On the existence and uniqueness of a slowly growing solution of singular linear functional differential systems

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This note deals with a class of linear functional differential equations which may involve singularities with respect to the independent variable. More precisely, we consider the system of linear functional differential equations

$$x_i'(t) = \sum_{k=1}^{n} p_{ik}(t) x_k(\omega_{ik}(t)) + q_i(t), \qquad t \in [a,b),\ i = 1,2,\dots,n, \tag{1}$$

subjected to the initial conditions

$$x_i(a) = \lambda_i, \qquad i = 1,2,\dots,n, \tag{2}$$

where $-\infty < a < b < \infty$ and $\{p_{ik}, q_i \mid i,k = 1,2,\dots,n\} \subset L_{1;\,\mathrm{loc}}([a,b),\mathbb{R})$. The argument deviations ω_k, $k = 1,2,\dots,n$, in (1) are arbitrary Lebesgue measurable functions that are supposed to transform the interval $[a,b)$ to itself. Similarly to [1, 2], our aim here is to find conditions sufficient for the existence and uniqueness of a slowly growing solution of the initial value problem (1), (2). The "slow growth" of a solution $x = (x_i)_{i=1}^{n} : [a,b) \to \mathbb{R}^n$ is understood in the sense that its components satisfy the conditions

$$\sup_{t \in [a,b)} h_i(t)|x_i(t)| < +\infty, \qquad i = 1,2,\dots,n, \tag{3}$$

where $h_i : [a,b) \to [0,+\infty)$, $i = 1,2,\dots,n$, are certain given continuous functions possessing the properties $\lim_{t \to b-} h_i(t) = 0$, $i = 1,2,\dots,n$. In addition, we assume that the functions $h_i : [a,b) \to [0,+\infty)$, $i = 1,2,\dots,n$ are non-increasing.

By a *solution* of the functional differential system (1), we mean a locally absolutely continuous vector function $x = (x_i)_{i=1}^{n} : [a,b) \to \mathbb{R}^n$ with components possessing the properties $h_i x_i' \in L_1\left([a,b),\mathbb{R}\right)$, $i = 1,2\dots,n$, and satisfying equalities (1) almost everywhere on the interval $[a,b)$.

Theorem 1. *Assume that the functions p_{ik}, $i,k = 1,2,\dots,n$, are non-negative almost everywhere on $[a,b)$. Moreover, assume that, for all $i,k = 1,2,\dots,n$,*

$$\int_a^b \frac{h_k(t) p_{ik}(t)}{h_k(\omega_{ik}(t))} dt < +\infty, \tag{4}$$

$$\operatorname*{ess\,sup}_{t \in [a,b)} h_k(\omega_{ik}(t)) \sum_{j=1}^{n} \int_a^{\omega_{ik}(t)} \frac{p_{kj}(s)}{h_j(\omega_{kj}(s))} ds < 1. \tag{5}$$

Then problem (1), (2), (3) *has a unique solution for arbitrary locally integrable functions $q_i : [a,b) \to \mathbb{R}$, $i = 1,2,\dots,n$, possessing the property*

$$\{h_i q_i \mid i = 1,2,\dots,n\} \subset L_1([a,b),\mathbb{R}). \tag{6}$$

and any $\{\lambda_i \mid i = 1, 2, \ldots, n\}$. *Furthermore, if* q_i *and* λ_i, $i = 1, 2, \ldots, n$, *for almost every* $t \in [a, b)$ *satisfy the condition*

$$-\sum_{k=1}^{n} \lambda_k p_{ik}(t) \le q_i(t), \qquad i = 1, 2, \ldots, n, \tag{7}$$

then the unique solution of problem (1), (2), (3) *has non-negative components.*

Note that condition (5) of Theorem 1 is unimprovable in the sense that it cannot be replaced by the corresponding non-strict inequality

$$\operatorname*{ess\,sup}_{t\in[a,b)} h_k(\omega_{ik}(t)) \sum_{j=1}^{n} \int_a^{\omega_{ik}(t)} \frac{p_{kj}(s)}{h_j(\omega_{kj}(s))} ds \le 1 \tag{8}$$

even for a single pair of indices i and k, because after such a replacement the assertion of Theorem 1 is not true any more.

Theorem 2. *Let* p_{ik}, $i, k = 1, 2, \ldots, n$, *satisfy relations* (4) *and the condition*

$$\operatorname*{ess\,sup}_{t\in[a,b)} h_k(\omega_{ik}(t)) \sum_{j=1}^{n} \int_a^{\omega_{ik}(t)} \frac{|p_{kj}(s)|}{h_j(\omega_{kj}(s))} ds < 1 \tag{9}$$

for all $i, k = 1, 2, \ldots, n$.

Then, for any locally integrable functions $q_i : [a, b) \to \mathbb{R}$, $i = 1, 2, \ldots, n$, *possessing property* (6) *and arbitrary real* λ_i, $i = 1, 2, \ldots, n$, *the initial value problem* (1), (2) *has a unique solution possessing property* (3).

It should be mentioned that, under the assumptions of the last theorem, the unique solution of problem (1), (2), (3) may not be non-negative even under condition (7). Note also that a remark similar to that on the non-strict inequality (8) is also true for the non-strict version of condition (9).

Acknowledgement

The research was supported in part by the Academy of Sciences of the Czech Republic through the Institutional Research Plan No. AV0Z10190503.

References

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