine is given according to intervals. Analogous results are known for systems of linear differential equations with constant delay and a constant matrix and delayed exponential of matrix, for more details see [5] and for difference systems [3], too. The purpose of this contribution is the description of asymptotic properties of delayed matrix cosine resp. sine by asymptotic properties of delayed exponential matrix which was studied in [7]. Here is find the matrix such that the exponential of this matrix has the same asymptotic properties as delayed exponential of given matrix. This problem is solved in a special case. If we suppose that there is the constant matrix C such that exponential of this matrix e^{Ct} is matrix solution of the equation (1), we obtain for unknown matrix C the equation

$$C^2 + \Omega^2 e^{-C\tau} = 0.$$

The asymptotic properties of solutions corresponds to the properties of roots of the characteristic equation, for more details see [1]. This fact accords with analogy of known Euler identity and for delayed matrix cosine resp. sine and delayed exponential of given matrix hold:

$$\mathrm{Cos}_\tau\,\Omega\left(t-\frac{\tau}{2}\right)=\frac{e_{\frac{\tau}{2}}^{i\Omega t}+e_{\frac{\tau}{2}}^{-i\Omega t}}{2},\qquad \mathrm{Sin}_\tau\,\Omega(t-\tau)=\frac{e_{\frac{\tau}{2}}^{i\Omega t}-e_{\frac{\tau}{2}}^{-i\Omega t}}{2i}.$$

The eigenvalues of the matrix for which exponential has the same asymptotic properties as delayed exponential e_τ^{At} i.e. for matrix satisfying the equation:

$$C=Ae^{-C\tau}$$

is possible in special cases expressed by eigenvalues of the given matrix A and known Lambert W function , named after Johann Heinrich Lambert, see [6]. This function is defined as the inverse function of $f(w)=we^w$. The function $W(z)$ satisfying $z=W(z)e^{W(z)}$ is a multivalued (except at 0), for more details see [2].

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