Singular second-order boundary value problem on an unbounded domain

Irena Rachůnková

Olomouc, Czech Republic

Asymptotic properties of solutions of the singular differential equation

$$(p(t)u'(t))' = p(t)f(u(t)), \quad t \in (0,\infty), \tag{1}$$

are described. The function $p : [0,\infty) \to [0,\infty)$ has a continuous first derivative which is positive on $(0,\infty)$ and $p(0) = 0$. Therefore equation (1) has a singularity at $t = 0$. The function f is supposed to be locally Lipschitz continuous on the real line, it has at least two zeros 0 and L, where $0 < L$, and $xf(x) < 0$ for $x \in (L_0, 0) \cup (0, L)$. We assume $L_0 \geq -\infty$.

Two types of boundary conditions on the positive half-line are investigated:

$$u(0) = B, \quad \lim_{t\to\infty} u(t) = L \tag{2}$$

and

$$u(0) = B, \quad \lim_{t\to\infty} u(t) = 0, \tag{3}$$

where $B \in (L_0, L]$. To prove the existence of solutions of problem (1), (2) and of problem (1), (3) in $C^1[0,\infty) \cap C^2(0,\infty)$, we introduce the following initial conditions

$$u(0) = B, \quad u'(0) = 0 \tag{4}$$

and investigate solutions of problem (1), (4).

Definition 1. Let u be a solution u of problem (1), (4). Denote $u_{sup} = \sup\{u(t) : t \in [0,\infty)\}$. If $u_{sup} < L$ ($u_{sup} = L$ or $u_{sup} > L$), then u is called a damped solution (a bounding homoclinic solution or an escape solution).

The set of all $B \in (L_0, 0)$ such that corresponding solutions of problem (1), (4) are damped (escape) solutions is denoted by $\mathcal{M}_d$ ($\mathcal{M}_e$). Conditions under which both sets $\mathcal{M}_d$ and $\mathcal{M}_e$ are open and nonempty are presented. The proofs are based on the Gronwall inequality, a priori estimates, special differential and integral inequalities and limit processes. Having such results, we get that the set $\mathcal{M}_h$=$(L_0, 0) \setminus (\mathcal{M}_d \cup \mathcal{M}_e)$ is nonempty and if $B_h \in \mathcal{M}_h$, then the corresponding solution of problem (1), (4) with $B = B_h$ satisfies $u_{sup} = L$, that is u is a bounding homoclinic solution. As a result, we give various conditions for f and p which yield the existence of all three types of solutions introduced in Definition 1.

It is shown that bounding homoclinic solutions of the initial problem (1), (4) are solutions of the boundary value problem (1), (2) and that these solutions are always increasing. Further, damped monotonous solutions of the initial problem (1), (4) are solutions of the boundary value problem (1), (3). An example demonstrating the existence of monotonous solutions of problem (1), (3) is shown here. But damped solutions can be also oscillatory and converge to zero. Then they are solutions of the boundary value problem (1), (3). Two theorems giving conditions for the existence of oscillatory solutions are presented here. Using a generalization of the Matell's theorem we derive an asymptotic formula for oscillatory solutions of problem (1), (3).

Problems of this type arise for example in hydrodynamics, in population genetics, in the homogenenous nucleation theory, or in the nonlinear field theory.

Acknowledgement

The research was supported by the Council of Czech Government MSM 6198959214

References

[1] I. Rachůnková, J. Tomeček, *Strictly increasing solutions of a nonlinear singular differential equation arising in hydrodynamics.* Nonliner Analysis 72 (2010), 2114–2118.

[2] I. Rachůnková, J. Tomeček, *Bubble-type solutions of nonlinear singular problem.* Math. Comp. Modelling 51 (2010), 658-669.

[3] I. Rachůnková, J. Tomeček, *Homoclinic solutions of singular nonautonomous second order differential equations.* Boundary Value Problems, Vol. 2009, Article ID 959636, 1-21.

[4] I. Rachůnková, J. Tomeček and J. Stryja, *Oscillatory solutions of singular equations arising in hydrodynamics.* Advances in Difference Equations, Recent Trends in Differential and Difference Equations, Vol. 2010, Article ID 872160, 1-13.