Boundary value problems on infinite intervals for second order differential equations

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Consider the second order nonlinear differential equation

$$\big(a(t)\Phi(x')\big)' + b(t)F(x) = 0, \quad (t \geq 0), \tag{1}$$

where Φ is an increasing odd homeomorphism defined on an open interval $(-\rho, \rho)$, $0 < \rho \leq \infty$, and $\mathrm{Im}\ \Phi = (-\sigma, \sigma)$, $0 < \sigma \leq \infty$, F is a real continuous nondecreasing function on $\mathbb{R}$ such that $F(u)u > 0$ for $u \neq 0$, and a, b are continuous functions for $t \geq 0$ such that $a(t) > 0$. The weight b is allowed to change its sign, that is, there exist t_1, $t_2 \geq 0$ satisfying $b(t_1)b(t_2) < 0$.

We are interested in solving the boundary value problem on the half-line associated to (1) with the boundary conditions

$$0 < \lim_{t\to\infty} x(t) < \infty, \quad \lim_{t\to\infty} x'(t) = 0, \tag{2}$$

and

$$x(0) = c > 0, \quad x(t) > 0 \quad \text{for } t \geq 0. \tag{3}$$

Kneser problem. When Φ is the classical Φ-Laplacian, i.e.,

$$\Phi(u) = \Phi_\alpha(u) = |u|^\alpha \mathrm{sgn}\, u, \ \ \alpha > 0, \tag{4}$$

and $b(t) \leq 0$ on $[0, \infty)$, it is well-known that (1) has nonnegative non-increasing solutions x such that $x(0) = c$ for any $c > 0$. These solutions are called *Kneser solutions* and are widely studied in the literature, e.g. [1, 5, 7]. Thus, the existence of these solutions, tending to a non-zero constant, is a special case of the boundary value problem (1)–(3).

Asymptotic problems. Many papers deal with the existence of eventually positive solutions of (1) satisfying (2) when b does not change its sign.

If b is positive, then asymptotic problems have been treated in [2, 3], where it has been showed that the asymptotic and oscillatory behavior of solutions of (1) is completely different in cases:

(a) $\liminf_{t\to\infty} a(t) > 0$, (b) $\liminf_{t\to\infty} a(t) = 0$, $\sigma < \infty$.

If case (a) holds and $b(t) > 0$, a necessary and sufficient condition for the existence of solutions of (1) satisfying (2) is the existence of $\mu > 0$ such that

$$J_\mu = \int_{t_0}^{\infty} \Phi^* \left(\mu \frac{1}{a(t)} \int_t^\infty b(s)ds \right) dt < \infty \tag{5}$$

where Φ^* is the inverse map of Φ.

If b is negative and Φ_α is defined by (4), then condition (5) is necessary and sufficient for the existence of solutions tending to a non-zero constant ([4]). Since Φ_α satisfies

$$\Phi_\alpha(u)\Phi_\alpha(v) = \Phi_\alpha(uv), \tag{6}$$

condition (5) can be considered with $\mu = 1$. Moreover, the asymptotics of solutions for equation (1) has been completely described in [4]. To this end, the extension of the Fubini theorem for integrals of type (5) has been stated.

BVP and Karamata functions.

It follows from above considerations, that the behavior of Φ near zero is fundamental for the problem (1), (2). Hence, we have used in [6] Karamata functions for solving the boundary value problem (1)–(3). Here we have proved some inequalities for regularly and rapidly varying functions at zero. These inequalities substitute the homogenity property (6) and enable us to apply the Tychonov fixed point theorem. To guarantee the existence of a globally positive solution on $[0, \infty)$, the assumptions on the behavior of F in a neighborhood of zero are needed.

It is worth to note that in general, condition (5) depends on the choice of μ. This is not longer case when Φ is regularly varying at zero.

Prototypes of equation (1) are the following equations:

$$\big(a(t)x'\big)' + b(t)F(x) = 0, \tag{7}$$

$$\big(a(t)\Phi_C(x')\big)' + b(t)F(x) = 0, \quad \Phi_C(u) = \frac{u}{\sqrt{1+|u|^2}}, \tag{8}$$

$$\big(a(t)\Phi_R(x')\big)' + b(t)F(x) = 0, \quad \Phi_R(u) = \frac{u}{\sqrt{1-|u|^2}}. \tag{9}$$

Equations (8) and (9) arises in studying radially symmetric solutions of partial differential equations with the mean curvature operator and the relativity operator, respectively.

For these equations the results of [6] read as follows.

Theorem 1. *Assume* $\liminf_{t\to\infty} a(t) > 0$,

$$\int_0^\infty \frac{1}{a(s)} \int_s^\infty |b(r)|\, dr\, ds < \infty,$$

and

$$\lim_{u\to 0^+} \frac{F(u)}{u} = 0.$$

Then the boundary value problems (2), (3) *associated to equations* (7), (8) *and* (9) *are solvable for any small positive c. Moreover, every solution is of bounded variation on* $[0, \infty)$.

References

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