A decomposition of Henstock-Kurzweil-Pettis integrable multifunctions

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Throughout X is a Banach space and X^* is the dual space of X. $[0,1]$ is the unit interval of the real line equipped with the usual topology and the Lebesgue measure. We denote by $\mathcal{L}$ the family of all Lebesgue measurable subsets of $[0,1]$ and by $\mathcal{I}$ the collection of all closed subintervals of the interval $[0,1]$. For $E \in \mathcal{I}$ the symbol $|E|$ denotes the Lebesgue measure of E.

Definition 1. A function $f\colon [0,1] \to X$ is said to be *Henstock-Kurzweil-Pettis integrable* (or *HKP-integrable*) on $[0,1]$ if for every $x^* \in X^*$ the function x^*f is Henstock-Kurzweil integrable and for each $I \in \mathcal{I}$ there exists a vector $w_I \in X$ such that $\langle x^*, w_I\rangle = (HK)\int_I x^*f\,dt$, for every $x^* \in X^*$. w_I is called the *Henstock-Kurzweil-Pettis integral* of f over I and we set $w_I := (HKP)\int_I f\,dt$.

Definition 2. A function $f\colon [0,1] \to X$ is said to be *Henstock-Kurzweil-Pettis integrable* (or *HKP-integrable*) on $[0,1]$ if for every $x^* \in X^*$ the function x^*f is Henstock-Kurzweil integrable and for each $I \in \mathcal{I}$ there exists a vector $w_I \in X$ such that $\langle x^*, w_I\rangle = (HK)\int_I x^*f\,dt$, for every $x^* \in X^*$. w_I is called the *Henstock-Kurzweil-Pettis integral* of f over I and we set $w_I := (HKP)\int_I f\,dt$.

Throughout $cwk(X)$ will denote the family of all nonempty convex weakly compact subsets of X. For every $C \in cwk(X)$ the *support function of* C is defined for each $x^* \in X^*$ by $s(x^*, C) = \sup\{\langle x^*, x\rangle : \ x \in C\}$. A map $\Gamma : [0,1] \to cwk(X)$ is called a *multifunction*. A multifunction Γ is said to be *scalarly measurable* if for every $x^* \in X^*$, the map $s(x^*, \Gamma(\cdot))$ is measurable. Γ is said to be *scalarly integrable* (resp. *scalarly HK-integrable*) if, for every $x^* \in X^*$, the function $s(x^*, \Gamma(\cdot))$ is integrable (resp. HK-integrable). A function $f : [0,1] \to X$ is called a *selector of Γ* if $f(t) \in \Gamma(t)$, for every $t \in [0,1]$.

Definition 3. A multifunction $\Gamma : [0,1] \to cwk(X)$ is said to be *Pettis integrable* in $cwk(X)$, if Γ is scalarly integrable and for each $A \in \mathcal{L}$ there exists a set $W_A \in$

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$cwk(X)$, respectively) such that

$$s(x^*, W_A) = \int_A s(x^*, \Gamma(t))\, dt \quad \text{for all } x^* \in X^*.$$

We call W_A the *Pettis integral of* Γ over A and we set $W_A := (P)\int_A \Gamma(t)dt$.

Replacing $\mathcal{L}$ by $\mathcal{I}$ and the Lebesgue integrability by the HK-integrability, we obtain the definitionnitions of HKP-integrability of Γ.
We set then $W_I =: (HKP)\int_I \Gamma(t)dt$.

Given a multifunction Γ, the symbol $\mathcal{S}_{HKP}(\Gamma)$ denotes the family of all selectors of Γ that are HKP-integrable.

Definition 4. A scalarly measurable multifunction $\Gamma : [0,1] \to cwk(X)$ is said to be *Aumann–Henstock–Kurzweil–Pettis integrable* if $\mathcal{S}_{HKP}(\Gamma) \neq \emptyset$. Then we definitionne for each $J \in \mathcal{I}$

$$(AHKP)\int_J \Gamma(t)\, dt := \overline{\{(HKP)\int_J f(t)dt : f \in \mathcal{S}_{HKP}(\Gamma)\}}\,.$$

The following theorem is the main result of my presentation:

Theorem 1. *Let $\Gamma : [0,1] \to cwk(X)$ be a scalarly measurable multifunction. Then the following conditions are equivalent:*

1. *Γ is HKP-integrable in $cwk(X)$;*
2. *$\mathcal{S}_{HKP}(\Gamma) \neq \emptyset$ and for every $f \in \mathcal{S}_{HKP}(\Gamma)$ the multifunction $G : [0,1] \to cwk(X)$ definitionned by $\Gamma(t) = G(t) + f(t)$, is Pettis integrable in $cwk(X)$;*
3. *there exists $f \in \mathcal{S}_{HKP}(\Gamma)$ such that the multifunction $G : [0,1] \to cwk(X)$ definitionned by $\Gamma(t) = G(t) + f(t)$ is Pettis integrable in $cwk(X)$;*
4. *for each interval $I \in \mathcal{I}$, the set $(AKHP)\int_I \Gamma(t)\, dt$ belongs to $cwk(X)$ and*

$$s\left(x^*, (AHKP)\int_I \Gamma(t)\, dt\right) = (HK)\int_I s(x^*, \Gamma(t))\, dt$$

for all $x^* \in X^*$;

5. *each scalarly measurable selector of Γ is HKP-integrable.*

References

[1] L. Di Piazza and K. Musiał, *Characterizations of Kurzweil-Henstock-Pettis integrable functions*, Studia Math. 176(2006), 159-176.

[2] L. Di Piazza and K. Musiał, *A Decomposition of Henstock-Kurzweil-Pettis Integrable Multifunctions*, Operator Theory: Advances and Appl. 201(2009), 171-182. Birkhäuser Verlag Basel.