Travelling waves

Zdeněk Pospíšil

Masaryk University, Faculty of science
Department of Mathematics and Statistics

Workshops *Finanční matematika v praxi III* **&** *Matematické modely a aplikace*

Podlesí, September 3–6, 2013

Motivation

Fisher equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \quad x \in \mathbb{R},\ t > 0$$

Fisher R.A. (1937) The wave of advance of adventagous genes. *Ann. Eugenics* **7**, 355–369

Fisher equation

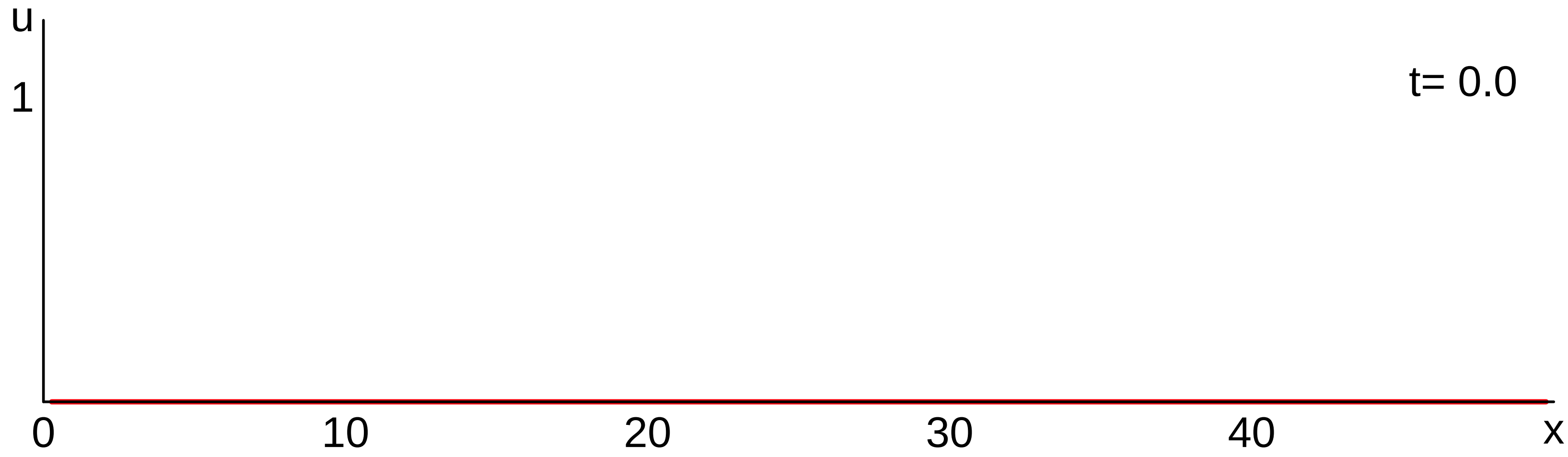
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \quad x \in \mathbb{R},\ t > 0$$

$$u(0,x) = 0.01\,\delta(x), \qquad x \in \mathbb{R}$$

Fisher equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \quad x \in \mathbb{R},\ t > 0$$

$$u(0,x) = 0.01\,\delta(x), \qquad x \in \mathbb{R}$$



Fisher equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \quad x \in \mathbb{R},\ t > 0$$

Travelling wave solution:

$$U_t(x) := u(t,x) = u\big(t_1, x - c(t - t_1)\big)$$

for all $t, x > 0$ and for some $t_1 > 0$, $c > 0$.
The wave propagates with a constant velocity c.

$$\lim_{t\to\infty} U_t(0) = 1, \qquad \lim_{x\to\infty} U_t(x) = 0 \text{ for all } t > 0.$$

Fisher equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \quad x \in \mathbb{R},\ t > 0$$

$$u(0,x) = \varphi(x), \qquad x \in \mathbb{R}$$

Fisher equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \quad x \in \mathbb{R},\ t > 0$$
$$u(0,x) = \varphi(x), \qquad\qquad x \in \mathbb{R}$$

Problems:

- possible wave speed c
- stability of the travelling wave solution
- shape of the wavefront

Fisher equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \quad x \in \mathbb{R},\ t > 0$$
$$u(0,x) = \varphi(x), \qquad x \in \mathbb{R}$$

Problems:

- possible wave speed c: $c \geq c_{\min} = 2$
- stability of the travelling wave solution:

$$\operatorname{supp}\varphi \text{ compact} \Rightarrow (\exists t_1) \lim_{t\to\infty} \sup\{|U_t(x) - u(t,x)| : x > 0\} = 0$$

- shape of the wavefront

Kolmogoroff A., Petrovsky I., Piscounoff N. (1937) Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Moscow Univ. Bull. Math.* **1**, 1–25

Fisher equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \quad x \in \mathbb{R},\ t > 0$$
$$u(0,x) = \varphi(x), \qquad x \in \mathbb{R}$$

Problems:

- possible wave speed c: $c \geq c_{\min} = 2$
- stability of the travelling wave solution:

$$\operatorname{supp}\varphi \text{ compact} \Rightarrow (\exists t_1) \lim_{t\to\infty} \sup\{|U_t(x) - u(t,x)| : x > 0\} = 0$$

- shape of the wavefront: asymptotic solution

Canosa J. (1973) On a nonlinear diffusion equation describing population growth. *IBM J. Res.&Dev.* **17**, 307–313

Time-discrete analogy

Reaction-diffusion equation: $\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x} D \dfrac{\partial u}{\partial x} + f(u)$

Time-discrete analogy

Reaction-diffusion equation: $\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x} D \dfrac{\partial u}{\partial x} + f(u)$

Interpretation:

$u = u(t, x)$ ... population density on location x at time t, i.e. population size on interval (a, b) at time t equals $\int_a^b u(t, x)\mathrm{d}x$

D ... diffusivity, rate of individual dispersal

f ... density dependent intensity of “demographic events”, i.e. intensity of “the new individuals production” (birth) and of “the old ones removal” (death)

Individuals constituting the modelled population can appear, die and move at any time.

Time-discrete analogy

Reaction-diffusion equation: $\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} D \frac{\partial u}{\partial x} + f(u)$

Modification of assumptions:

- The life cycle of the modelled population consists of two phases – the demographic and the dispersal ones – separated in time.
- Demographic events occur at time instants $t = t_1, t_2, t_3, \ldots$ and individuals move at the remaining time

Time-discrete analogy

Reaction-diffusion equation: $\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x} D \dfrac{\partial u}{\partial x} + f(u)$

Modification of assumptions:

- The life cycle of the modelled population consists of two phases – the demographic and the dispersal ones – separated in time.
- Demographic events occur at time instants $t = t_1, t_2, t_3, \ldots$ and individuals move at the remaining time

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} D \frac{\partial u}{\partial x}, \qquad && x \in \mathbb{R},\ t \in (t_i, t_{i+1}), \quad i = 1, 2, 3, \ldots \\ \lim_{t \to t_i +} u(t,x) &= f\big(u(t_i, x)\big), && x \in \mathbb{R}. \end{aligned}$$

Time-discrete analogy

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} D \frac{\partial u}{\partial x}, \qquad x \in \mathbb{R},\ t \in (t_i, t_{i+1}), \quad i = 1, 2, 3, \ldots$$

$$\lim_{t \to t_i +} u(t, x) = f\big(u(t_i, x)\big), \qquad x \in \mathbb{R}.$$

Time-discrete analogy

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} D \frac{\partial u}{\partial x}, \qquad x \in \mathbb{R},\ t \in (t_i, t_{i+1}), \quad i = 1, 2, 3, \ldots$$

$$\lim_{t \to t_i +} u(t,x) = f\big(u(t_i, x)\big), \qquad x \in \mathbb{R}.$$

Solution (for $D = const$):

$$u(t,x) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi D(t - t_i)}} \exp\left(-\frac{(x-y)^2}{4D(t-t_i)}\right) f\big(u(t_i, y)\big) \mathrm{d}y$$

Time-discrete analogy

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} D \frac{\partial u}{\partial x}, \qquad x \in \mathbb{R},\ t \in (t_i, t_{i+1}), \quad i = 1, 2, 3, \ldots$$

$$\lim_{t \to t_i+} u(t,x) = f\big(u(t_i, x)\big), \qquad x \in \mathbb{R}.$$

Solution (for $D = const$):

$$u(t,x) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi D(t - t_i)}} \exp\left(-\frac{(x-y)^2}{4D(t-t_i)}\right) f\big(u(t_i, y)\big) \mathrm{d}y$$

$$\lim_{t \to t_{i+1}-} u(t,x) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi D(t_{i+1} - t_i)}} \exp\left(-\frac{(x-y)^2}{4D(t_{i+1}-t_i)}\right) f\big(u(t_i, y)\big) \mathrm{d}y$$

Time-discrete analogy

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} D \frac{\partial u}{\partial x}, \qquad x \in \mathbb{R},\ t \in (t_i, t_{i+1}), \quad i = 1, 2, 3, \ldots$$

$$\lim_{t \to t_i +} u(t,x) = f\big(u(t_i,x)\big), \qquad x \in \mathbb{R}.$$

Solution (for $D = const$):

$$u(t,x) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi D (t - t_i)}} \exp\left(-\frac{(x-y)^2}{4D(t-t_i)}\right) f\big(u(t_i,y)\big) \mathrm{d}y$$

$$\lim_{t \to t_{i+1} -} u(t,x) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi D (t_{i+1} - t_i)}} \exp\left(-\frac{(x-y)^2}{4D(t_{i+1}-t_i)}\right) f\big(u(t_i,y)\big) \mathrm{d}y$$

In particular, for $t_i = i \in \mathbb{N}$:

$$\lim_{t \to (i+1) -} u(t,x) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi D}} \exp\left(-\frac{(x-y)^2}{4D}\right) f\big(u(i,y)\big) \mathrm{d}y$$

Time-discrete analogy

$$u(t+1,x) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi D}} \exp\left(-\frac{(x-y)^2}{4D}\right) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0, 1, 2, \ldots$$

Time-discrete analogy

$$u(t+1,x) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi D}} \exp\left(-\frac{(x-y)^2}{4D}\right) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0,1,2,\dots$$

$$k(s) := \frac{1}{2\sqrt{\pi D}} \exp\left(-\frac{s^2}{4D}\right)$$

probability density function of normally distributed random variable: "distance of individual relocation in the direction $\mathrm{sgn}(x)$ during a unit time interval".

Time-discrete analogy

$$u(t+1,x) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi D}} \exp\left(-\frac{(x-y)^2}{4D}\right) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0, 1, 2, \ldots$$

$$k(s) := \frac{1}{2\sqrt{\pi D}} \exp\left(-\frac{s^2}{4D}\right)$$

probability density function of normally distributed random variable: "distance of individual relocation in the direction $\mathrm{sgn}(x)$ during a unit time interval".

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0, 1, 2, \ldots$$

Time-discrete analogy

$$u(t+1,x) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi D}} \exp\left(-\frac{(x-y)^2}{4D}\right) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0,1,2,\dots$$

$$k(s) := \frac{1}{2\sqrt{\pi D}} \exp\left(-\frac{s^2}{4D}\right)$$

probability density function of normally distributed random variable: "distance of individual relocation in the direction $\operatorname{sgn}(x)$ during a unit time interval".

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0,1,2,\dots$$

$$u(t+1,\,\cdot\,) = k * f\big(u(t,\,\cdot\,)\big), \quad t = 0,1,2,\dots$$

Time-discrete analogy

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0,1,2,\dots$$

Kot M., Schaffer W.M. (1986) Discrete-time growth-dispersal models. *Math.Biosci.* **80**, 109–136

Time-discrete reaction-dispersion equation

The equation and the travelling wave solution

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big) \mathrm{d}y, \quad t = 0, 1, 2, 3, \dots$$

The equation and the travelling wave solution

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0,1,2,3,\dots$$

k . . . integrable function, $k \geq 0$

$$\int_{-\infty}^{\infty} k(s)\mathrm{d}s \leq 1, \quad \int_{-\infty}^{\infty} sk(s)\mathrm{d}s \geq 0, \quad (\exists \mu_0)\,(\forall \mu)\left(|\mu| \leq \mu_0 \Rightarrow \int_{-\infty}^{\infty} \mathrm{e}^{\mu s} k(s)\mathrm{d}s < \infty\right)$$

f . . . continuous function, $f \geq 0$

$$f(0) = 0$$

The equation and the travelling wave solution

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \quad t = 0,1,2,3,\dots$$

Travelling wave solution:

$$u(t,x) = u(0,x-ct)$$

The equation and the travelling wave solution

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0,1,2,3,\dots$$

Travelling wave solution:

$$u(t,x) = u(0, x-ct) =: U_t(x)$$

$$U_{t+1} = \int_{-\infty}^{\infty} k(x-y) f\big(U_t(y)\big)\mathrm{d}y$$

The equation and the travelling wave solution

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0,1,2,3,\ldots$$

Travelling wave solution:

$$u(t,x) = u(0, x-ct) =: U_t(x)$$

$$U_{t+1} = \int_{-\infty}^{\infty} k(x-y) f\big(U_t(y)\big)\mathrm{d}y$$

$$U_{t+1}(x) = u\big(0, x - c(t+1)\big) = u\big(0, (x-c) - ct\big) = U_t(x-c)$$

Wave speed

$$U_{t+1}(x) = \int_{-\infty}^{\infty} k(x-y) f\big(U_t(y)\big) \mathrm{d}y$$

Wave speed

$$U_{t+1}(x) = \int_{-\infty}^{\infty} k(x-y) f\big(U_t(y)\big)\mathrm{d}y$$

Additional assumption: $f'(0) \int_{-\infty}^{\infty} k(s)\mathrm{d}s > 1.$

Wave speed

$$U_{t+1}(x) = \int_{-\infty}^{\infty} k(x-y) f\big(U_t(y)\big) \mathrm{d}y$$

Additional assumption: $f'(0) \int_{-\infty}^{\infty} k(s) \mathrm{d}s > 1.$

Linearization of the equation ($f(U) \approx f(0) + f'(0)U = f'(0)U$):

$$U_{t+1}(x) = \int_{-\infty}^{\infty} k(x-y) f'(0) U_t(y) \mathrm{d}y$$

Wave speed

$$U_{t+1}(x) = \int_{-\infty}^{\infty} k(x-y) f\big(U_t(y)\big) \mathrm{d}y$$

Additional assumption: $f'(0) \int_{-\infty}^{\infty} k(s)\mathrm{d}s > 1$.

Linearization of the equation ($f(U) \approx f(0) + f'(0)U = f'(0)U$):

$$U_{t+1}(x) = \int_{-\infty}^{\infty} k(x-y) f'(0) U_t(y) \mathrm{d}y$$

We can try to find a solution in the form $U_t(x) = A\mathrm{e}^{-\mu x}$, where $A, \mu > 0$:

Wave speed

$$U_{t+1}(x) = \int_{-\infty}^{\infty} k(x-y) f\big(U_t(y)\big) \mathrm{d}y$$

Additional assumption: $f'(0) \int_{-\infty}^{\infty} k(s)\mathrm{d}s > 1$.

Linearization of the equation ($f(U) \approx f(0) + f'(0)U = f'(0)U$):

$$U_{t+1}(x) = \int_{-\infty}^{\infty} k(x-y) f'(0) U_t(y) \mathrm{d}y$$

We can try to find a solution in the form $U_t(x) = A\mathrm{e}^{-\mu x}$, where $A, \mu > 0$:

$$A\mathrm{e}^{-\mu(x-c)} = A f'(0) \int_{-\infty}^{\infty} k(x-y) \mathrm{e}^{-\mu y} \mathrm{d}y$$

Wave speed

$$U_{t+1}(x) = \int_{-\infty}^{\infty} k(x-y) f\big(U_t(y)\big)\mathrm{d}y$$

Additional assumption: $f'(0) \int_{-\infty}^{\infty} k(s)\mathrm{d}s > 1.$

Linearization of the equation ($f(U) \approx f(0) + f'(0)U = f'(0)U$):

$$U_{t+1}(x) = \int_{-\infty}^{\infty} k(x-y) f'(0) U_t(y)\mathrm{d}y$$

We can try to find a solution in the form $U_t(x) = A\mathrm{e}^{-\mu x}$, where $A, \mu > 0$:

$$\mathrm{e}^{\mu c} = f'(0) \int_{-\infty}^{\infty} k(x-y)\mathrm{e}^{-\mu(y-x)}\mathrm{d}y$$

Wave speed

$$U_{t+1}(x) = \int_{-\infty}^{\infty} k(x-y) f\big(U_t(y)\big) \mathrm{d}y$$

Additional assumption: $f'(0) \int_{-\infty}^{\infty} k(s) \mathrm{d}s > 1$.

Linearization of the equation ($f(U) \approx f(0) + f'(0)U = f'(0)U$):

$$U_{t+1}(x) = \int_{-\infty}^{\infty} k(x-y) f'(0) U_t(y) \mathrm{d}y$$

We can try to find a solution in the form $U_t(x) = A\mathrm{e}^{-\mu x}$, where $A, \mu > 0$:

$$\mathrm{e}^{\mu c} = f'(0) \int_{-\infty}^{\infty} k(x-y) \mathrm{e}^{-\mu(y-x)} \mathrm{d}y$$

Denote $F(\mu) = f'(0) \int_{-\infty}^{\infty} k(x-y) \mathrm{e}^{\mu(x-y)} \mathrm{d}y = f'(0) \int_{-\infty}^{\infty} k(s) \mathrm{e}^{\mu s} \mathrm{d}s$

Wave speed

$$\mathrm{e}^{\mu c} = F(\mu) = f'(0) \int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s$$

Wave speed

$$e^{\mu c} = F(\mu) = f'(0) \int_{-\infty}^{\infty} k(s) e^{\mu s} \mathrm{d}s$$

$$F(0) = f'(0) \int_{-\infty}^{\infty} k(s) \mathrm{d}s > 1$$

Wave speed

$$e^{\mu c} = F(\mu) = f'(0) \int_{-\infty}^{\infty} k(s) e^{\mu s} ds$$

$$F(0) = f'(0) \int_{-\infty}^{\infty} k(s) ds > 1$$

$$F'(\mu) = f'(0) \int_{-\infty}^{\infty} s k(s) e^{\mu s} ds = f'(0) \int_{0}^{\infty} s \left(k(s) e^{\mu s} - k(-s) e^{-\mu s} \right) ds \geq$$

$$\geq f'(0) \int_{0}^{\infty} s \left(k(s) - k(-s) \right) ds = f'(0) \int_{0}^{\infty} s k(s) ds \geq 0$$

Wave speed

$$e^{\mu c} = F(\mu) = f'(0) \int_{-\infty}^{\infty} k(s) e^{\mu s} ds$$

$$F(0) = f'(0) \int_{-\infty}^{\infty} k(s) ds > 1$$

$$F'(\mu) = f'(0) \int_{-\infty}^{\infty} s k(s) e^{\mu s} ds = f'(0) \int_{0}^{\infty} s \left(k(s) e^{\mu s} - k(-s) e^{-\mu s} \right) ds \geq$$

$$\geq f'(0) \int_{0}^{\infty} s \left(k(s) - k(-s) \right) ds = f'(0) \int_{0}^{\infty} s k(s) ds \geq 0$$

$$F''(\mu) = f'(0) \int_{-\infty}^{\infty} s^2 k(s) e^{\mu s} ds \geq 0$$

Wave speed

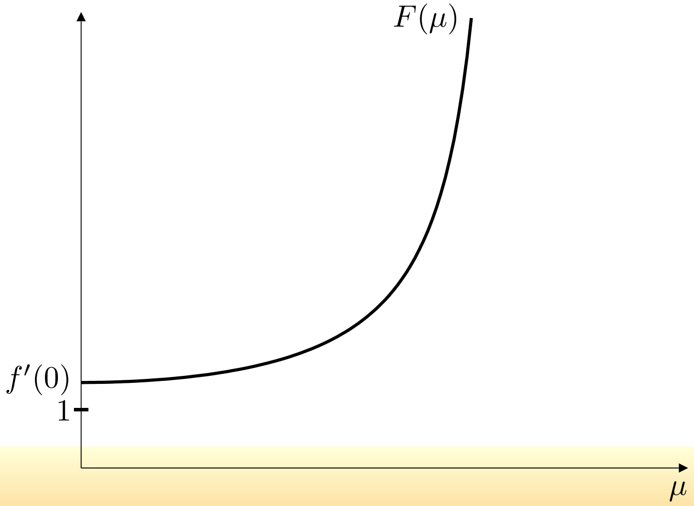
$$e^{\mu c} = F(\mu) = f'(0) \int_{-\infty}^{\infty} k(s) e^{\mu s} ds$$

$$F(0) = f'(0) \int_{-\infty}^{\infty} k(s) ds > 1$$

$$F'(\mu) = f'(0) \int_{-\infty}^{\infty} s k(s) e^{\mu s} ds = f'(0) \int_{0}^{\infty} s \left(k(s) e^{\mu s} - k(-s) e^{-\mu s}\right) ds \geq$$

$$\geq f'(0) \int_{0}^{\infty} s \left(k(s) - k(-s)\right) ds = f'(0) \int_{0}^{\infty} s k(s) ds \geq 0$$

$$F''(\mu) = f'(0) \int_{-\infty}^{\infty} s^2 k(s) e^{\mu s} ds \geq 0$$



Wave speed

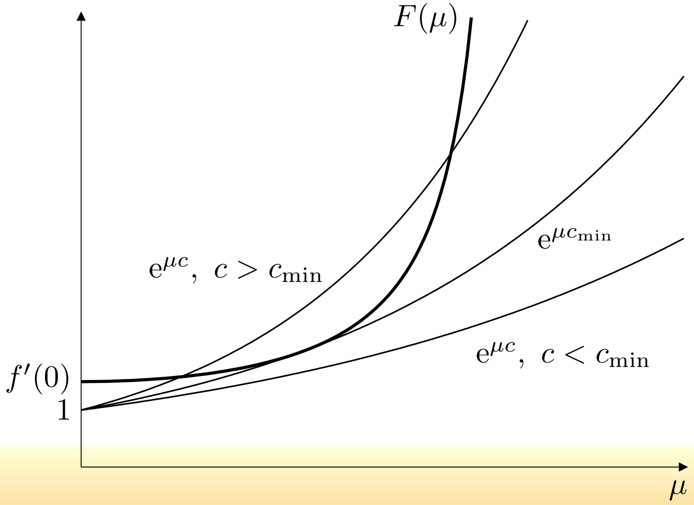
$$\mathrm{e}^{\mu c} = F(\mu) = f'(0) \int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s$$

$$F(0) = f'(0) \int_{-\infty}^{\infty} k(s)\mathrm{d}s > 1$$

$$F'(\mu) = f'(0) \int_{-\infty}^{\infty} sk(s)\mathrm{e}^{\mu s}\mathrm{d}s = f'(0) \int_{0}^{\infty} s\left(k(s)\mathrm{e}^{\mu s} - k(-s)\mathrm{e}^{-\mu s}\right)\mathrm{d}s \geq$$

$$\geq f'(0) \int_{0}^{\infty} s\left(k(s) - k(-s)\right)\mathrm{d}s = f'(0) \int_{0}^{\infty} sk(s)\mathrm{d}s \geq 0$$

$$F''(\mu) = f'(0) \int_{-\infty}^{\infty} s^2k(s)\mathrm{e}^{\mu s}\mathrm{d}s \geq 0$$



Wave speed

The minimal wave speed $c = c_{\min}$ is the solution of the system of equations

$$\mathrm{e}^{\mu c} = f'(0) \int\limits_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s, \qquad c\mathrm{e}^{\mu c} = f'(0) \int\limits_{-\infty}^{\infty} sk(s)\mathrm{e}^{\mu s}\mathrm{d}s$$

Wave speed

The minimal wave speed $c = c_{\min}$ is the solution of the system of equations

$$\mathrm{e}^{\mu c} = f'(0) \int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s, \qquad c\mathrm{e}^{\mu c} = f'(0) \int_{-\infty}^{\infty} sk(s)\mathrm{e}^{\mu s}\mathrm{d}s$$

Hence, μ solves the equation

$$\exp\left\{\mu \frac{\int_{-\infty}^{\infty} sk(s)\mathrm{e}^{\mu s}\mathrm{d}s}{\int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s}\right\} = f'(0) \int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s$$

and

$$c_{\min} = \frac{\int_{-\infty}^{\infty} sk(s)\mathrm{e}^{\mu s}\mathrm{d}s}{\int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s}$$

Wave speed

The minimal wave speed $c = c_{\min}$ is the solution of the system of equations

$$\mathrm{e}^{\mu c} = f'(0) \int\limits_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s, \qquad c\mathrm{e}^{\mu c} = f'(0) \int\limits_{-\infty}^{\infty} sk(s)\mathrm{e}^{\mu s}\mathrm{d}s$$

Equivalently

$$c_{\min} = \min_{\mu>0} \left\{ \frac{1}{\mu} \ln \left(f'(0) \int\limits_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s \right) \right\}$$

Kot M. (1992) Discrete-time travelling waves: Ecological examples. *J.Math.Biol.* **30**, 413–436

Existence and stability of the travelling wave solution

$$u(t+1,x)=\int\limits_{-\infty}^{\infty} k(x-y)f\big(u(t,y)\big)\mathrm{d}y, \quad t=0,1,2,\dots,\ x\in\mathbb{R}$$

Existence and stability of the travelling wave solution

$$u(t+1,x)=\int_{-\infty}^{\infty}k(x-y)f\big(u(t,y)\big)\mathrm{d}y,\quad t=0,1,2,\dots,\ x\in\mathbb{R}$$

Theorem 1. Let $f\in C^2[0,\infty)$ and there exist $K>0$, $\varrho\in(0,1)$ such that

$$f(0)=0, f(K)=K,\quad 0\le f'(u)\le f'(0) \text{ for } u\in[0,K],$$

$$f'(u)<1 \text{ for } u\in[K-\varrho,K].$$

Let $c=c_{\min}$, μ be numbers defined in the previous slide. Then there exists a solution $U_t(x)=u(t,x)$ of the equation such that

$$\lim_{x\to-\infty}U_t(x)=K,\quad \lim_{x\to\infty}U_t(x)\mathrm{e}^{\mu x}=\varrho,\quad U_{t+1}(x)=U_t(x-ct).$$

Existence and stability of the travelling wave solution

$$
\begin{aligned}
u(t+1,x) &= \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big) \mathrm{d}y, && t = 0, 1, 2, \dots, \ x \in \mathbb{R} \\
u(0,x) &= \varphi(x), && x \in \mathbb{R}
\end{aligned}
$$

Existence and stability of the travelling wave solution

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0,1,2,\ldots, \; x \in \mathbb{R}$$
$$u(0,x) = \varphi(x), \qquad x \in \mathbb{R}$$

Theorem 2. Let the assumptions of Theorem 1 hold. Let the function φ satisfy

$$\varphi(x) \geq 0 \text{ for } x \in \mathbb{R}, \qquad \lim_{|x|\to\infty} \varphi(x)\mathrm{e}^{-\mu|x|} < \varrho.$$

Let $U_t(\,\cdot\,)$ be the solution of the equation introduced in Theorem 1 and $u = u(t,x)$ be the solution of initial value problem.
Then there exists $x_0 \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \sup_{x>0} \left| \frac{u(t,x)}{U_t(x-x_0)} - 1 \right| = 0.$$

Existence and stability of the travelling wave solution

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0,1,2,\dots, \; x \in \mathbb{R}$$
$$u(0,x) = \varphi(x), \quad x \in \mathbb{R}$$

Theorem 2. Let the assumptions of Theorem 1 hold. Let the function φ satisfy

$$\varphi(x) \geq 0 \text{ for } x \in \mathbb{R}, \qquad \lim_{|x|\to\infty} \varphi(x)\mathrm{e}^{-\mu|x|} < \varrho.$$

Let $U_t(\,\cdot\,)$ be the solution of the equation introduced in Theorem 1 and $u = u(t,x)$ be the solution of initial value problem.
Then there exists $x_0 \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \sup_{x>0} \left| \frac{u(t,x)}{U_t(x-x_0)} - 1 \right| = 0.$$

Lin G., Li WT., Ruan SG. (2010) Asymptotic stability of monostable wavefronts in discrete-time integral recursions. *Sci.China.Math.* **53**, 1185–1194

Special cases

Functions f and k

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y$$

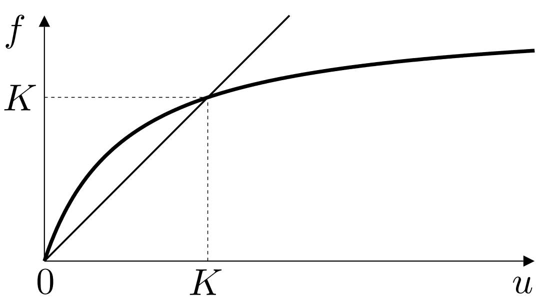
Functions f and k

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y$$

Particular reaction terms:

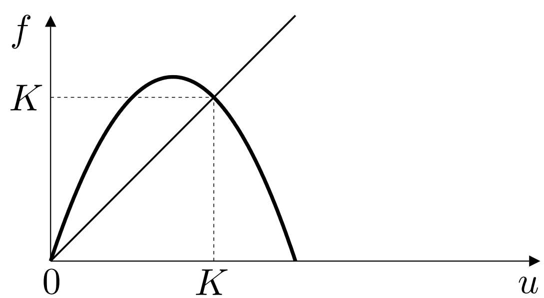
$$f(u) = \frac{\lambda u}{1+(\lambda-1)u/K}$$

Beverton-Holt



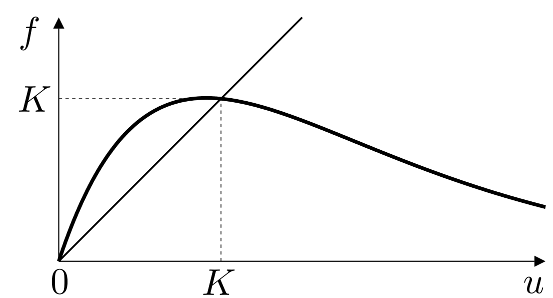
$$f(u) = (1+r)u - \frac{r}{K}u^2$$

logistic



$$f(u) = u \exp\left\{r\left(1-\frac{u}{K}\right)\right\}$$

Ricker



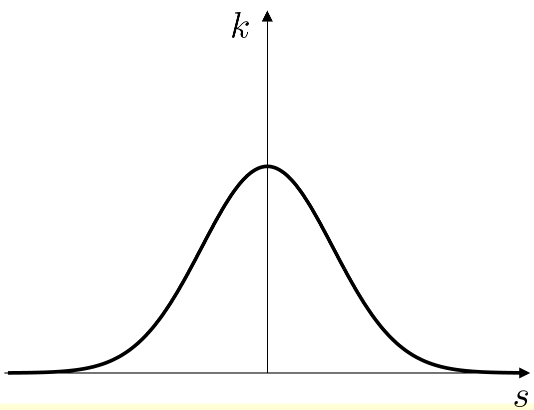
Functions f and k

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y$$

Particular dispersion kernels:

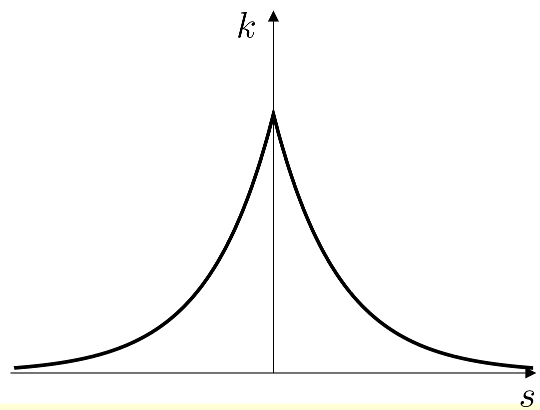
$$k(s) = \frac{1}{2\sqrt{\pi D}} \exp\left\{-\frac{s^2}{4D}\right\}$$

Gauss
normal



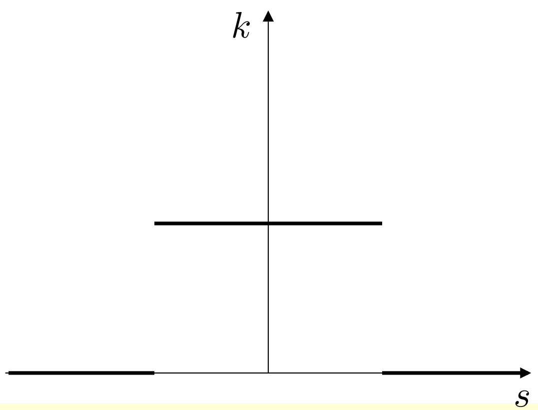
$$k(s) = \tfrac{1}{2}\alpha \mathrm{e}^{-\alpha|s|}$$

Laplace
double exponential



$$k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

top-hat
uniform



Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed:

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed:

$$\int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \tfrac{1}{2}\alpha \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|s|}\mathrm{e}^{\mu s}\mathrm{d}s = \frac{\alpha^2}{\alpha^2-\mu^2} \quad \text{for } 0<\mu<\alpha$$

$$\lim_{\mu\to\alpha-} \int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \infty$$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed:

$$\int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \tfrac{1}{2}\alpha \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|s|}\mathrm{e}^{\mu s}\mathrm{d}s = \frac{\alpha^2}{\alpha^2-\mu^2} \quad \text{for } 0<\mu<\alpha$$

$$\lim_{\mu\to\alpha-} \int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \infty$$

Hence $c = \min\limits_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln\left(f'(0)\frac{\alpha^2}{\alpha^2-\mu^2} \right) \right\}$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min\limits_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln\left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln \left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Shape of wavefront:

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min\limits_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln \left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Shape of wavefront:

$$U_t(x-c) = \int_{-\infty}^{\infty} k(x-y) f\big(U_t(y)\big)\,\mathrm{d}y$$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min\limits_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln \left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Shape of wavefront:

$$U_t(x-c) = \frac{1}{2}\alpha \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|x-y|} f\big(U_t(y)\big)\,\mathrm{d}y$$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln \left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Shape of wavefront:

$$U(x-c) = \frac{1}{2}\alpha \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|x-y|} f\big(U(y)\big)\mathrm{d}y$$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min\limits_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln\left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Shape of wavefront:

$$U(x-c) = \frac{1}{2}\alpha \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|x-y|} f\big(U(y)\big)\,\mathrm{d}y$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} U(x-c) = \alpha^2 \Big(U(x-c) - f\big(U(x)\big) \Big)$$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min\limits_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln \left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Shape of wavefront:

$$U(x-c) = \frac{1}{2}\alpha \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|x-y|} f\big(U(y)\big)\mathrm{d}y$$

$$\frac{1}{\alpha^2} U''(x-c) + f\big(U(x)\big) - U(x-c) = 0$$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln \left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Shape of wavefront:

$$U(x-c) = \frac{1}{2}\alpha \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|x-y|} f\big(U(y)\big)\mathrm{d}y$$

$$\frac{1}{\alpha^2} U''(x-c) + f\big(U(x)\big) - U(x-c) = 0$$

Transformation: $z = \frac{x}{c}$, $V(z) = U(cz)$, $\varepsilon = \left(\frac{1}{\alpha\varepsilon}\right)^2$

Laplace dispersion

$$u(t+1,x) = \int\limits_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min\limits_{0<\mu<\alpha} \left\{ \dfrac{1}{\mu} \ln\left(f'(0) \dfrac{\alpha^2}{\alpha^2-\mu^2} \right) \right\}$

Shape of wavefront:

$$U(x-c) = \frac{1}{2}\alpha \int\limits_{-\infty}^{\infty} \mathrm{e}^{-\alpha|x-y|} f\big(U(y)\big)\mathrm{d}y$$

$$\frac{1}{\alpha^2} U''(x-c) + f\big(U(x)\big) - U(x-c) = 0$$

Transformation: $z = \dfrac{x}{c}$, $V(z) = U(cz)$, $\varepsilon = \left(\dfrac{1}{\alpha\varepsilon}\right)^2$

$$\varepsilon V''(z-1) + f\big(V(z)\big) - V(z-1) = 0$$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln\left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Shape of wavefront:

$$\varepsilon V''(z-1) + f\big(V(z)\big) - V(z-1) = 0$$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min\limits_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln\left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Shape of wavefront:

$$\varepsilon V''(z-1) + f\big(V(z)\big) - V(z-1) = 0$$

Let V_0 be the solution of the equation with $\varepsilon = 0$: $V_0(z-1) = f\big(V_0(z)\big)$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min\limits_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln\left(f'(0)\frac{\alpha^2}{\alpha^2-\mu^2} \right) \right\}$

Shape of wavefront:

$$\varepsilon V''(z-1) + f\big(V(z)\big) - V(z-1) = 0$$

Let V_0 be the solution of the equation with $\varepsilon = 0$: $V_0(z-1) = f\big(V_0(z)\big)$

Suppose $V(z) = V_0(z) + \sum\limits_{i=1}^{\infty} \varepsilon^i W_i(z)$

Put $V_j(z) = V_0(z) + \sum\limits_{i=1}^{j} \varepsilon^i W_i(z) = V_{j-1}(z) + \varepsilon^j W_j(z)$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min\limits_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln\left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Shape of wavefront:

$$\varepsilon V''(z-1) + f\big(V(z)\big) - V(z-1) = 0$$

Let V_0 be the solution of the equation with $\varepsilon = 0$: $V_0(z-1) = f\big(V_0(z)\big)$

Suppose $V(z) = V_0(z) + \sum\limits_{i=1}^{\infty} \varepsilon^i W_i(z)$

Put $V_j(z) = V_0(z) + \sum\limits_{i=1}^{j} \varepsilon^i W_i(z) = V_{j-1}(z) + \varepsilon^j W_j(z)$

Then $V(z) = V_j(z) + o(\varepsilon^j) = V_{j-1}(z) + \varepsilon^j W_j(z) + o(\varepsilon^j)$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big) \mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln\left(f'(0) \frac{\alpha^2}{\alpha^2-\mu^2} \right) \right\}$

Shape of wavefront:

$$\varepsilon V''(z-1) + f\big(V(z)\big) - V(z-1) = 0$$

Let V_0 be the solution of the equation with $\varepsilon = 0$: $V_0(z-1) = f\big(V_0(z)\big)$

Suppose $V(z) = V_0(z) + \sum_{i=1}^{\infty} \varepsilon^i W_i(z)$

Put $V_j(z) = V_0(z) + \sum_{i=1}^{j} \varepsilon^i W_i(z) = V_{j-1}(z) + \varepsilon^j W_j(z)$

Then $V(z) = V_j(z) + o(\varepsilon^j) = V_{j-1}(z) + \varepsilon^j W_j(z) + o(\varepsilon^j)$

$O(\varepsilon)$: $W_1(z-1) = V_0''(z) + f'\big(V_0(z)\big) W_1(z)$

Laplace dispersion

$$u(t+1,x)=\int_{-\infty}^{\infty} k(x-y)f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s)=\frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c=\min_{0<\mu<\alpha}\left\{\frac{1}{\mu}\ln\left(f'(0)\frac{\alpha^2}{\alpha^2-\mu^2}\right)\right\}$

Shape of wavefront:

$$\varepsilon V''(z-1)+f\big(V(z)\big)-V(z-1)=0$$

Let V_0 be the solution of the equation with $\varepsilon=0$: $V_0(z-1)=f\big(V_0(z)\big)$

Suppose $V(z)=V_0(z)+\sum_{i=1}^{\infty}\varepsilon^i W_i(z)$

Put $V_j(z)=V_0(z)+\sum_{i=1}^{j}\varepsilon^i W_i(z)=V_{j-1}(z)+\varepsilon^j W_j(z)$

Then $V(z)=V_j(z)+o(\varepsilon^j)=V_{j-1}(z)+\varepsilon^j W_j(z)+o(\varepsilon^j)$

$O(\varepsilon)$: $W_1(z-1)=V_0''(z)+f'\big(V_0(z)\big)W_1(z)$
$O(\varepsilon^j)$: $W_j(z-1)=V_{j-1}''(z)+f'\big(V_{j-1}(z)\big)W_j(z)$, $j=1,2,3,\dots$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min\limits_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln\left(f'(0) \frac{\alpha^2}{\alpha^2-\mu^2} \right) \right\}$

Shape of wavefront:

$$\varepsilon V''(z-1) + f\big(V(z)\big) - V(z-1) = 0$$

Let V_0 be the solution of the equation with $\varepsilon = 0$: $V_0(z-1) = f\big(V_0(z)\big)$

Suppose $V(z) = V_0(z) + \sum\limits_{i=1}^{\infty} \varepsilon^i W_i(z)$

Put $V_j(z) = V_0(z) + \sum\limits_{i=1}^{j} \varepsilon^i W_i(z) = V_{j-1}(z) + \varepsilon^j W_j(z)$

$O(\varepsilon^j)$: $W_j(z-1) = V''_{j-1}(z) + f'\big(V_{j-1}(z)\big) W_j(z)$, $j = 1, 2, 3, \ldots$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min\limits_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln\left(f'(0) \frac{\alpha^2}{\alpha^2-\mu^2} \right) \right\}$

Shape of wavefront:

$$\varepsilon V''(z-1) + f\big(V(z)\big) - V(z-1) = 0$$

Let V_0 be the solution of the equation with $\varepsilon = 0$: $V_0(z-1) = f\big(V_0(z)\big)$

Suppose $V(z) = V_0(z) + \sum\limits_{i=1}^{\infty} \varepsilon^i W_i(z)$

Put $V_j(z) = V_0(z) + \sum\limits_{i=1}^{j} \varepsilon^i W_i(z) = V_{j-1}(z) + \varepsilon^j W_j(z)$

$O(\varepsilon^j)$: $W_j(z-1) = V''_{j-1}(z) + f'\big(V_{j-1}(z)\big) W_j(z)$, $j = 1, 2, 3, \ldots$

We can put $V_0(z) = \eta$ and compute $V_0(z-1), V_0(z-2), \ldots$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln \left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Shape of wavefront:

$$\varepsilon V''(z-1) + f\big(V(z)\big) - V(z-1) = 0$$

Let V_0 be the solution of the equation with $\varepsilon = 0$: $V_0(z-1) = f\big(V_0(z)\big)$

Suppose $V(z) = V_0(z) + \sum_{i=1}^{\infty} \varepsilon^i W_i(z)$

Put $V_j(z) = V_0(z) + \sum_{i=1}^{j} \varepsilon^i W_i(z) = V_{j-1}(z) + \varepsilon^j W_j(z)$

$O(\varepsilon^j)$: $W_j(z-1) = V''_{j-1}(z) + f'\big(V_{j-1}(z)\big)W_j(z)$, $j = 1, 2, 3, \ldots$

We can put $V_0(z) = \eta$ and compute $V_0(z-1), V_0(z-2), \ldots$

Then put $W_1(z) = 0$ and compute $W_1(z-1), W_1(z-2), \ldots,$

Laplace dispersion

$$u(t+1,x)=\int\limits_{-\infty}^{\infty} k(x-y)f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s)=\frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c=\min\limits_{0<\mu<\alpha}\left\{\frac{1}{\mu}\ln\left(f'(0)\frac{\alpha^2}{\alpha^2-\mu^2}\right)\right\}$

Shape of wavefront:

$$\varepsilon V''(z-1)+f\big(V(z)\big)-V(z-1)=0$$

Let V_0 be the solution of the equation with $\varepsilon=0$: $V_0(z-1)=f\big(V_0(z)\big)$

Suppose $V(z)=V_0(z)+\sum\limits_{i=1}^{\infty}\varepsilon^i W_i(z)$

Put $V_j(z)=V_0(z)+\sum\limits_{i=1}^{j}\varepsilon^i W_i(z)=V_{j-1}(z)+\varepsilon^j W_j(z)$

$O(\varepsilon^j)$: $W_j(z-1)=V''_{j-1}(z)+f'\big(V_{j-1}(z)\big)W_j(z)$, $j=1,2,3,\ldots$

We can put $V_0(z)=\eta$ and compute $V_0(z-1),V_0(z-2),\ldots$
Then put $W_1(z)=0$ and compute $W_1(z-1),W_1(z-2),\ldots,$

$$V_1(z-i),\ i=1,2,\ldots$$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln\left(f'(0) \frac{\alpha^2}{\alpha^2 - \mu^2} \right) \right\}$

Shape of wavefront:

$$\varepsilon V''(z-1) + f\big(V(z)\big) - V(z-1) = 0$$

Let V_0 be the solution of the equation with $\varepsilon = 0$: $V_0(z-1) = f\big(V_0(z)\big)$

Suppose $V(z) = V_0(z) + \sum_{i=1}^{\infty} \varepsilon^i W_i(z)$

Put $V_j(z) = V_0(z) + \sum_{i=1}^{j} \varepsilon^i W_i(z) = V_{j-1}(z) + \varepsilon^j W_j(z)$

$O(\varepsilon^j)$: $W_j(z-1) = V''_{j-1}(z) + f'\big(V_{j-1}(z)\big) W_j(z)$, $j = 1, 2, 3, \ldots$

We can put $V_0(z) = \eta$ and compute $V_0(z-1), V_0(z-2), \ldots$

Then put $W_1(z) = 0$ and compute $W_1(z-1), W_1(z-2), \ldots,$

$$V_1(z-i),\ i = 1, 2, \ldots$$

e.t.c

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Wave speed: $c = \min_{0<\mu<\alpha} \left\{ \frac{1}{\mu} \ln\left(f'(0) \frac{\alpha^2}{\alpha^2-\mu^2} \right) \right\}$

Shape of wavefront: Perturbation problem

Kot M. (1992) Discrete-time travelling waves: Ecological examples. *J.Math.Biol.* **30**, 413-436

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big) \mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Example: $\alpha = \sqrt{24}$, $f(u) = \dfrac{1.5u}{1+0.5u}$ (Beverton-Holt)

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big) \mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\, \mathrm{e}^{-\alpha|s|}$$

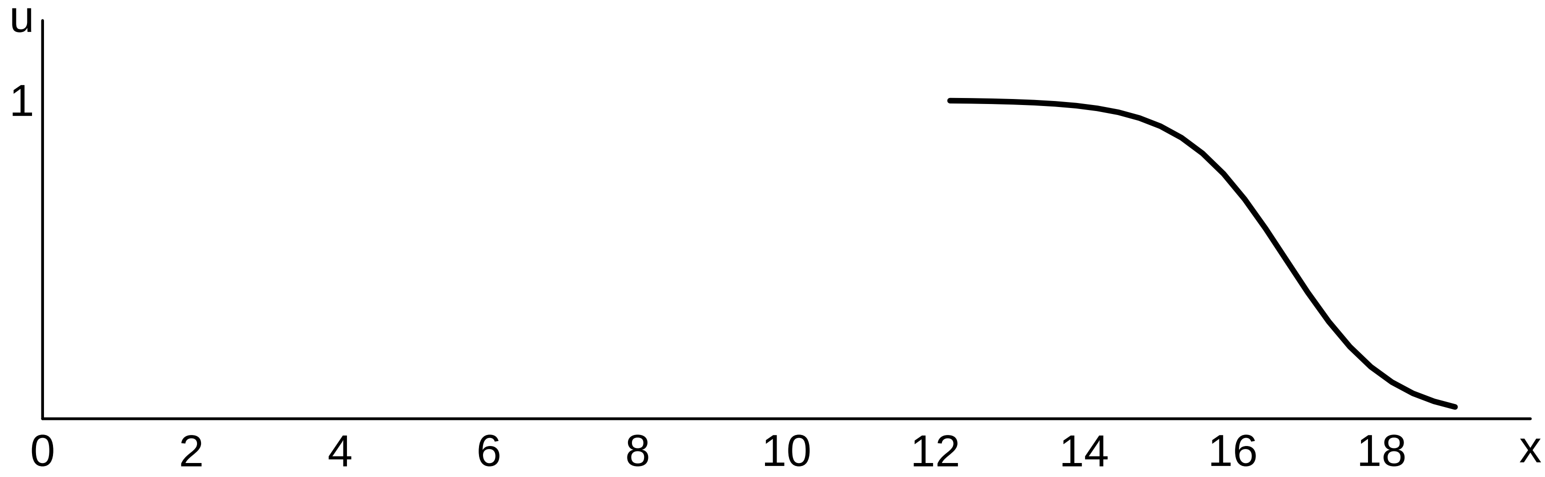
Example: $\alpha = \sqrt{24}$, $f(u) = \dfrac{1.5u}{1+0.5u}$ (Beverton-Holt); $c = 0.2825$

Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Example: $\alpha = \sqrt{24}$, $f(u) = \dfrac{1.5u}{1+0.5u}$ (Beverton-Holt); $c = 0.2825$

V_0

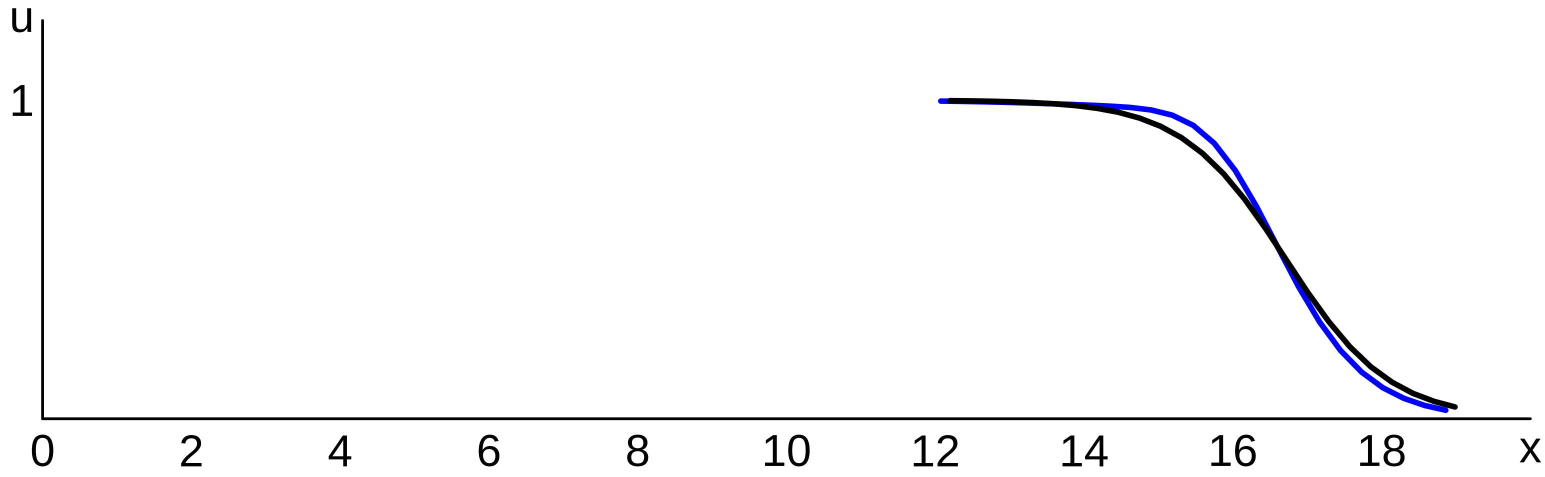


Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Example: $\alpha = \sqrt{24}$, $f(u) = \dfrac{1.5u}{1+0.5u}$ (Beverton-Holt); $c = 0.2825$

V_0, V_1

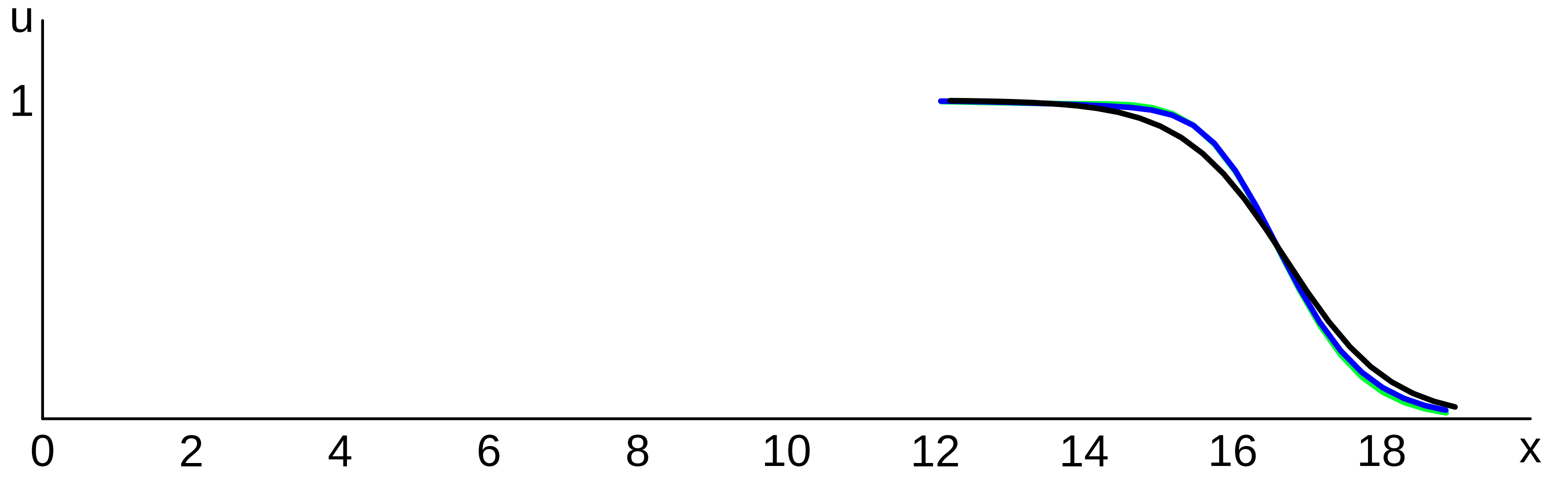


Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Example: $\alpha = \sqrt{24}$, $f(u) = \dfrac{1.5u}{1+0.5u}$ (Beverton-Holt); $c = 0.2825$

V_0, V_1, V_{11}

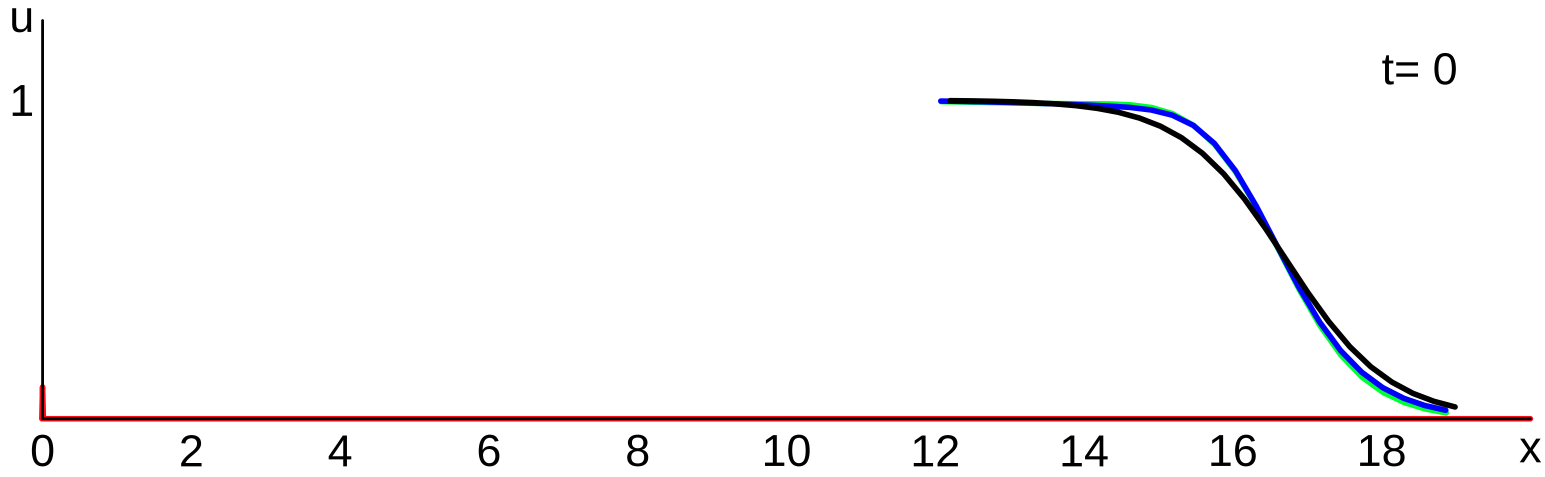


Laplace dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|s|}$$

Example: $\alpha = \sqrt{24}$, $f(u) = \dfrac{1.5u}{1+0.5u}$ (Beverton-Holt); $c = 0.2825$

V_0, V_1, V_{11}



Top-hat dispersion

$$u(t+1,x)=\int_{-\infty}^{\infty} k(x-y)f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s)=\begin{cases}1/2\beta, & -\beta<s<\beta\\ 0, & \text{else}\end{cases}$$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed:

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed:

$$\int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \frac{\sinh\beta\mu}{\beta\mu}, \quad \int_{-\infty}^{\infty} sk(s)\mathrm{e}^{\mu s}\mathrm{d}s = \frac{1}{\beta\mu}\frac{\beta\mu\cosh\beta\mu - \sinh\beta\mu}{\mu}$$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed:

$$\int_{-\infty}^{\infty} k(s) \mathrm{e}^{\mu s} \mathrm{d}s = \frac{\sinh \beta\mu}{\beta\mu}, \qquad \int_{-\infty}^{\infty} s k(s) \mathrm{e}^{\mu s} \mathrm{d}s = \frac{1}{\beta\mu} \frac{\beta\mu \cosh \beta\mu - \sinh \beta\mu}{\mu}$$

$$c_{\min} = \frac{\beta\mu \cosh \beta\mu - \sinh \beta\mu}{\mu \sinh \beta\mu} = \beta \left(\frac{\cosh \beta\mu}{\sinh \beta\mu} - \frac{1}{\beta\mu} \right)$$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed:

$$\int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \frac{\sinh \beta\mu}{\beta\mu}, \quad \int_{-\infty}^{\infty} s k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \frac{1}{\beta\mu}\frac{\beta\mu\cosh\beta\mu - \sinh\beta\mu}{\mu}$$

$$c_{\min} = \frac{\beta\mu\cosh\beta\mu - \sinh\beta\mu}{\mu\sinh\beta\mu} = \beta\left(\frac{\cosh\beta\mu}{\sinh\beta\mu} - \frac{1}{\beta\mu}\right)$$

$$\frac{c_{\min}}{\beta} = \frac{\cosh\beta\mu}{\sinh\beta\mu} - \frac{1}{\beta\mu}$$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed:

$$\int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \frac{\sinh \beta\mu}{\beta\mu}, \qquad \int_{-\infty}^{\infty} s k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \frac{1}{\beta\mu}\frac{\beta\mu\cosh\beta\mu - \sinh\beta\mu}{\mu}$$

$$c_{\min} = \frac{\beta\mu\cosh\beta\mu - \sinh\beta\mu}{\mu\sinh\beta\mu} = \beta\left(\frac{\cosh\beta\mu}{\sinh\beta\mu} - \frac{1}{\beta\mu}\right)$$

$$\frac{c_{\min}}{\beta} = \frac{\cosh\beta\mu}{\sinh\beta\mu} - \frac{1}{\beta\mu}$$

Top-hat dispersion

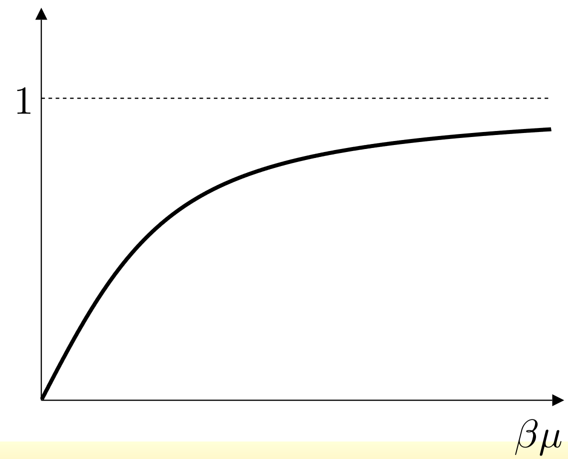
$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed:

$$\int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \frac{\sinh \beta\mu}{\beta\mu}, \qquad \int_{-\infty}^{\infty} s k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \frac{1}{\beta\mu}\frac{\beta\mu\cosh\beta\mu - \sinh\beta\mu}{\mu}$$

$$c_{\min} = \frac{\beta\mu\cosh\beta\mu - \sinh\beta\mu}{\mu\sinh\beta\mu} = \beta\left(\frac{\cosh\beta\mu}{\sinh\beta\mu} - \frac{1}{\beta\mu}\right)$$

$$\frac{c_{\min}}{\beta} = \frac{\cosh\beta\mu}{\sinh\beta\mu} - \frac{1}{\beta\mu} < 1$$



Top-hat dispersion

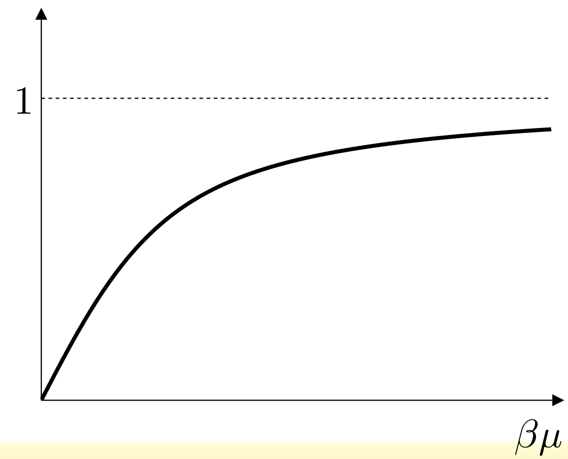
$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed:

$$\int_{-\infty}^{\infty} k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \frac{\sinh \beta\mu}{\beta\mu}, \qquad \int_{-\infty}^{\infty} s k(s)\mathrm{e}^{\mu s}\mathrm{d}s = \frac{1}{\beta\mu}\frac{\beta\mu\cosh\beta\mu - \sinh\beta\mu}{\mu}$$

$$c_{\min} = \frac{\beta\mu\cosh\beta\mu - \sinh\beta\mu}{\mu\sinh\beta\mu} = \beta\left(\frac{\cosh\beta\mu}{\sinh\beta\mu} - \frac{1}{\beta\mu}\right)$$

$$\frac{c_{\min}}{\beta} = \frac{\cosh\beta\mu}{\sinh\beta\mu} - \frac{1}{\beta\mu} < 1$$



Hence $c_{\min} < \beta$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed: $c = c_{\min} < \beta$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed: $c = c_{\min} < \beta$

Shape of wavefront:

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big) \mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed: $c = c_{\mathrm{min}} < \beta$

Shape of wavefront:

$$U_t(x-c) = \int_{-\infty}^{\infty} k(x-y) f\big(U_t(y)\big) \mathrm{d}y$$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed: $c = c_{\min} < \beta$

Shape of wavefront:

$$U_t(x-c) = \frac{1}{2\beta} \int_{x-\beta}^{x+\beta} f\big(U_t(y)\big)\mathrm{d}y$$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed: $c = c_{\min} < \beta$

Shape of wavefront:

$$U(x-c) = \frac{1}{2\beta} \int_{x-\beta}^{x+\beta} f\big(U(y)\big)\mathrm{d}y$$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed: $c = c_{\min} < \beta$

Shape of wavefront:

$$U(x-c) = \frac{1}{2\beta} \int_{x-\beta}^{x+\beta} f\big(U(y)\big)\mathrm{d}y$$

$$U'(x-c) = \frac{1}{2\beta}\Big(f\big(U(x+\beta)\big) - f\big(U(x-\beta)\big)\Big)$$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed: $c = c_{\min} < \beta$

Shape of wavefront:

$$U(x-c) = \frac{1}{2\beta} \int_{x-\beta}^{x+\beta} f\big(U(y)\big)\mathrm{d}y$$

$$U'(x-c) = \frac{1}{2\beta}\Big(f\big(U(x+\beta)\big) - f\big(U(x-\beta)\big)\Big)$$

Linear approximation $U'(x-c) \approx \dfrac{U(x+\beta) - U(x-\beta)}{2\beta}$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed: $c = c_{\min} < \beta$

Shape of wavefront:

$$U(x-c) = \frac{1}{2\beta} \int_{x-\beta}^{x+\beta} f\big(U(y)\big)\mathrm{d}y$$

$$U'(x-c) = \frac{1}{2\beta}\Big(f\big(U(x+\beta)\big) - f\big(U(x-\beta)\big)\Big)$$

Linear approximation $U'(x-c) \approx \dfrac{U(x+\beta) - U(x-\beta)}{2\beta}$

$$U(x+\beta) - U(x-\beta) = f\big(U(x+\beta)\big) - f\big(U(x-\beta)\big)$$

Top-hat dispersion

$$u(t+1,x)=\int_{-\infty}^{\infty} k(x-y)f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s)=\begin{cases}1/2\beta, & -\beta<s<\beta\\ 0, & \text{else}\end{cases}$$

Wave speed: $c=c_{\min}<\beta$

Shape of wavefront:

$$U(x-c)=\frac{1}{2\beta}\int_{x-\beta}^{x+\beta} f\big(U(y)\big)\mathrm{d}y$$

$$U'(x-c)=\frac{1}{2\beta}\Big(f\big(U(x+\beta)\big)-f\big(U(x-\beta)\big)\Big)$$

Linear approximation $U'(x-c)\approx\dfrac{U(x+\beta)-U(x-\beta)}{2\beta}$

$$U(x+\beta)-U(x-\beta)=f\big(U(x+\beta)\big)-f\big(U(x-\beta)\big)$$

We can put $U(x+\beta)=\eta$ and compute $U(x-\beta)$. Hence, we obtain

$$U(x+\beta),\ U(x-\beta),\ U(x-3\beta),\ U(x-5\beta),\dots$$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Wave speed: $c = c_{\min} < \beta$

Shape of wavefront: stepwise linear approximation

Pospíšil Z. (2013?) Shape of a travelling wave in a time-discrete reaction diffusion equation. *Adv.Dyn.Syst.Appl*, to appear

Top-hat dispersion

$$u(t+1,x) = \int\limits_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Example: $\beta = 0.5$, $f(u) = \dfrac{1.5u}{1+0.5u}$ (Beverton-Holt)

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

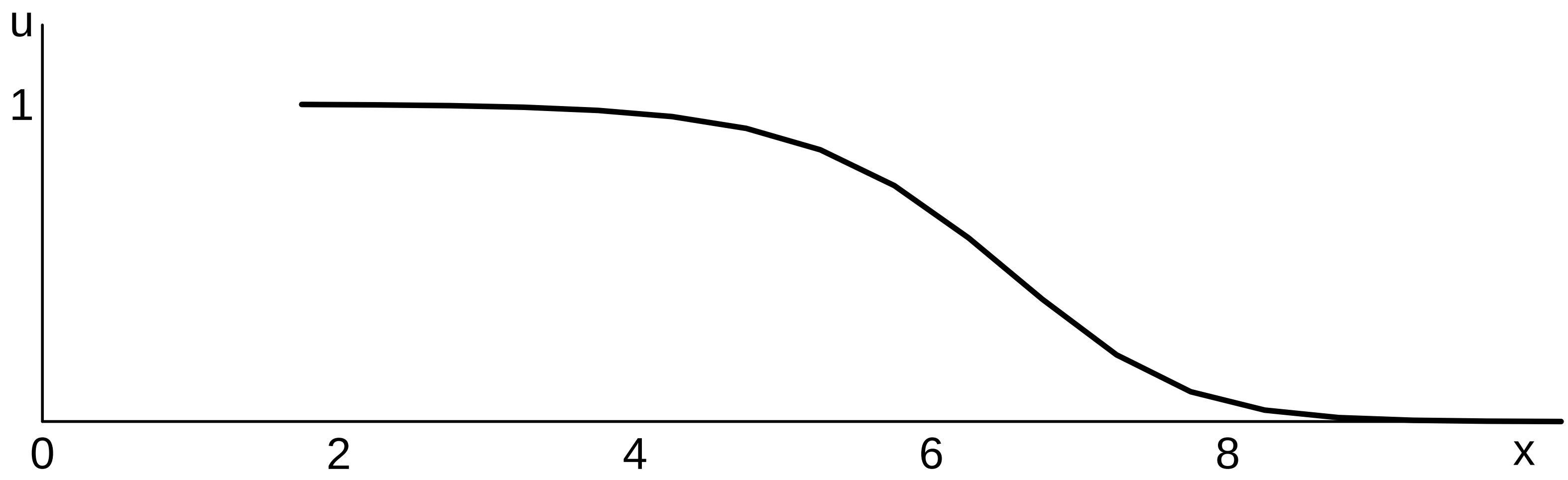
Example: $\beta = 0.5$, $f(u) = \dfrac{1.5u}{1+0.5u}$ (Beverton-Holt); $c = 0.2491$

Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\,\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Example: $\beta = 0.5,\ f(u) = \dfrac{1.5u}{1+0.5u}$ (Beverton-Holt); $c = 0.2491$

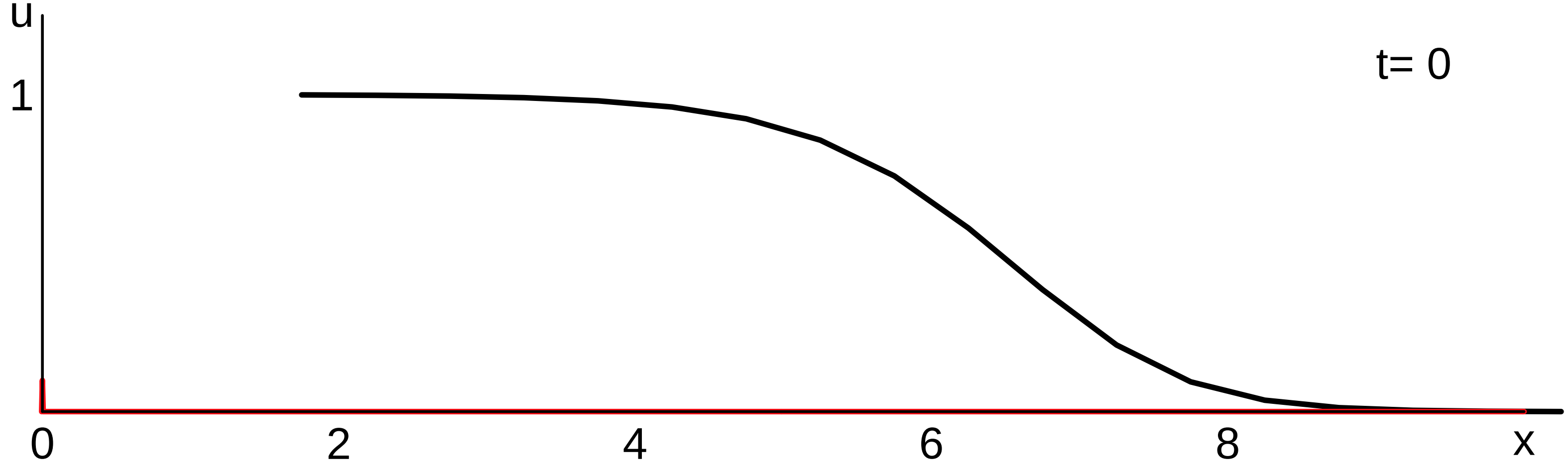
$U(9.75) = 0.001$



Top-hat dispersion

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \qquad k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}$$

Example: $\beta = 0.5,\ f(u) = \dfrac{1.5u}{1+0.5u}$ (Beverton-Holt); $c = 0.2491$
$U(9.75) = 0.001$



Thank you