Picone type inequality for half-linear differential operators with anisotropic p-Laplacian

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The Picone identity

$$\begin{aligned}\Big[\frac{u}{v}(vru' - uRv')\Big]' &= (r-R)u'^2 + (Q-q)u^2 + R\big(u' - \frac{u}{v}v'\big)^2 \\ &\quad + \frac{u}{v}\big[v\big((ru')' + qu\big) - u\big((Rv')' + qv\big)\big]\end{aligned} \tag{1}$$

appears to be a useful tool in qualitative theory of differential equations. M. Picone [6] used this identity for a proof of Sturmian comparison theorem for linear second order ODE and other related results. This identity has been extended in several aspects to more general operators than second order linear differential operator. Picone identity is used not only to derive important results in comparison and oscillation theory of related differential equations, but can be also used to get uniqueness or nonexistence results, monotonicity of eigenvalue in domain, results for various eigenvalue problems and inequalities and other results. See [1, 2, 4, 5] for more details.

On the poster we present a suitable replacement for Picone identity in the theory of half-linear partial differential operators

$$l(u) := \operatorname{div}\Big(a(x)\,\|\nabla u\|^{p-2}\,\nabla u\Big) + c(x)|u|^{p-2}u, \tag{2}$$

$$L(u) := \operatorname{div}\Big(A(x)\,\|\nabla u\|^{p-2}\,\nabla u\Big) + C(x)|u|^{p-2}u, \tag{3}$$

with anisotropic p-Laplacian, where $\Omega \in \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$ for which the Gauss-Ostrogradskii divergence theorem holds, $a \in C^1(\overline{\Omega}, \mathbb{R}^{n\times n})$ and $A \in C^1(\overline{\Omega}, \mathbb{R}^{n\times n})$ are smooth elliptic matrix valued functions, $c \in C^{0,\alpha}(\overline{\Omega})$ and $C \in C^{0,\alpha}(\overline{\Omega})$ are Hölder continuous functions, $\operatorname{div}(\cdot)$ and ∇ are the usual divergence and nabla operators, $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^n$ and $p > 1$ is a real constant. By $\Lambda_{\max}(x)$ and $\Lambda_{\min}(x)$ we denote the maximal and minimal eigenvalues of the matrix $A(x)$ and similarly $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$ denote the maximal and minimal eigenvalues of the matrix $a(x)$.

The domain $D_l(\Omega)$ of operator l is the set of all functions $u(x) \in C^1(\overline{\Omega})$ such that $a(x)\,\|\nabla u\|^{p-2}\,\nabla u \in C^1(\Omega) \cap C(\overline{\Omega})$. In a similar way we define domain $D_L(\Omega)$ of the operator L.

The following version of Picone identity can be proved (see [3] for the proof and some cosequences).

Theorem 1. *Let $u \in D_l(\Omega)$ and $v \in D_L(\Omega)$, $v \neq 0$ on Ω. Denote*

$$K(x) = \begin{cases} \left(\frac{\Lambda_{\max}(x)}{\Lambda_{\min}(x)}\right)^{p-1} \Lambda_{\max}(x) & \text{for } p > 2, \\ \Lambda_{\max}(x) & \text{for } 1 < p \le 2. \end{cases} \tag{4}$$

The inequality

$$\begin{aligned}
&\operatorname{div}\left(\frac{u}{|v|^{p-2}v}\left[|v|^{p-2}va(x)\,\|\nabla u\|^{p-2}\,\nabla u-|u|^{p-2}uA(x)\|\nabla v\|^{p-2}\nabla v\right]\right)\\
&\geq\left[\lambda_{\min}(x)-K(x)\right]\|\nabla u\|^{p}+\left[C(x)-c(x)\right]|u|^{p}\\
&\quad+\frac{u}{|v|^{p-2}v}\left[|v|^{p-2}vl(u)-|u|^{p-2}uL(v)\right]
\end{aligned}\tag{5}$$

holds for every $x\in\Omega$. The inequality becomes equality if and only if the following conditions hold

(i) $\nabla u(x)$ *is an eigenvector of the matrix* $a(x)$ *associated with the eigenvalue* $\lambda_{\min}(x)$,

(ii) $\nabla v(x)$ *is an eigenvector of the matrix* $A(x)$ *associated with the eigenvalue* $\Lambda_{\max}(x)$,

(iii) if $p>2$ *then* $\Lambda_{\max}(x)=\Lambda_{\min}(x)$,

(iv) $u(x)$ *is a constant multiple of* $v(x)$.

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References

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