Spectral Density Estimation via Autoregressive Modeling

Marie Forbelská

Masaryk University Brno, Department of Mathematics and Statistics

Podlesí, 3. 9. – 6. 9. 2013

evropský sociální fond v ČR

EVROPSKÁ UNIE

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

OP Vzdělávání pro konkurenceschopnost

AMathNet
síť pro transfer znalostí v aplikované matematice

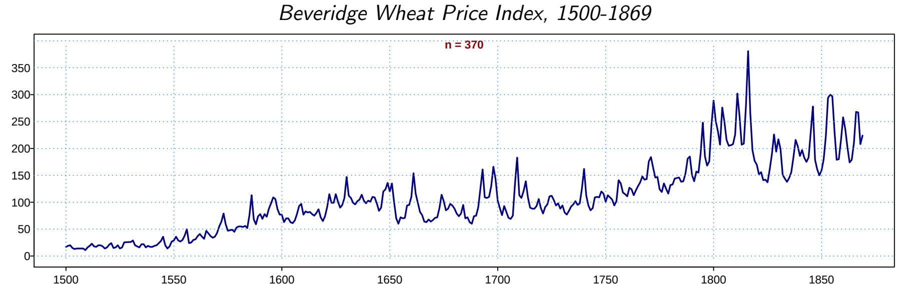
INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Characteristics of Time Series

Time Series (Discrete–Time Stochastic Processes)

- A time series is a sequence of random variables

$$\{Y_t, t = 0, \pm 1, \pm 2, \ldots\}$$



Second Order Statistical Description

Definition

Stochastic process $\{Y_t, t \in T\}$ is said to be a **second-order process** if $EY_t^2 < \infty$ for all $t \in T$.

Mean

- Mean $EY_t = \mu_t < \infty$ for all $t \in T$.

Autocovariance and Autocorelation

- Autocovariance $C_Y(t,s)$ of a random process $\{Y_t, t \in \mathbb{Z}\}$ is defined as the covariance of Y_t and Y_s:
$$C_Y(t,s) = E(Y_t - EY_t)(Y_s - EY_s)$$
- In particular, when $t = s$, we have
$$C_Y(t,t) = E(Y_t - EY_t)^2 = DY_t$$
- Autocorrelation coefficient is defined as
$$R_Y(t,s) = \frac{C_Y(t,s)}{\sqrt{DY_t DY_s}}$$

Weak Stationarity

- We introduce weak stationarity which require that time series exhibit certain time-invariant behavior.

Definition

- A time series $\{Y_t, t \in \mathbb{Z}\}$ is (weak) stationary if $EY_t < \infty$ for each t, and
  (i) $EY_t = \mu$ is a constant, independent of t, and
  (ii) $C_Y(t, t+k)$ is independent of t for each k.

Notation

- If $\{Y_t, t \in \mathbb{Z}\}$ is (weak) stationary denote by

$$\begin{aligned} \gamma_Y(k) &= C_Y(t, t+k) \\ \rho_Y(k) &= R_Y(t, t+k) \end{aligned} \quad \text{for all } t.$$

Spectral theory

Spectral density

Let $\{Y_t, t \in \mathbb{Z}\}$ be a zero mean stationary random sequence with the autocovariance function satisfying

$$\sum_{t=-\infty}^{\infty} |\gamma(t)| < \infty.$$

Then the spectral density function is the continuous function $f(\lambda)$ given by the uniformly convergent series

$$f(\lambda) = \sum_{t=-\infty}^{\infty} \gamma(t) e^{-i\lambda t}$$

(see Doob 1953, p. 476).

White Noise

Definition

The process $\{\varepsilon_t, t \in T\}$ is said to be an **White Noise**

- if ε_t are uncorrelated random variables,
- each with zero mean and variance $\sigma_\varepsilon^2 > 0$

Notation: $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$.

Definition

If ε_t are also independent and identically distributed, then the process $\{\varepsilon_t, t \in T\}$ is said to be an **IID process**.

Notation: $\varepsilon_t \sim IID(0, \sigma_\varepsilon^2)$.

Gaussian White Noise

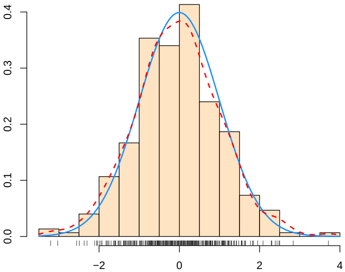
$$\varepsilon_t = \eta_t - \mu \sim WN(0, \sigma_\varepsilon^2)$$

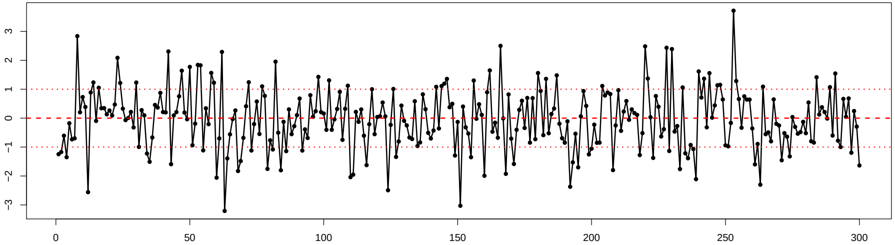
where $\eta_t \sim N(\mu = 1, \sigma^2 = 1)$

density: $f_\eta(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\}$ pro $x \in \mathbb{R}$

mean: $E\eta_t = \mu$

variance: $D\eta_t = \sigma^2 = \sigma_\varepsilon^2$





Exponential White Noise

$$\varepsilon_t = \eta_t - \mu \quad \sim \quad WN(0, \sigma_\varepsilon^2)$$

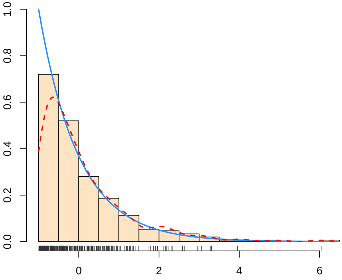
---

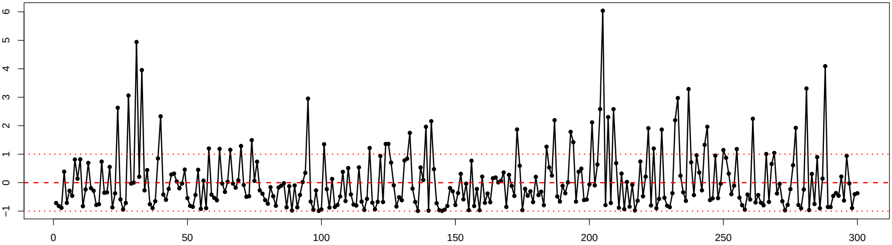
where $\eta_t \sim Exp(\mu = 1)$

density: $f_\eta(x) = \frac{1}{\mu} \exp\{-\frac{1}{\mu}x\}$ pro $x \geq 0$

mean: $E\eta_t = \mu$

variance: $D\eta_t = \mu^2 = \sigma_\varepsilon^2$





Beta-distributed White Noise

$$\varepsilon_t = \eta_t - \mu \quad \sim \quad WN(0, \sigma_\varepsilon^2)$$

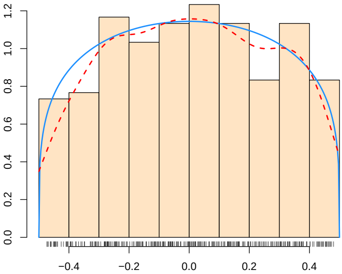
---

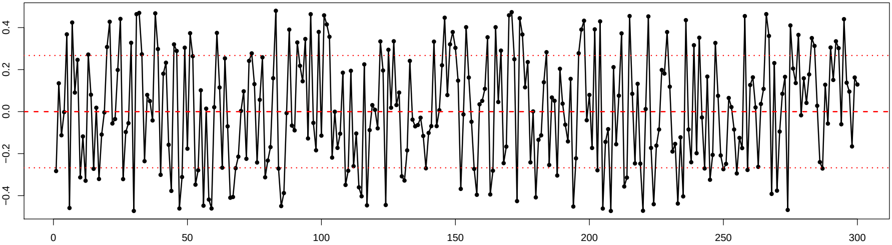
where $\quad \eta_t \sim Beta(a = 1.25, b = 1.25)$

density: $\quad f_\eta(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$ pro $x \in (0,1)$

mean: $\quad E\eta_t = \mu = \frac{a}{a+b}$

variance: $\quad D\eta_t = \frac{ab}{(a+b)^2(a+b+1)} = \sigma_\varepsilon^2$





ARMA Process

Definition

The process $\{Y_t, t \in \mathbb{Z}\}$ is said to be an *ARMA*(p,q) process

- if $\{Y_t, t \in \mathbb{Z}\}$ is stationary and
- if for every $t \in \mathbb{Z}$,

$$Y_t - \varphi_1 Y_{t-1} - \cdots - \varphi_p Y_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

  where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$.

We say that $\{Y_t, t \in \mathbb{Z}\}$ is an $ARMA(p,q)$ process with mean μ

- if $\{Y_t - \mu, t \in \mathbb{Z}\}$ is an $ARMA(p,q)$ process.

Special cases

- If $p = 0$ then Y_t is said to be moving average process $MA(q)$.
- If $q = 0$ then Y_t is said to be autoregressive $AR(p)$.

Backshift Operators and Characteristic Polynomials

Backshift Operator B such that

$$BY_t = BY_{t-1} \quad \text{and} \quad B^k Y_t = BY_{t-k} \quad \text{for all} \quad k \in \mathbb{Z}$$

ARMA notation using backshift operators

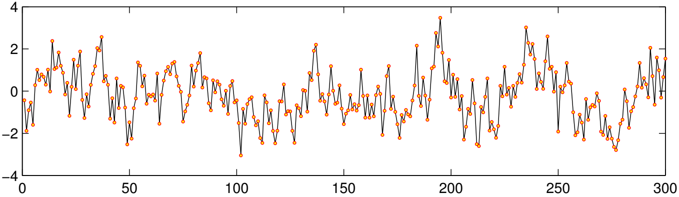
$$Y_t \sim ARMA(p,q): \ \Phi(B)Y_t = \Theta(B)\varepsilon_t$$

Characteristic polynomials

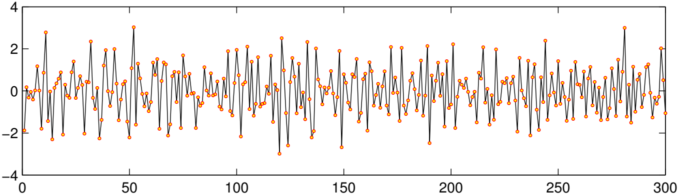
$$\begin{array}{ll} AR \text{ part} & \Phi(z) = 1 - \varphi_1 z - \cdots - \varphi_p z^p \\ MA \text{ part} & \Theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^p \end{array}$$

defined on $|z| < 1$.

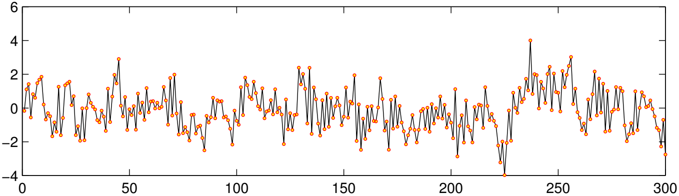
$AR(2): \quad Y_t = 0.5Y_{t-1} + 0.2Y_{t-2} + \varepsilon_t, \qquad \varepsilon_t \sim N(0,1)$



$MA(2): \quad Y_t = \varepsilon_t - 0.5\varepsilon_{t-1} - 0.2\varepsilon_{t-1}, \qquad \varepsilon_t \sim N(0,1)$



$ARMA(2,2): \quad Y_t = 0.5Y_{t-1} + 0.2Y_{t-2} + \varepsilon_t - 0.4\varepsilon_{t-1} + 0.3\varepsilon_{t-1}, \qquad \varepsilon_t \sim N(0,1)$



Causality and Invertibility of ARMA Processes

Definition

An $ARMA(p, q)$ process is said to be **causal** (relative to $\{\varepsilon_t\}$) if there exists a sequence of constants $\{\psi_i\}$ such that $\sum_{i=0}^{\infty} |\psi_i| < \infty$ and

$$Y_t = \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}, \ \ t \in \mathbb{Z}$$

Which is equivalent to the condition

$$\Phi(z) = 1 - \varphi_1 z - \ldots - \varphi_p z^p \neq 0, \ \forall \ |z| < 1$$

A similar definition for the invertibility of an $ARMA(p, q)$ process relative to ε_t can be presented if we interchange the role of $\{Y_t\}$ with $\{\varepsilon_t\}$. Then the invertibility is equivalent to the condition

$$\Theta(z) = 1 + \theta_1 z + \ldots + \theta_p z^q \neq 0, \ \forall \ |z| < 1$$

Spectral density of ARMA process

Spectral density of a $MA(q)$ process

$$f_Y(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \left|\Theta\left(e^{-i\omega}\right)\right|^2 \qquad \text{for} \qquad \omega \in \langle -\pi, \pi \rangle$$

Spectral density of a $AR(p)$ process

$$f_Y(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{1}{|\Phi(e^{-i\omega})|^2} \qquad \text{for} \qquad \omega \in \langle -\pi, \pi \rangle$$

Spectral density of a $ARMA(p, q)$ process

$$f_Y(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{\left|\Theta\left(e^{-i\omega}\right)\right|^2}{|\Phi(e^{-i\omega})|^2} \qquad \text{for} \qquad \omega \in \langle -\pi, \pi \rangle$$

where

$$\Theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q \quad \text{and} \quad \Phi(z) = 1 - \varphi_1 z - \ldots - \varphi_p z^p.$$

Moments of the $AR(p)$ process

To calculate the mean we need causal $AR(p)$ process:

$$EY_t = E\sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j} = \sum_{j=0}^{\infty}\psi_j E\varepsilon_{t-j} = 0.$$

Calculation of the autocovariance function is complicated:
first equation $Y_t = \varphi_1 Y_{t-1} + \cdots + \varphi_p Y_{t-p} + \varepsilon_t$
multiplied by a term Y_{t-k} and calculate the mean values of both sides, i.e.

$$\underbrace{EY_t Y_{t-k}}_{=\gamma(k)} = \varphi_1 \underbrace{EY_{t-1}Y_{t-k}}_{=\gamma(k-1)} + \cdots + \varphi_p \underbrace{EY_{t-p}Y_{t-k}}_{=\gamma(k-p)} + E\varepsilon_t Y_{t-k}.$$

then we compute

$$\begin{aligned} EY_{t-k}\varepsilon_t &= E(\sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j-k})\varepsilon_t = \sum_{j=0}^{\infty}\psi_j E\varepsilon_{t-j-k}\varepsilon_t = \sum_{j=0}^{\infty}\psi_j\sigma_\varepsilon^2\delta_{j+k} \\ &= \begin{cases} \sigma_\varepsilon^2 & k=0, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Yule–Walker equations

By simple modifications of the previous equations we get Yule–Walker equations

for the autocovariance function

$$\text{for } k=0: \quad \gamma(0) - \varphi_1\gamma(1) - \cdots - \varphi_p\gamma(p) = \sigma_\varepsilon^2$$

$$\text{for } k\neq 0: \quad \gamma(k) - \varphi_1\gamma(k-1) - \cdots - \varphi_p\gamma(k-p) = 0$$

for the autocorrelation function

$$\text{for } k=0: \quad \underbrace{\rho(0)}_{=1} - \varphi_1\rho(1) - \cdots - \varphi_p\rho(p) = \frac{\sigma_\varepsilon^2}{\gamma(0)}$$

$$\text{for } k\neq 0: \quad \rho(k) - \varphi_1\rho(k-1) - \cdots - \varphi_p\rho(k-p) = 0 \qquad (\mathrm{YW}_*)$$

Yule–Walker equation is a widely used method to estimate the coefficients of the $AR(p)$ models.

Limit properties of the $\rho(k)$ of the $AR(p)$ process

Solution of the homogeneous differential equation, which we marked with a (YW$_*$), we get in addition to recurrent relationship too explicit form of the autocorrelation function

$$\rho_{AR(p)}(k) = \sum_{j=1}^{m} \left(\sum_{s=0}^{p_j-1} c_{js} k^s \right) \lambda_j^k = \sum_{j=1}^{m} \left(\sum_{s=0}^{p_j-1} c_{js} k^s \right) r_j^k e^{ik\theta_j},$$

where c_{js} are constants determined by the initial conditions and $\lambda_j = r_j e^{i\theta_j}$ are the inverse of the roots of the $\Phi(z) = 1 - \varphi_1 z - \ldots - \varphi_p z^p$ with multiplicities p_j. Because holds

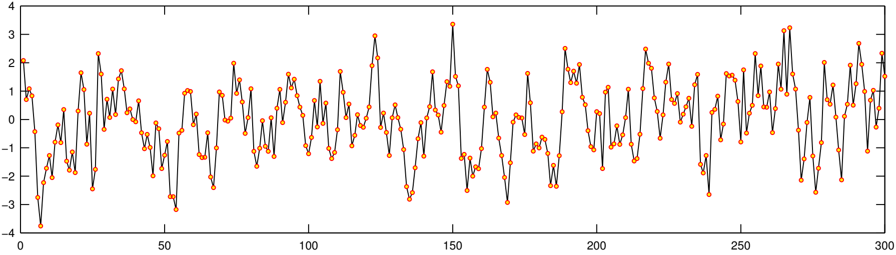
$$|\lambda_j| = r_j < 1, \quad \text{kde} \quad \Phi(z_{0j}) = 0 \quad \text{pro} \quad z_{0j} = \frac{1}{\lambda_j},$$

we get here, that $\rho(k)$ decreases for $k \to \infty$ exponentially to zero, i.e.

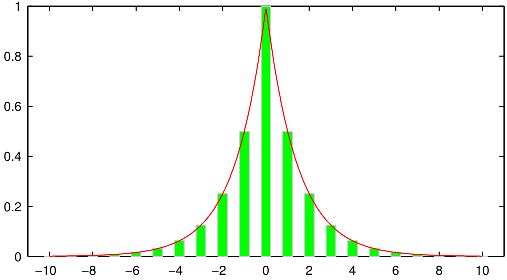
$$\rho(k) \xrightarrow[k\to\infty]{} 0,$$

which is a very important property identification autoregressive $AR(p)$ processes.

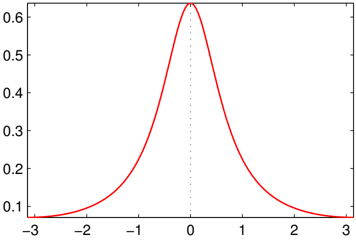
$$AR(1): \quad Y_t = 0.5Y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0,1)$$



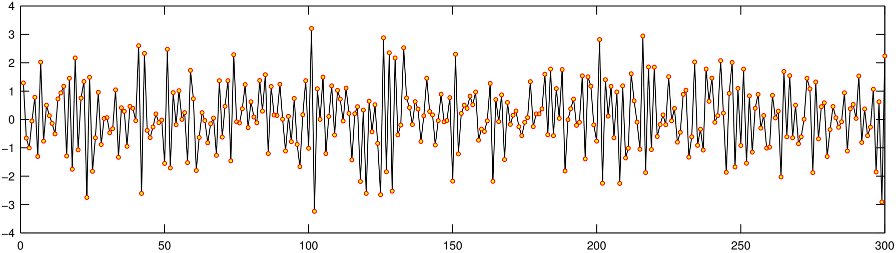
ACF



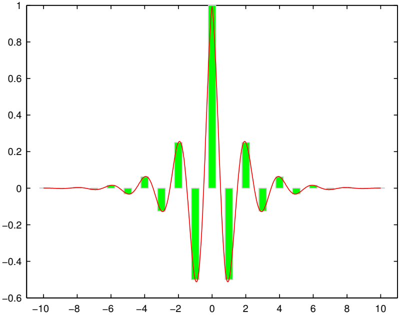
$$f_{AR}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{1}{|\Phi(e^{-i\omega})|^2}$$



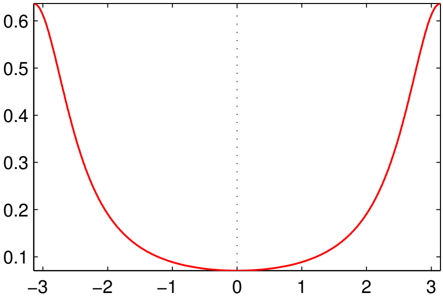
$$AR(1): \quad Y_t = -0.5Y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0,1)$$



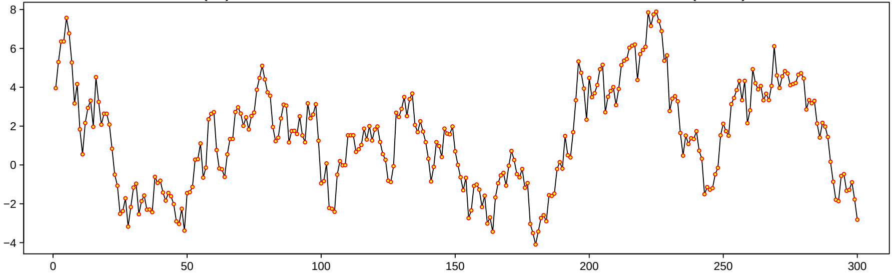
ACF



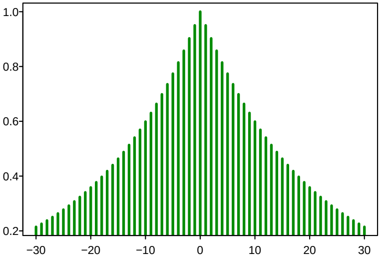
$$f_{AR}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{1}{|\Phi(e^{-i\omega})|^2}$$



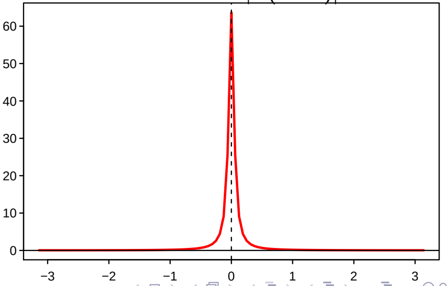
$$AR(1): \quad Y_t = 0.95 Y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0,1)$$



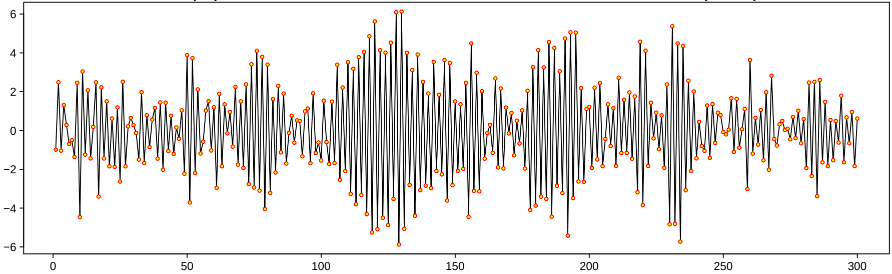
$$ACF$$



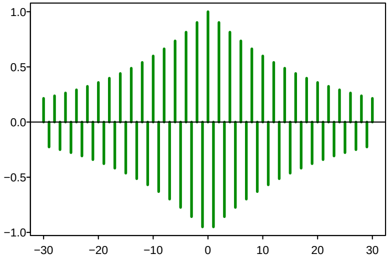
$$f_{AR}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{1}{|\Phi(e^{-i\omega})|^2}$$



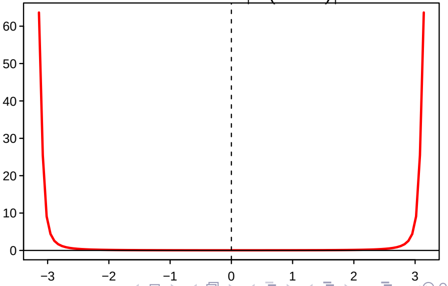
$$AR(1): \quad Y_t = -0.95\,Y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0,1)$$



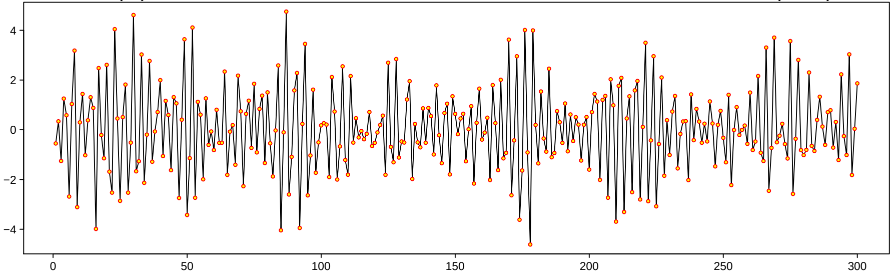
ACF



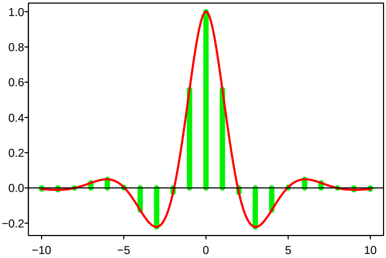
$$f_{AR}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\,\frac{1}{|\Phi(e^{-i\omega})|^2}$$



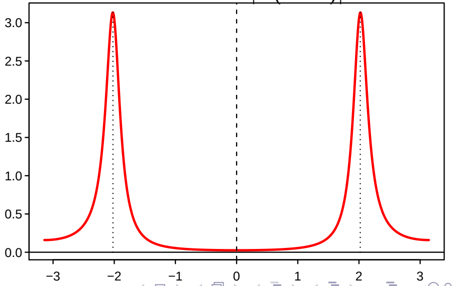
$$AR(2): \quad Y_t = -0.75Y_{t-1} - 0.75Y_{t-2} + \varepsilon_t, \qquad \varepsilon_t \sim N(0,1)$$



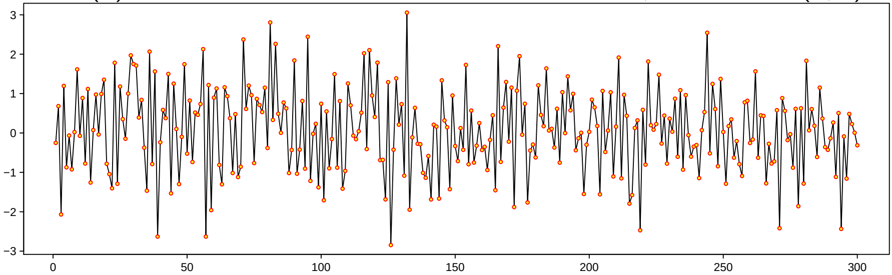
ACF



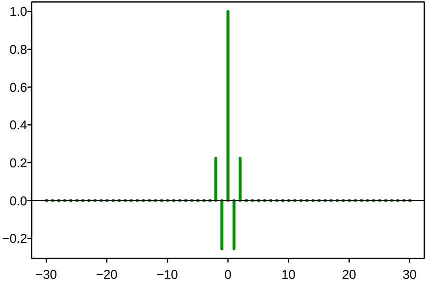
$$f_{AR}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{1}{|\Phi(e^{-i\omega})|^2}$$



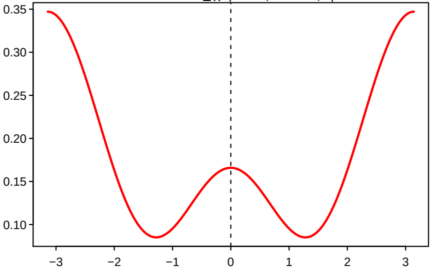
$$MA(2): \qquad Y_t = \varepsilon_t - 0.2279\varepsilon_{t-1} + 0.2488\varepsilon_{t-2}, \qquad \varepsilon_t \sim N(0,1)$$



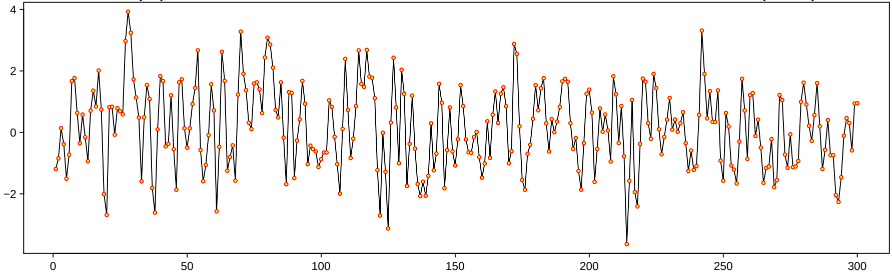
ACF



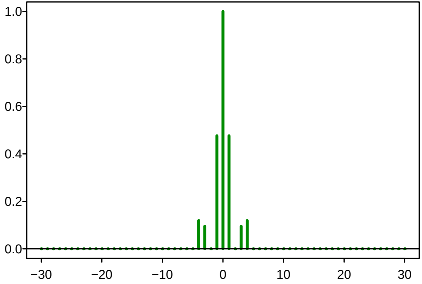
$$f_{MA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\left|\Theta\left(e^{-i\omega}\right)\right|^2$$



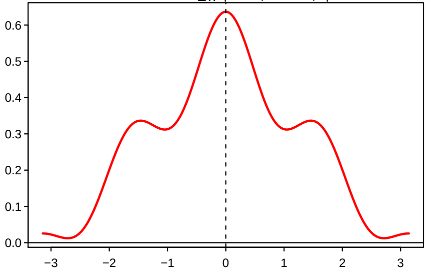
$$MA(4): \qquad Y_t = \varepsilon_t + 0.8\varepsilon_{t-1} + 0.2\varepsilon_{t-4}, \qquad \varepsilon_t \sim N(0,1)$$



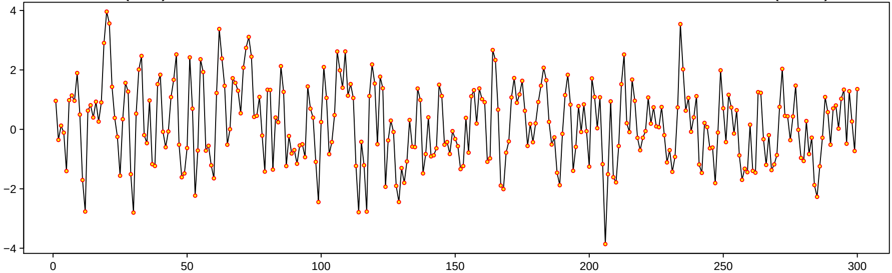
ACF



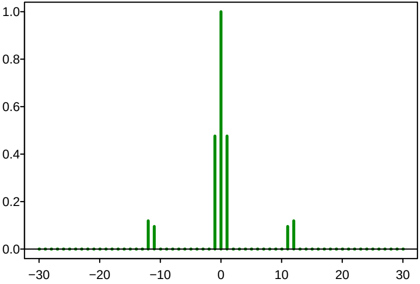
$$f_{MA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\left|\Theta\left(e^{-i\omega}\right)\right|^2$$



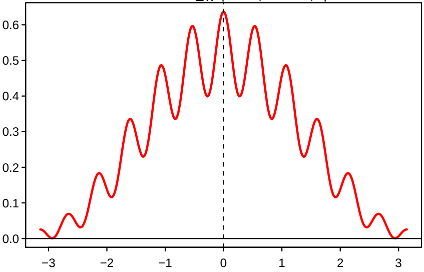
$$MA(12): \qquad Y_t = \varepsilon_t + 0.8\varepsilon_{t-1} + 0.2\varepsilon_{t-12}, \qquad \varepsilon_t \sim N(0,1)$$



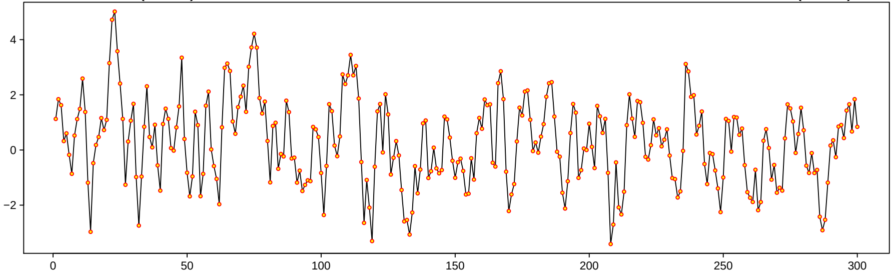
ACF



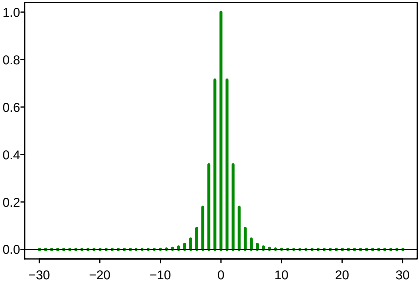
$$f_{MA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\left|\Theta\left(e^{-i\omega}\right)\right|^2$$



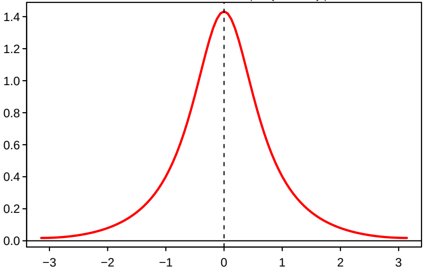
$$ARMA(1,1): \quad Y_t = 0.5Y_{t-1} + \varepsilon_t + 0.5\varepsilon_{t-1}, \quad \varepsilon_t \sim N(0,1)$$



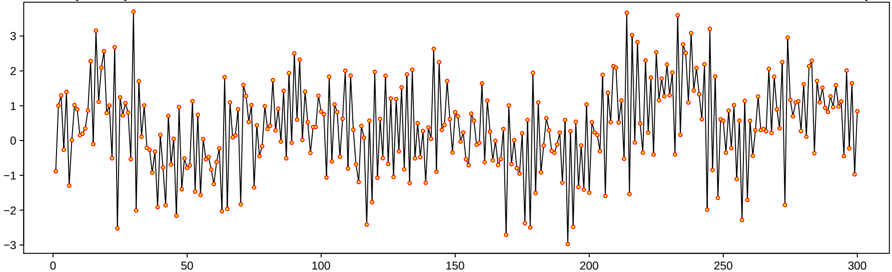
ACF



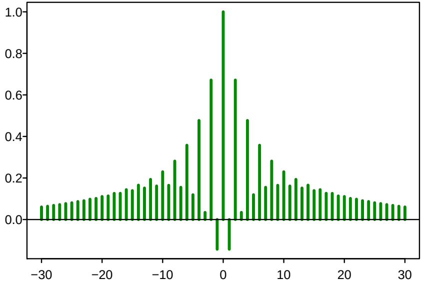
$$f_{ARMA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\frac{\left|\Theta\left(e^{-i\omega}\right)\right|^2}{\left|\Phi\left(e^{-i\omega}\right)\right|^2}$$



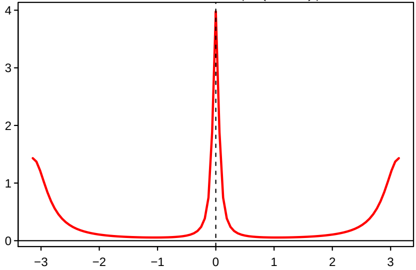
$$ARMA(2,1): \qquad Y_t = 0.2Y_{t-1} + 0.7Y_{t-1} + \varepsilon_t + 0.5\varepsilon_{t-1}, \qquad \varepsilon_t \sim N(0,1)$$



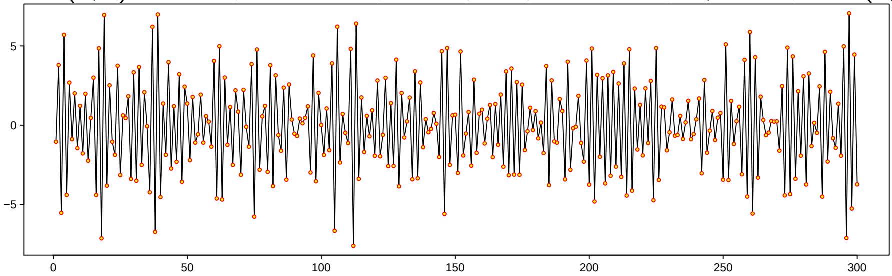
ACF

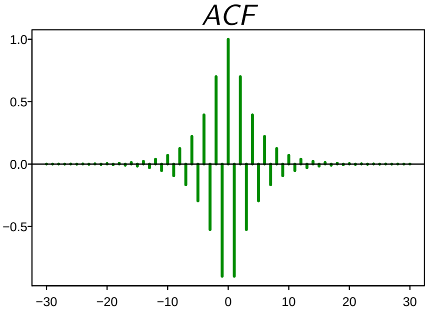


$$f_{ARMA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\frac{\left|\Theta\left(e^{-i\omega}\right)\right|^2}{\left|\Phi\left(e^{-i\omega}\right)\right|^2}$$

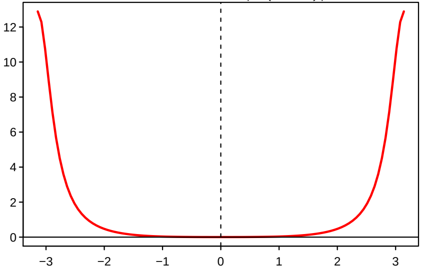


$$ARMA(1,2): \qquad Y_t = -0.75Y_{t-1} + \varepsilon_t - \varepsilon_{t-1} + 0.25\varepsilon_{t-2}, \qquad \varepsilon_t \sim N(0,1)$$

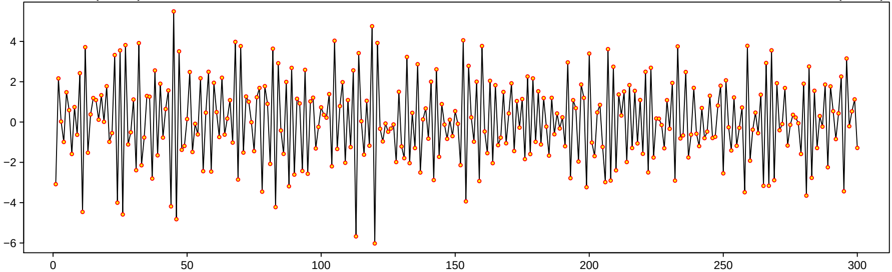




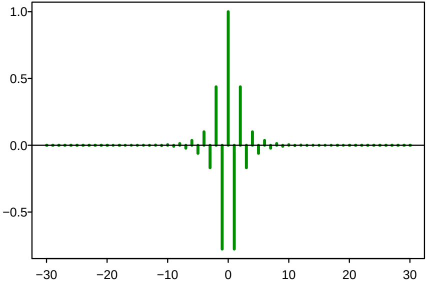
$$f_{ARMA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{\left|\Theta\left(e^{-i\omega}\right)\right|^2}{\left|\Phi\left(e^{-i\omega}\right)\right|^2}$$



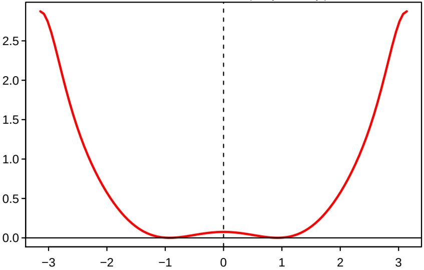
$$ARMA(1,3): \quad Y_t = -0.6Y_{t-1} + \varepsilon_t - 0.7\varepsilon_{t-1} + 0.4\varepsilon_{t-2} + 0.4\varepsilon_{t-3}, \quad \varepsilon_t \sim N(0,1)$$



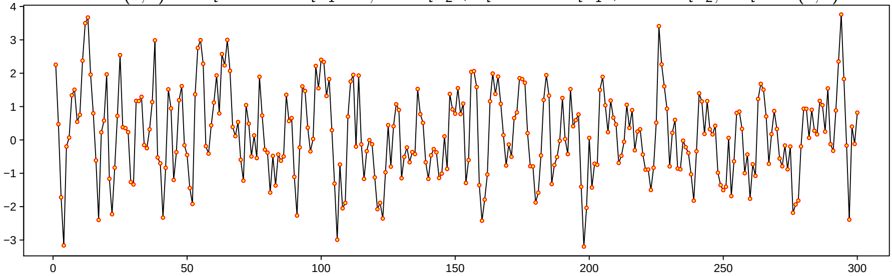
ACF



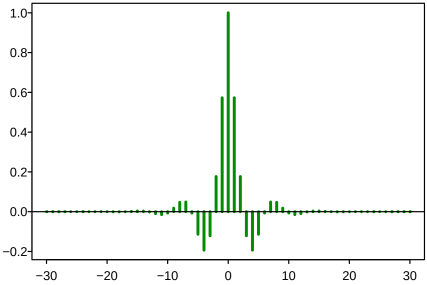
$$f_{ARMA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{\left|\Theta\left(e^{-i\omega}\right)\right|^2}{\left|\Phi\left(e^{-i\omega}\right)\right|^2}$$



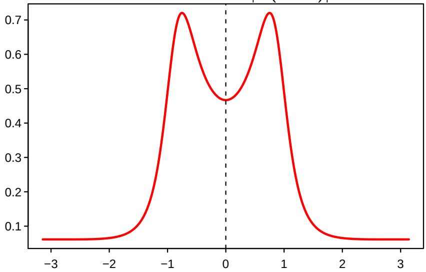
$ARMA(2,2): \quad Y_t = 0.8897 Y_{t-1} - 0,4858 Y_{t-2} + \varepsilon_t - 0.2279\varepsilon_{t-1} + 0.2488\varepsilon_{t-2}, \quad \varepsilon_t \sim N(0,1)$



ACF



$$f_{ARMA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{\left|\Theta\left(e^{-i\omega}\right)\right|^2}{\left|\Phi\left(e^{-i\omega}\right)\right|^2}$$



Seasonal linear models

So far we have discussed the links between neighboring random variables

$$\ldots, Y_t, Y_{t+1}, Y_{t+2}, \ldots.$$

If a random process also includes seasonal fluctuations, it is necessary to notice the dependencies between random variables, which divides season length L.

$$\ldots, Y_t, Y_{t+L}, Y_{t+2L}, \ldots.$$

First, we introduce **seasonal differential operator of length** $L > 0$:

$$\begin{aligned}
\Delta_L Y_t &= Y_t - Y_{t-L} = (1 - B^L) Y_t \\
\Delta_L^2 Y_t &= \Delta_L(\Delta_L Y_t) = \Delta_L(Y_t - Y_{t-L}) \\
&= (Y_t - Y_{t-L}) - (Y_{t-L} - Y_{t-2L}) \\
&= Y_t - 2Y_{t-L} + Y_{t-2L} = (1 - B^L)^2 Y_t \\
&\vdots \\
\Delta_L^D Y_t &= (1 - B^L)^D Y_t
\end{aligned}$$

Construction seasonal models

- To better understand the structure of seasonal patterns in the B–J methodology, divide, for example, monthly data ($L = 12$) for $\boxed{r}$ years in the following table.

Year	January	February	$\cdots$	December
1	Y_1	Y_2	$\cdots$	Y_{12}
2	Y_{13}	Y_{14}	$\cdots$	Y_{24}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	$Y_{1+12(r-1)}$	$Y_{2+12(r-1)}$	$\cdots$	$Y_{12+12(r-1)}$

- For each column $j \in \{1, \ldots, 12\}$ separately consider a $ARMA(P, Q)$ model of the same type:

$$\begin{aligned} Y_{j+12t} &= \pi_1 Y_{j+12(t-1)} + \cdots + \pi_P Y_{j+12(t-1)} + \\ & \eta_{j+12t} + \psi_1 \eta_{j+12(t-1)} + \cdots + \psi_Q \eta_{j+12(t-1)} \end{aligned}$$

- Because all 12 random processes is of the same type, we can write

$$\pi(B^{12}) Y_t = \Psi(B^{12}) \eta_t.$$

Remap white noise to the new process

When 12 white noise of the same type

$$\begin{array}{ccc} \{\eta_{1+12t}\} & \sim & WN(0,\sigma_\eta^2) \\ \{\eta_{2+12t}\} & \sim & WN(0,\sigma_\eta^2) \\ \vdots & \vdots & \vdots \\ \{\eta_{12+12t}\} & \sim & WN(0,\sigma_\eta^2) \end{array}$$

sequentially assemble in time and create a single random process

$$\{\eta_t^*, t = 0, \pm 1, \pm 2, \ldots\},$$

we do not get white noise, it is recalled that:

$$\begin{array}{ll} E\eta_t^*\eta_{t+h}^* = 0 & \text{only where } h \text{ that are multiples of } 12 \\ E\eta_t^*\eta_{t+h} \neq 0 & \text{may occur for any other } h, \end{array}$$

therefore model the process $\boxed{\eta_t}$ as a general $ARMA(pq)$ process

$$\Phi(B)\eta_t^* = \Theta(B)\varepsilon_t, \qquad \varepsilon_t \sim WN(0,\sigma_\varepsilon^2).$$

Stationary *SARMA* models

- **General stationary seasonal mixed SARMA model**:

  $$\Phi(B)\pi(B^L)Y_t = \Theta(B)\Psi(B^L)\varepsilon_t \sim SARMA(p,q) \times (P,Q)_L$$

  kde

  - $\Phi(B) = 1 - \varphi_1 B - \cdots - \varphi_p B^p$
  - $\pi(B^L) = 1 - \pi_1 B^L - \cdots - \pi_P B^{PL}$
  - $\Theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^p$
  - $\Psi(B^L) = 1 + \psi_1 B^L + \cdots + \psi_Q B^{QL}$

- *MA* homogeneous seasonal models

  $$Y_t = \Theta(B)\Psi(B^L)\varepsilon_t \sim SARMA(0,q) \times (0,Q)_L.$$

- *AR* homogeneous seasonal models

  $$\Phi(B)\pi(B^L)Y_t = \varepsilon_t \sim SARMA(p,0) \times (P,0)_L$$

SARMA model as a special type of *ARMA* model

- Consider a simple example $SARMA(1,0) \times (1,0)_{12}$ model:

$$\begin{aligned} \Phi(B)\pi(B^{12})Y_t &= \varepsilon_t \\ (1-\varphi_1 B)(1-\pi_1 B^{12})Y_t &= \varepsilon_t \\ (1-\varphi_1 B - \pi_1 B^{12} + \varphi_1\pi_1 B^{13})Y_t &= \varepsilon_t \\ Y_t - \varphi_1 Y_{t-1} - \pi_1 Y_{t-12} + \varphi_1\pi_1 Y_{t-13} &= \varepsilon_t \end{aligned}$$

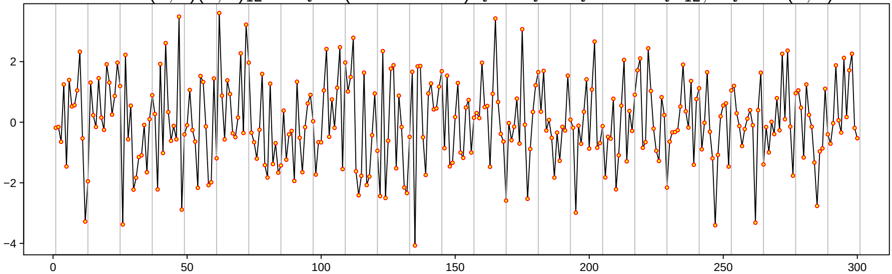
- We see that it is a special case of $AR(13)$ model in which:
  - 10 coefficients are zero,
  - three remaining non-zero coefficients were created on two parameters:.

Relationship between *SARMA* and *ARMA* models

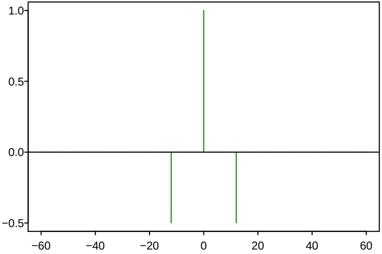
Model $SARMA(p,q) \times (P,Q)_L$ is actually $ARMA(p+PL, QL+q)$ model with additional conditions on *AR* and *MA* coefficients.

Pure seasonal homogeneous model with MA parts: $Y_t = \Psi(B^{12})\varepsilon_t$

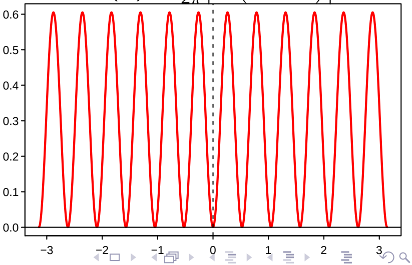
$SARMA(0,0)(0,1)_{12}: \quad Y_t = (1 - 0.95B^{12})\varepsilon_t: \quad Y_t = \varepsilon_t - 0.95\varepsilon_{t-12}, \quad \varepsilon_t \sim N(0,1)$



ACF

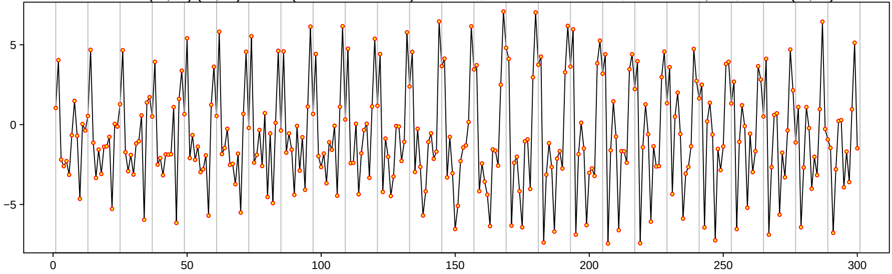


$f_{SMA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\left|\Psi\left(e^{-i12\omega}\right)\right|^2$

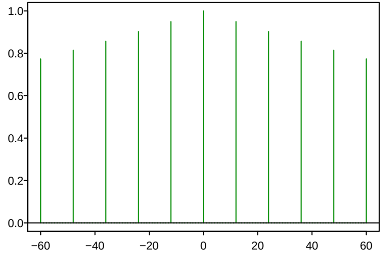


Pure seasonal homogeneous model with AR parts: $\pi(B^{12})Y_t = \varepsilon_t$

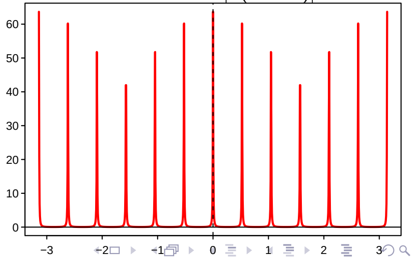
$SARMA(0,0)(0,1)_{12}: \quad (1-0.95B^{12})Y_t = \varepsilon_t: \quad Y_t = 0.95Y_{t-12} + \varepsilon_t, \quad \varepsilon_t \sim N(0,1)$



ACF

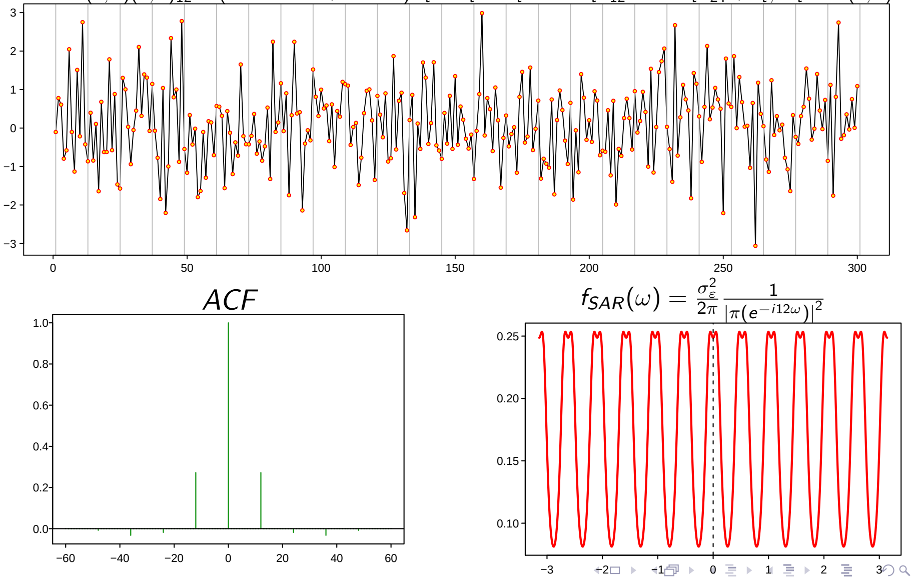


$f_{SAR}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{1}{|\pi(e^{-i12\omega})|^2}$



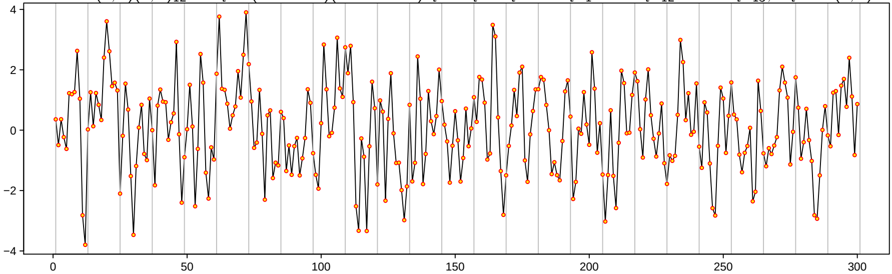
Pure seasonal homogeneous model with AR parts: $\pi(B^{12})Y_t = \varepsilon_t$

$SARMA(0,0)(2,0)_{12}: \quad (1 - 0.3B^{12} + 0.1B^{24})Y_t = \varepsilon_t: \quad Y_t = 0.3Y_{t-12} - 0.1Y_{t-24} + \varepsilon_t, \;\; \varepsilon_t \sim N(0,1)$

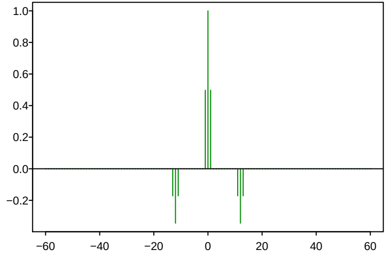


Seasonal homogeneous model with *MA* parts: $Y_t = \Theta(B)\Psi(B^{12})\varepsilon_t$

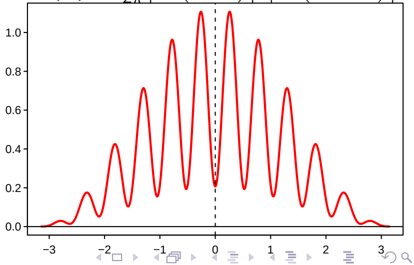
$SARMA(0,1)(0,1)_{12}: \quad Y_t = (1+0.9B)(1-0.4B^{12})\varepsilon_t: \quad Y_t = \varepsilon_t + 0.9\varepsilon_{t-1} - 0.4\varepsilon_{t-12} - 0.36\varepsilon_{t-13}, \quad \varepsilon_t \sim N(0,1)$



ACF

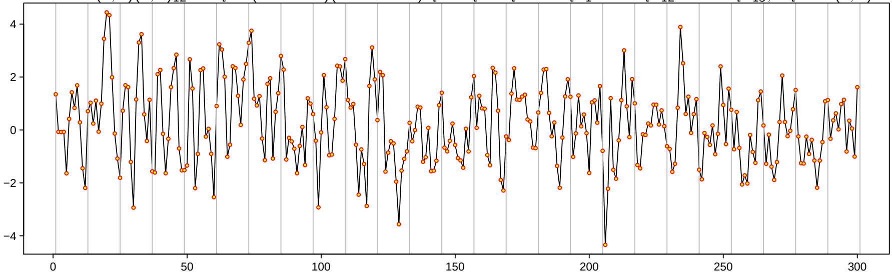


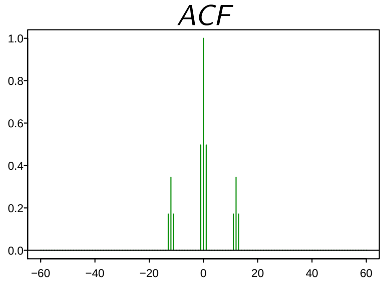
$$f_{SMA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\left|\Theta\left(e^{-i\omega}\right)\right|^2\left|\Psi\left(e^{-i12\omega}\right)\right|^2$$



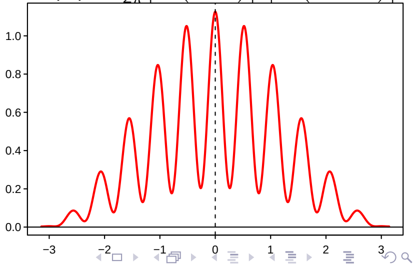
Seasonal homogeneous model with *MA* parts: $Y_t = \Theta(B)\Psi(B^{12})\varepsilon_t$

$SARMA(0,1)(0,1)_{12}: \quad Y_t = (1+0.9B)(1+0.4B^{12})\varepsilon_t: \quad Y_t = \varepsilon_t + 0.9\varepsilon_{t-1} + 0.4\varepsilon_{t-12} + 0.36\varepsilon_{t-13}, \quad \varepsilon_t \sim N(0,1)$



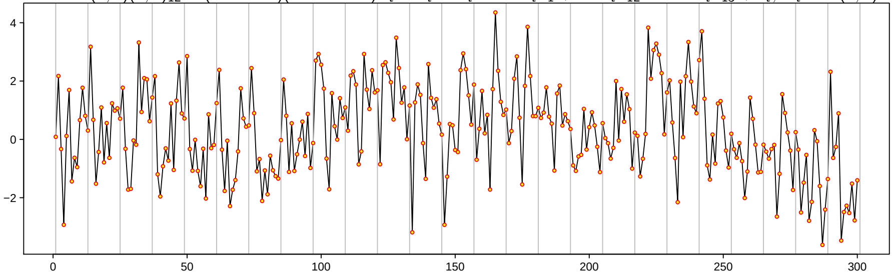


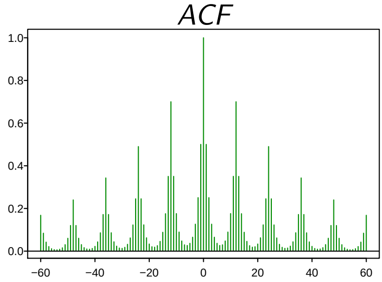
$f_{SMA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\left|\Theta\left(e^{-i\omega}\right)\right|^2\left|\Psi\left(e^{-i12\omega}\right)\right|^2$



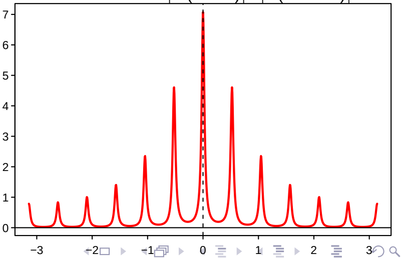
Seasonal homogeneous model with AR parts: $\Phi(B)\pi(B^{12})Y_t = \varepsilon_t$

$SARMA(1,0)(1,0)_{12}$: $(1-0.5B)(1-0.7B^{12})Y_t = \varepsilon_t$: $Y_t = 0.5Y_{t-1} + 0.7Y_{t-12} - 0.35Y_{t-13} + \varepsilon_t$, $\varepsilon_t \sim N(0,1)$



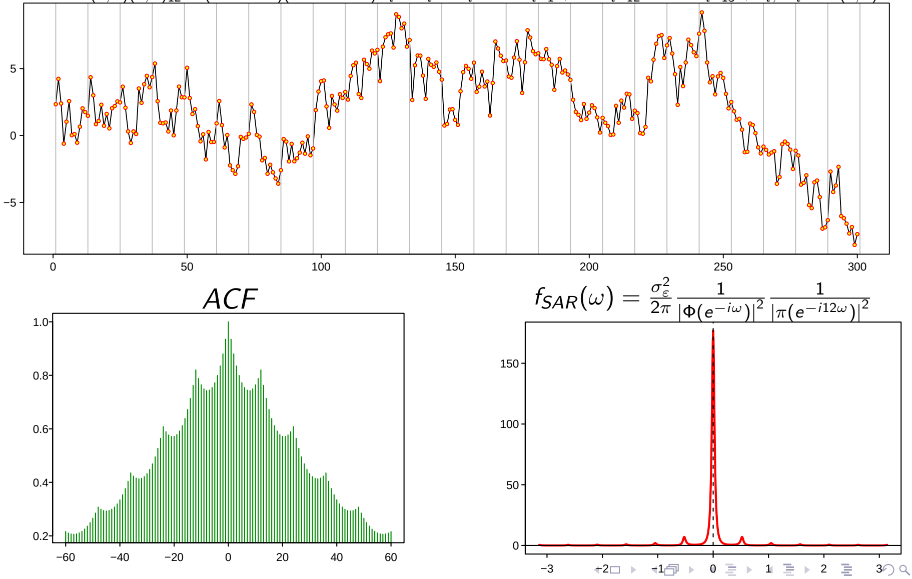


$$f_{SAR}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\frac{1}{|\Phi(e^{-i\omega})|^2}\frac{1}{|\pi(e^{-i12\omega})|^2}$$



Seasonal homogeneous model with AR parts: $\Phi(B)\pi(B^{12})Y_t = \varepsilon_t$

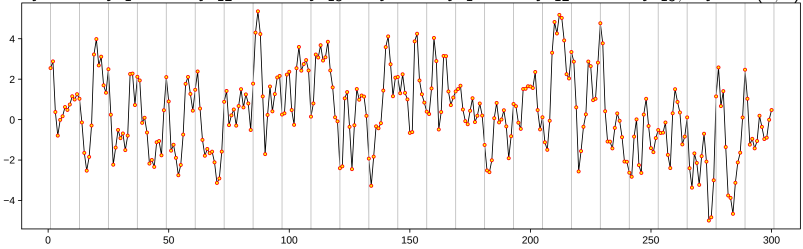
$SARMA(1,0)(1,0)_{12}: \quad (1-0.9B)(1-0.7B^{12})Y_t = \varepsilon_t: \quad Y_t = 0.9Y_{t-1} + 0.7Y_{t-12} - 0.63Y_{t-13} + \varepsilon_t, \quad \varepsilon_t \sim N(0,1)$



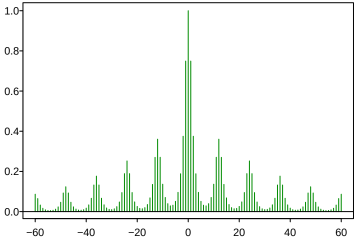
Seasonal mixed model: $\Phi(B)\pi(B^{12})Y_t = \Theta(B)\Psi(B^{12})\varepsilon_t$

$$SARMA(1,1)(1,1)_{12}: \quad (1-0.5B)(1-0.7B^{12})Y_t = (1+0.9B)(1-0.4B^{12})\varepsilon_t$$

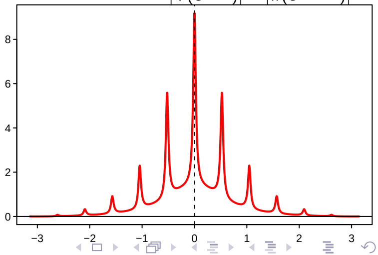
$$Y_t = 0.5Y_{t-1} + 0.7Y_{t-12} - 0.35Y_{t-13} + \varepsilon_t + 0.9\varepsilon_{t-1} - 0.4\varepsilon_{t-12} - 0.36\varepsilon_{t-13}, \quad \varepsilon_t \sim N(0,1)$$



ACF



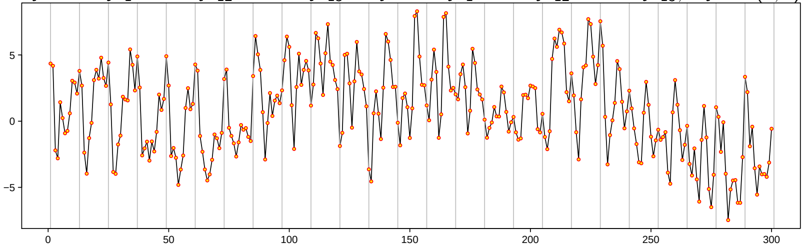
$$f_{SARMA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\frac{\left|\Theta\left(e^{-i\omega}\right)\right|^2}{\left|\Phi\left(e^{-i\omega}\right)\right|^2}\frac{\left|\Psi\left(e^{-i12\omega}\right)\right|^2}{\left|\pi\left(e^{-i12\omega}\right)\right|^2}$$



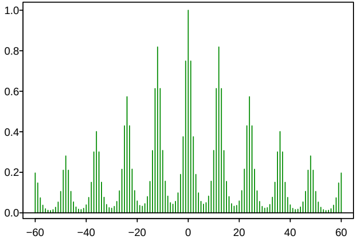
Seasonal mixed model: $\Phi(B)\pi(B^{12})Y_t = \Theta(B)\Psi(B^{12})\varepsilon_t$

$$SARMA(1,1)(1,1)_{12}: \quad (1-0.5B)(1-0.7B^{12})Y_t = (1+0.9B)(1+0.4B^{12})\varepsilon_t$$

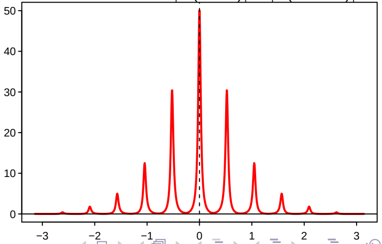
$$Y_t = 0.5Y_{t-1} + 0.7Y_{t-12} - 0.35Y_{t-13} + \varepsilon_t + 0.9\varepsilon_{t-1} + 0.4\varepsilon_{t-12} + 0.36\varepsilon_{t-13}, \quad \varepsilon_t \sim N(0,1)$$



ACF



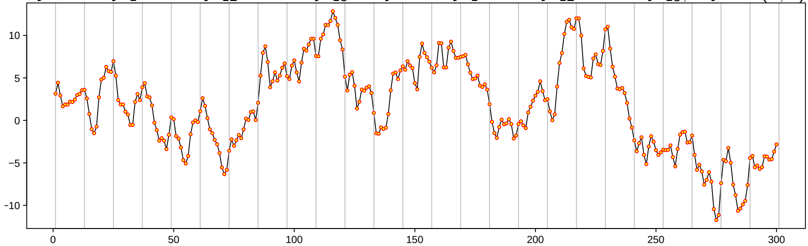
$$f_{SARMA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{\left|\Theta\left(e^{-i\omega}\right)\right|^2}{\left|\Phi\left(e^{-i\omega}\right)\right|^2} \frac{\left|\Psi\left(e^{-i12\omega}\right)\right|^2}{\left|\pi\left(e^{-i12\omega}\right)\right|^2}$$



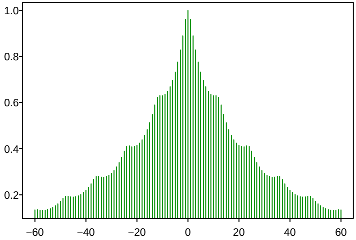
Seasonal mixed model: $\Phi(B)\pi(B^{12})Y_t = \Theta(B)\Psi(B^{12})\varepsilon_t$

$$SARMA(1,1)(1,1)_{12}: \quad (1-0.9B)(1-0.7B^{12})Y_t = (1+0.9B)(1-0.4B^{12})\varepsilon_t$$

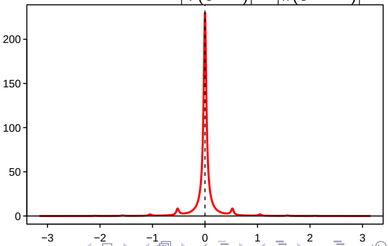
$$Y_t = 0.9Y_{t-1} + 0.7Y_{t-12} - 0.63Y_{t-13} + \varepsilon_t + 0.9\varepsilon_{t-1} - 0.4\varepsilon_{t-12} - 0.36\varepsilon_{t-13}, \quad \varepsilon_t \sim N(0,1)$$



ACF



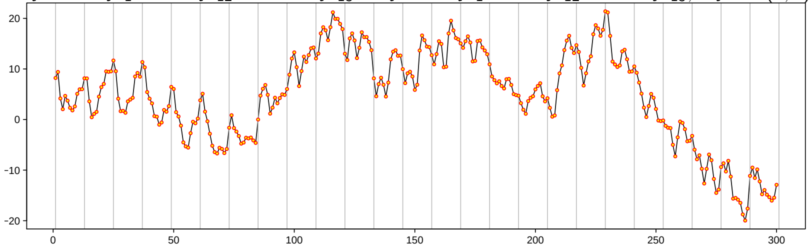
$$f_{SARMA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\frac{|\Theta(e^{-i\omega})|^2}{|\Phi(e^{-i\omega})|^2}\frac{|\Psi(e^{-i12\omega})|^2}{|\pi(e^{-i12\omega})|^2}$$

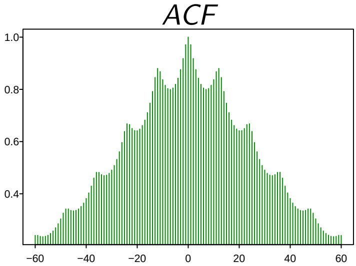


Seasonal mixed model: $\Phi(B)\pi(B^{12})Y_t = \Theta(B)\Psi(B^{12})\varepsilon_t$

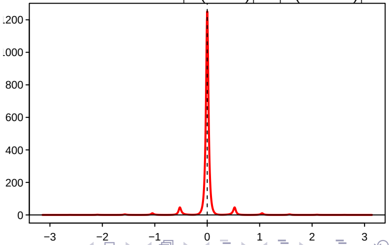
$$SARMA(1,1)(1,1)_{12}: \quad (1-0.9B)(1-0.7B^{12})Y_t = (1+0.9B)(1+0.4B^{12})\varepsilon_t$$

$$Y_t = 0.9Y_{t-1} + 0.7Y_{t-12} - 0.635Y_{t-13} + \varepsilon_t + 0.9\varepsilon_{t-1} + 0.4\varepsilon_{t-12} + 0.36\varepsilon_{t-13}, \quad \varepsilon_t \sim N(0,1)$$



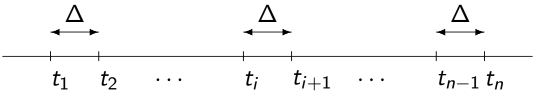


$$f_{SARMA}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\frac{\left|\Theta\left(e^{-i\omega}\right)\right|^2}{\left|\Phi\left(e^{-i\omega}\right)\right|^2}\frac{\left|\Psi\left(e^{-i12\omega}\right)\right|^2}{\left|\pi\left(e^{-i12\omega}\right)\right|^2}$$



Estimation of moments

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)^{\mathrm{T}}$ be a time series observed at equally-spaced time points $t_1, \ldots, t_n$. We consider the problem of using these data to forecast Y_{n+1} at time t_{n+1}.



Denote by $\Delta = t_{i+1} - t_i$. Then

$$\begin{aligned} t_i &= t_1 + (i-1)\Delta \quad \text{for } i = 2, \ldots, n \\ i &= \frac{t_i - t_1}{\Delta} + 1 \end{aligned}$$

Without loss of generality, we can therefore assume that $t_i = i$.

Estimation of the second order moments

Suppose we have data $Y_1, \ldots, Y_n$ from a stationary time series. We can estimate

Empirical Mean Estimator

$$\hat{Y} = \frac{1}{n} \sum_{t=1}^{n} Y_t$$

Empirical Autocovariance Function Estimator

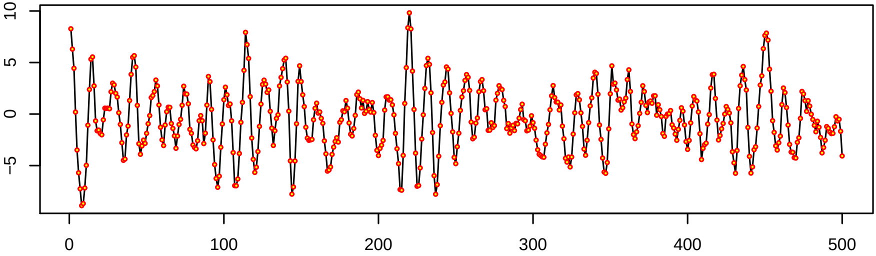
$$C_k = \widehat{\gamma}(k) = \frac{1}{n-k} \sum_{t=1}^{n-k} (Y_t - \bar{Y})(Y_{t+k} - \bar{Y}) \text{ for } k = 0, 1, \ldots, n-1$$

Empirical Autocorrelation Function Estimator

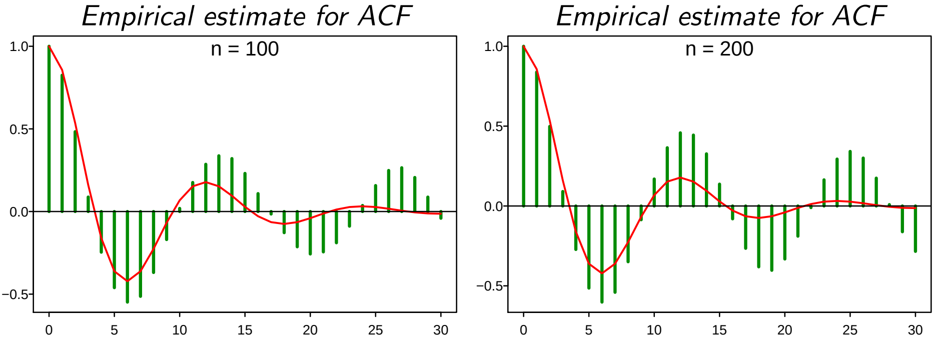
$$\widehat{\rho}(k) = \frac{\widehat{\gamma}(k)}{\widehat{\gamma}(0)}$$

Example with ACF estimates for $AR(2)$

$$AR(2): \quad (1 - 1.5B + 0.75B^2)Y_t = \varepsilon_t : \quad Y_t = 1.5Y_{t-1} - 0.75Y_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim N(0,1)$$

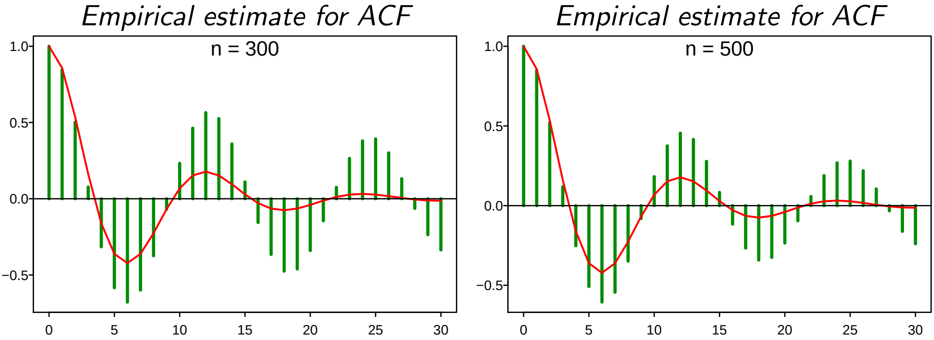


Example with ACF estimates for $AR(2)$



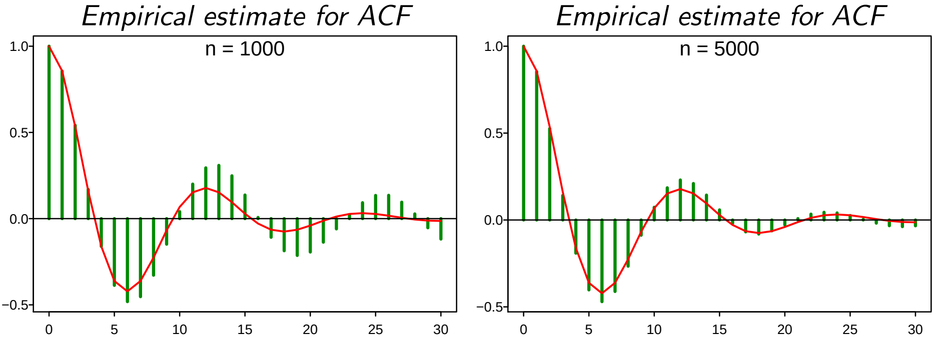
red . . . theoretical values

Example with ACF estimates for $AR(2)$ (cont.)



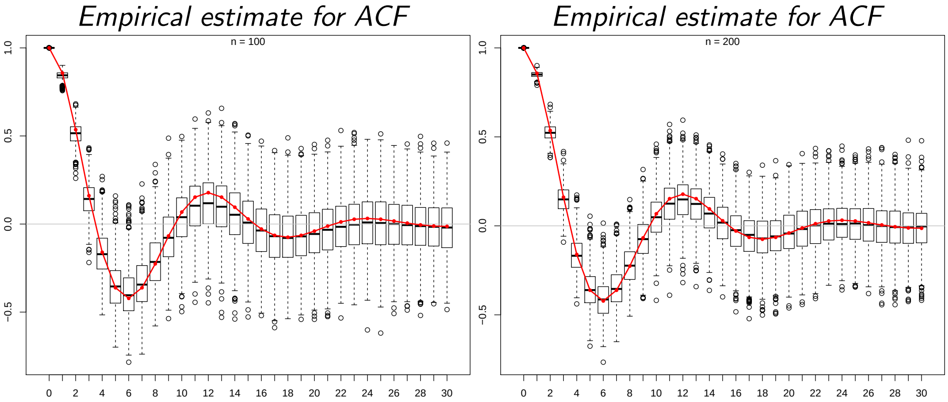
red . . . theoretical values

Example with ACF estimates for $AR(2)$ (cont.)



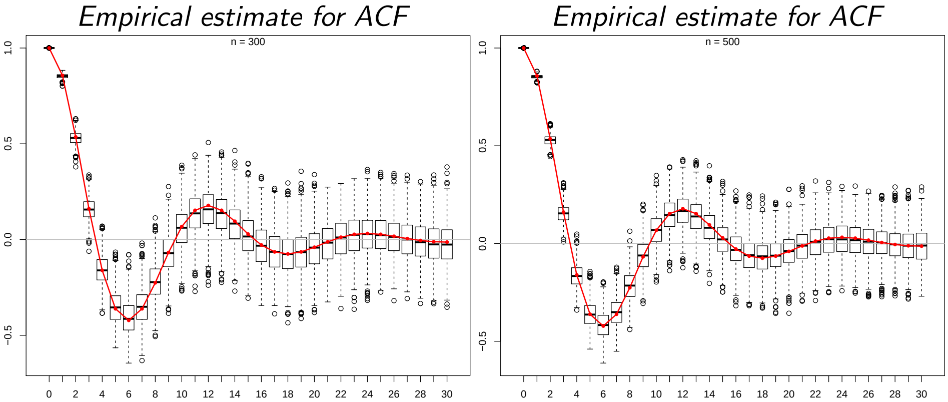
red . . . theoretical values

Monte Carlo study for the 1000 replication



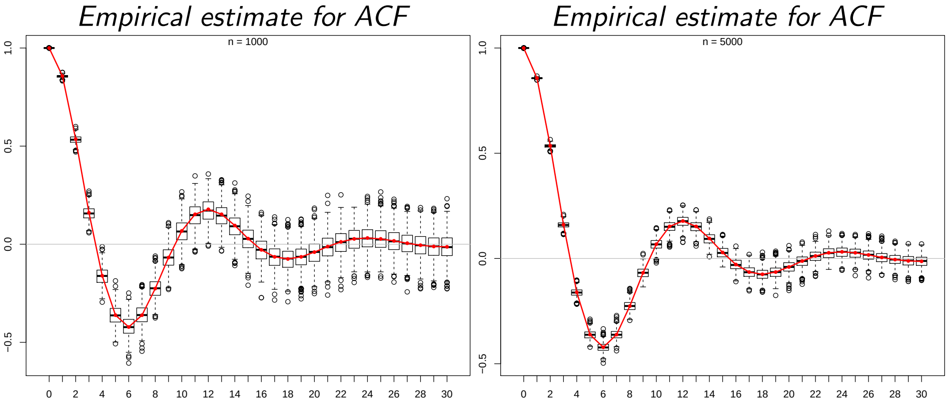
red . . . theoretical values

Monte Carlo study for the 1000 replication (cont. 1)



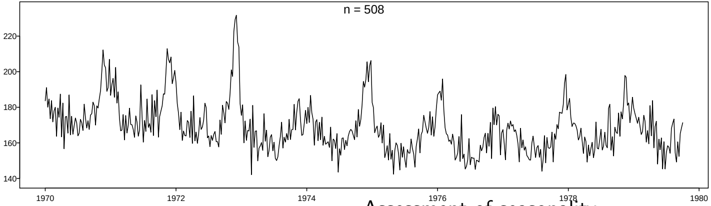
red . . . theoretical values

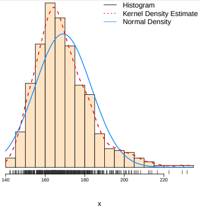
Monte Carlo study for the 1000 replication (cont. 2)



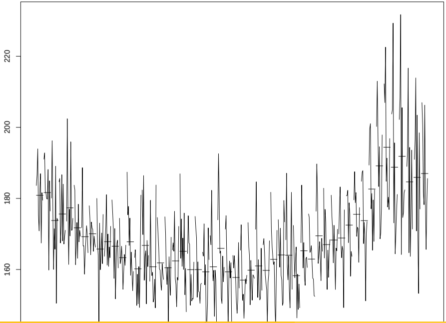
red . . . theoretical values

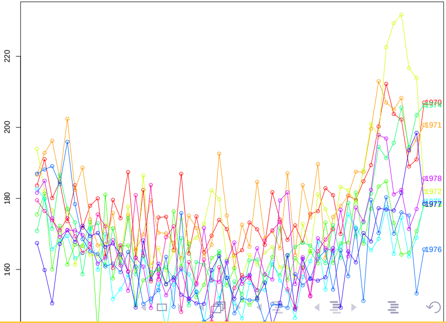
LA Pollution-Mortality Study: Total Mortality (weekly data)



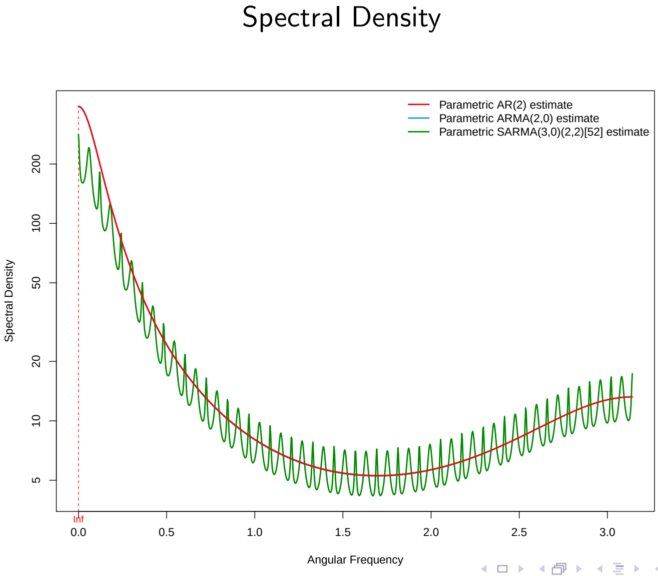


Assessment of seasonality

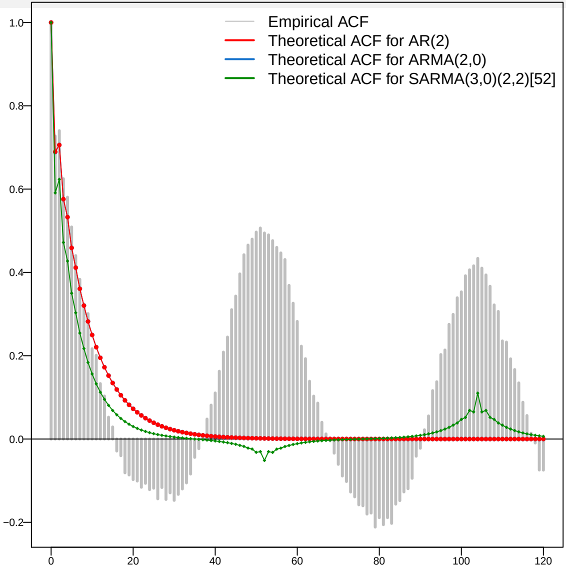


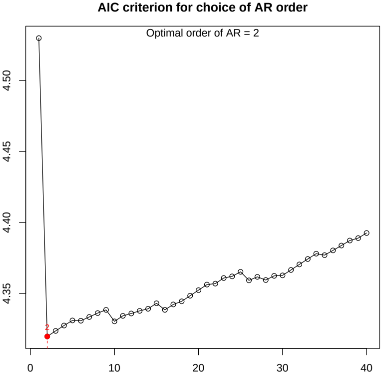


LA Pollution-Mortality Study: Total Mortality



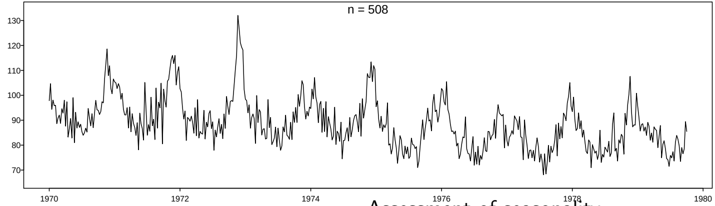
LA Pollution-Mortality Study: Total Mortality

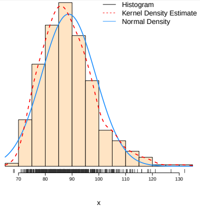




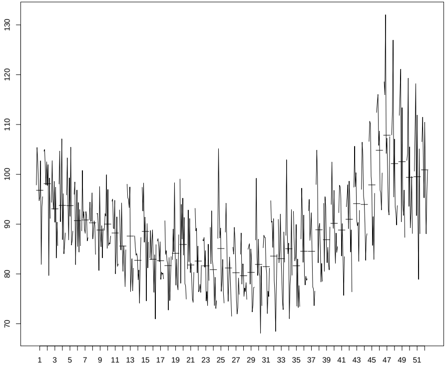
$$AIC = n(\log \widehat{\sigma_\varepsilon^2} + 1) + 2(p+1)$$

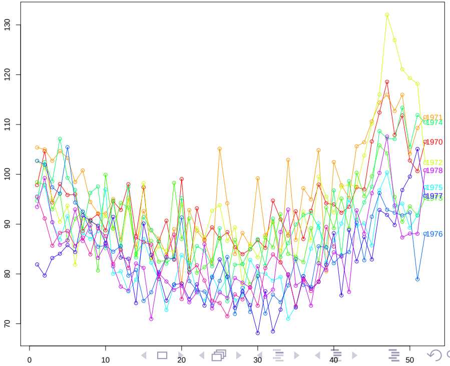
LA Pollution-Mortality Study: Cardiovascular Mortality





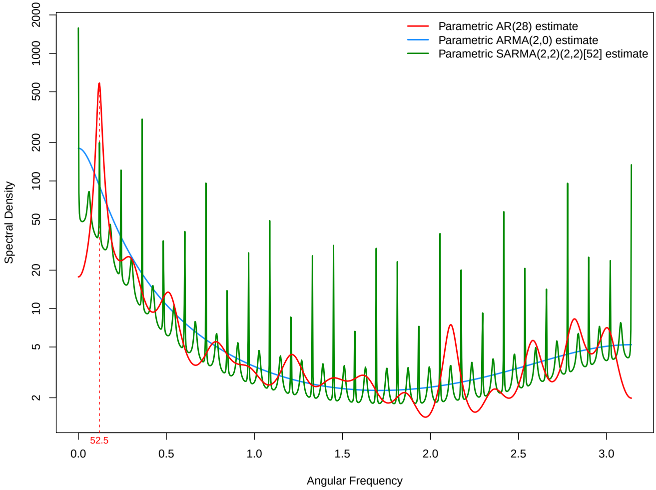
Assessment of seasonality



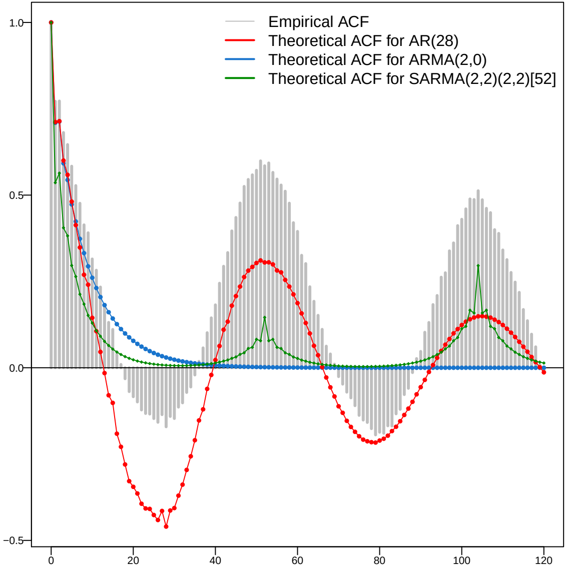


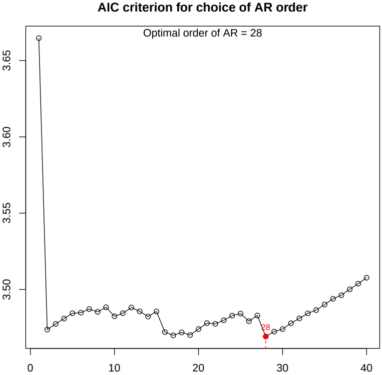
LA Pollution-Mortality Study: Cardiovascular Mortality

Spectral Density



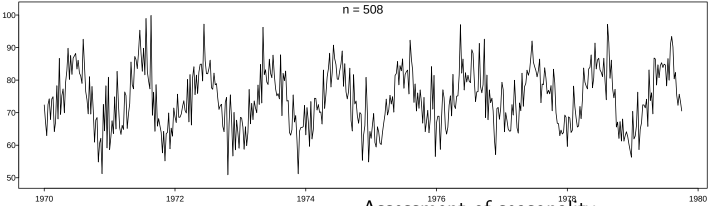
LA Pollution-Mortality Study: Cardiovascular Mortality

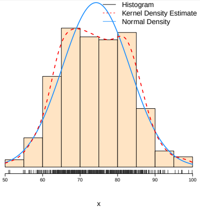




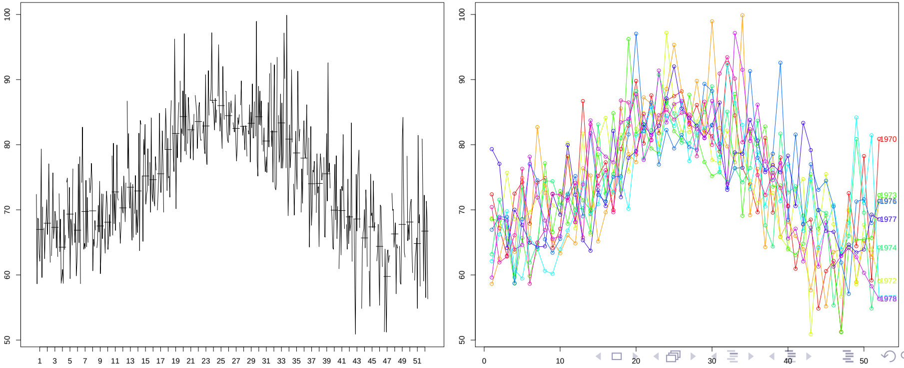
$$AIC = n(\log \widehat{\sigma_\varepsilon^2} + 1) + 2(p+1)$$

LA Pollution-Mortality Study: Temperature (weekly data)

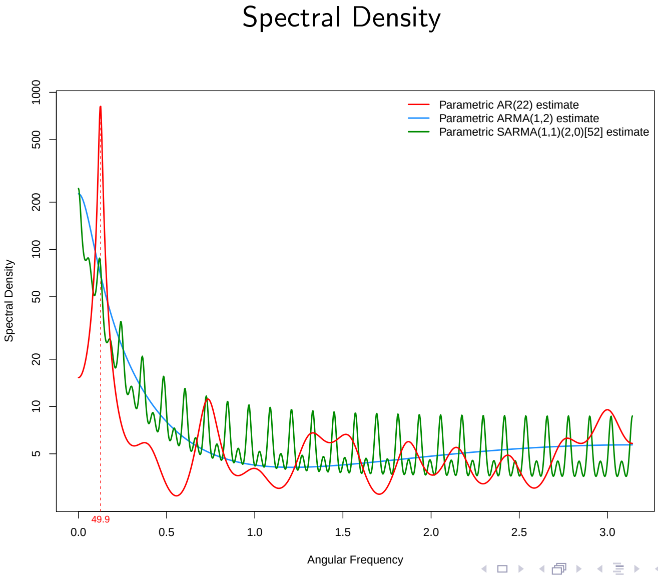




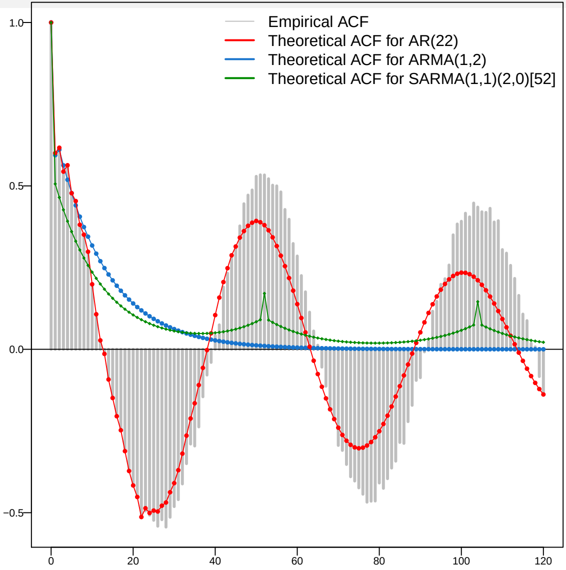
Assessment of seasonality

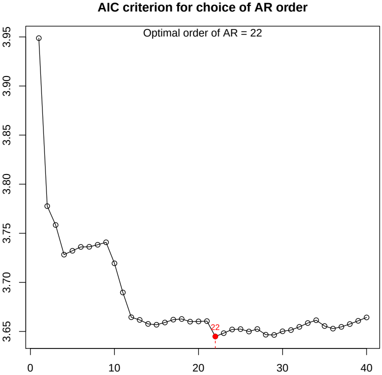


LA Pollution-Mortality Study: Temperature



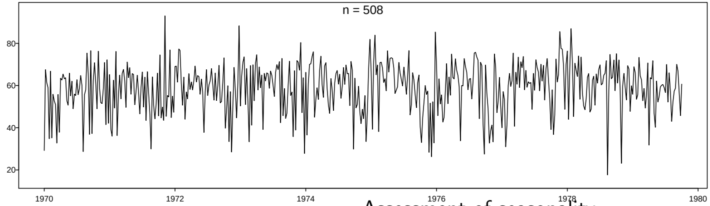
LA Pollution-Mortality Study: Temperature

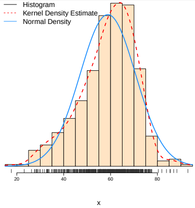




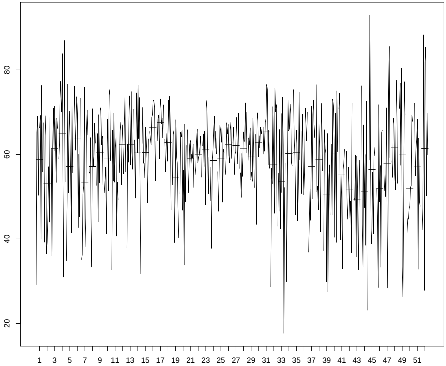
$$AIC = n(\log \widehat{\sigma_\varepsilon^2} + 1) + 2(p+1)$$

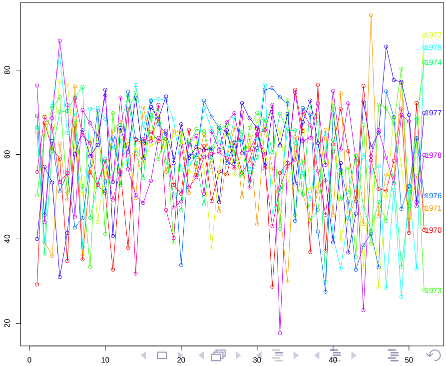
LA Pollution-Mortality Study: Relative Humidity



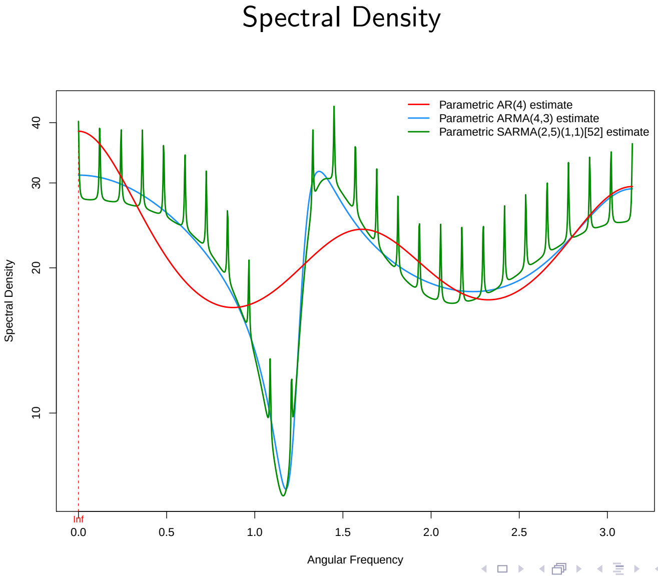


Assessment of seasonality

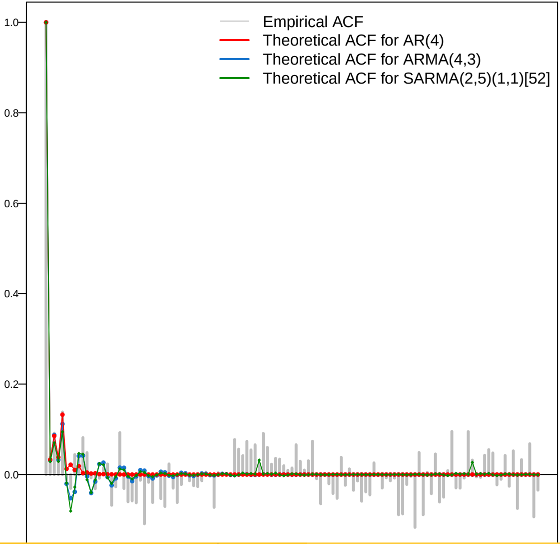


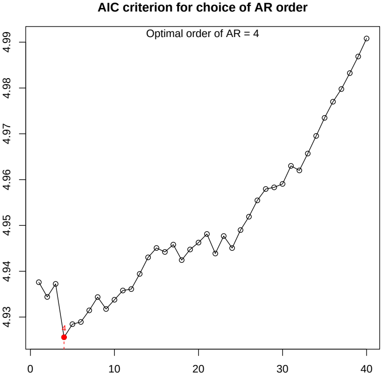


LA Pollution-Mortality Study: Relative Humidity



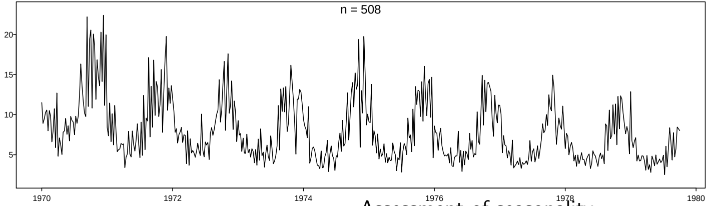
LA Pollution-Mortality Study: Relative Humidity (weekly data)

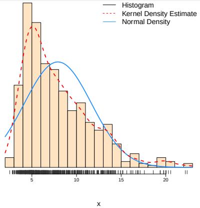




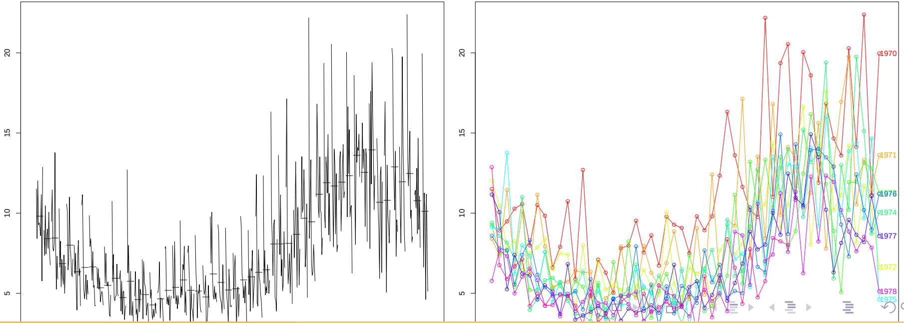
$$AIC = n(\log \widehat{\sigma_\varepsilon^2} + 1) + 2(p + 1)$$

LA Pollution-Mortality Study: Carbon Monoxide (weekly data)

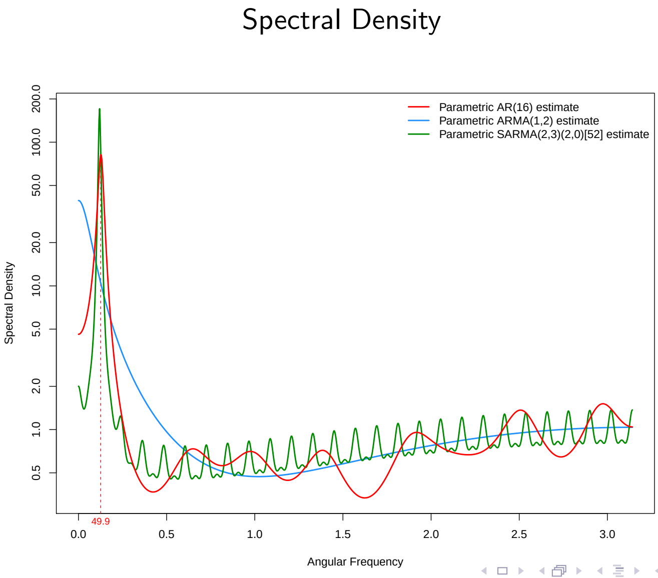




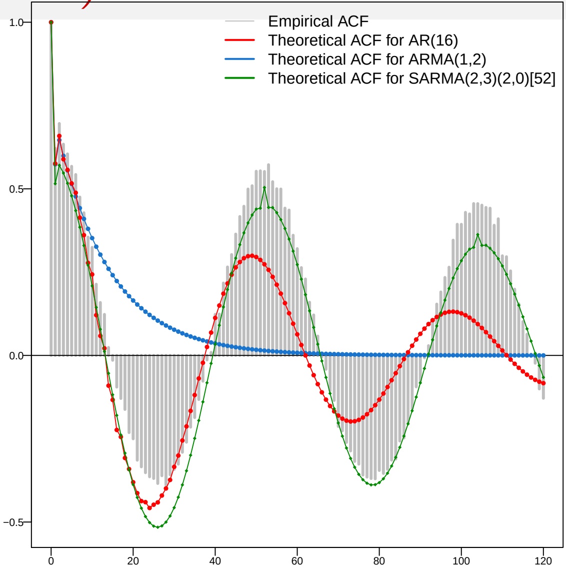
Assessment of seasonality

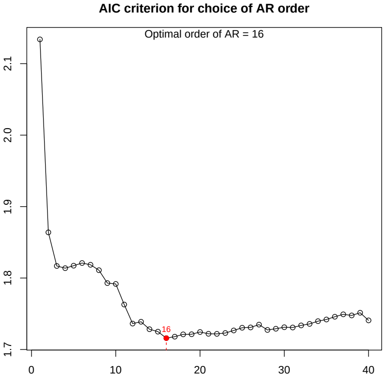


LA Pollution-Mortality Study: Carbon Monoxide (weekly data)



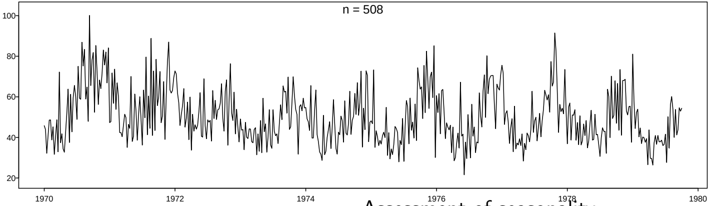
LA Pollution-Mortality Study: Carbon Monoxide (weekly data)

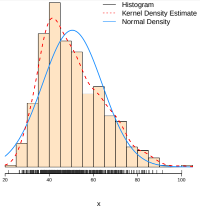




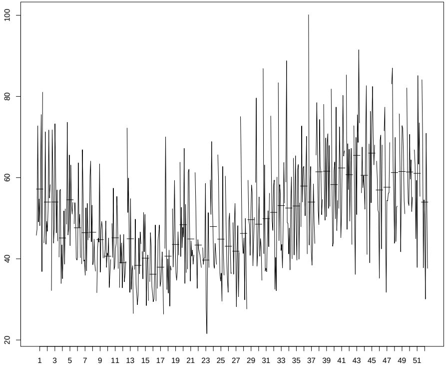
$$AIC = n(\log \widehat{\sigma_\varepsilon^2} + 1) + 2(p+1)$$

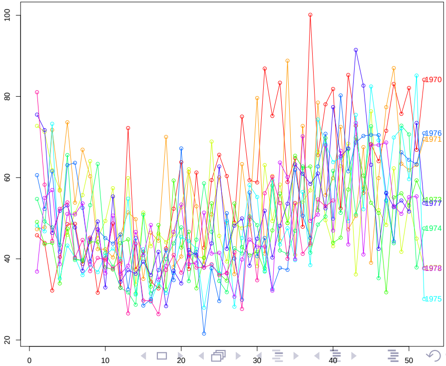
LA Pollution-Mortality Study: Hydrocarbons





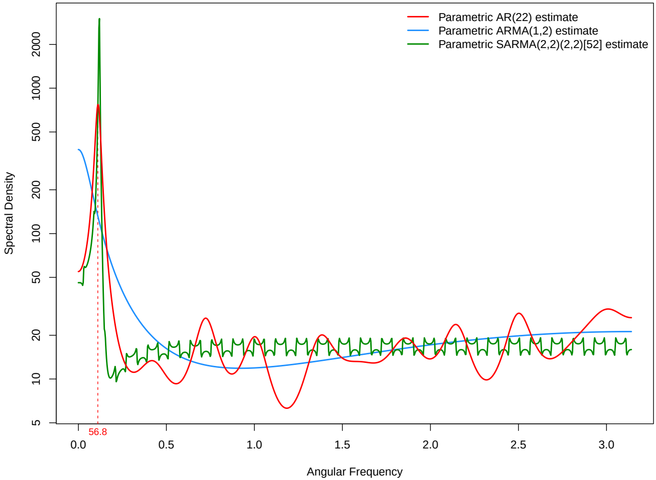
Assessment of seasonality



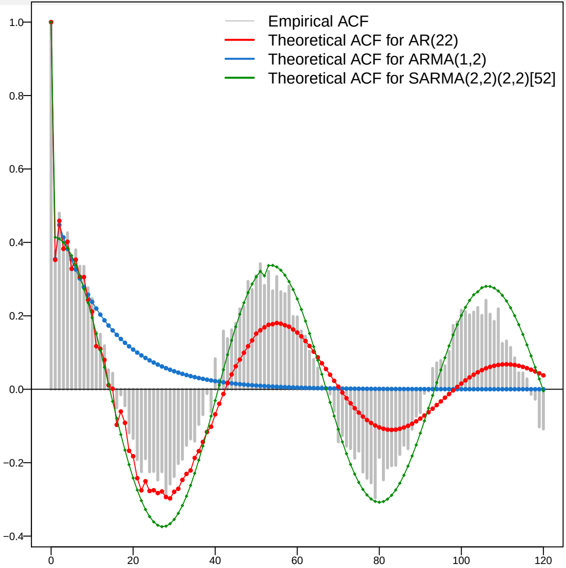


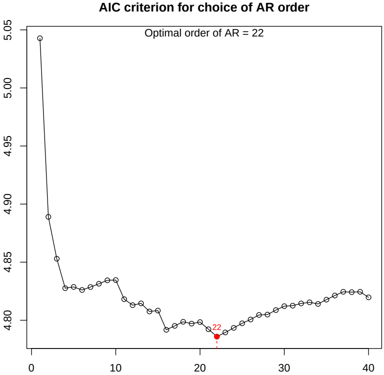
LA Pollution-Mortality Study: Hydrocarbons

Spectral Density



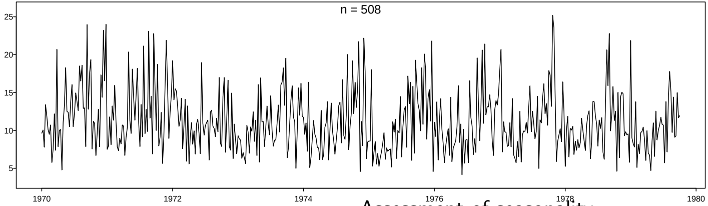
LA Pollution-Mortality Study: Hydrocarbons

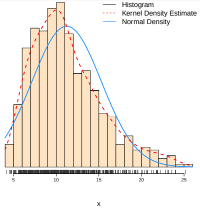




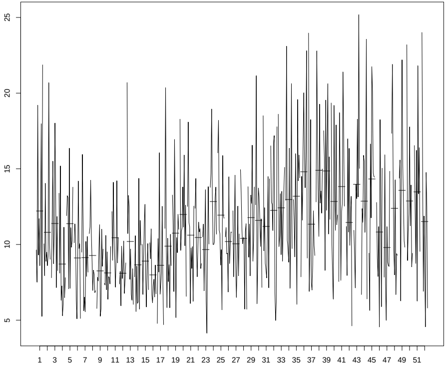
$$AIC = n(\log \widehat{\sigma_\varepsilon^2} + 1) + 2(p+1)$$

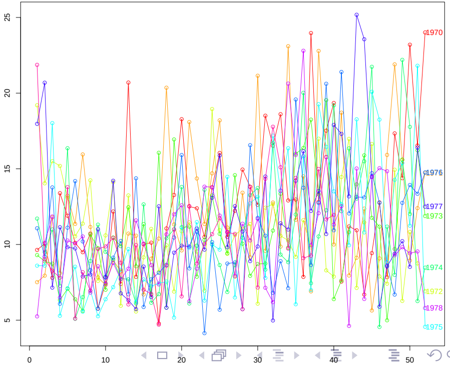
LA Pollution-Mortality Study: Nitrogen Dioxide



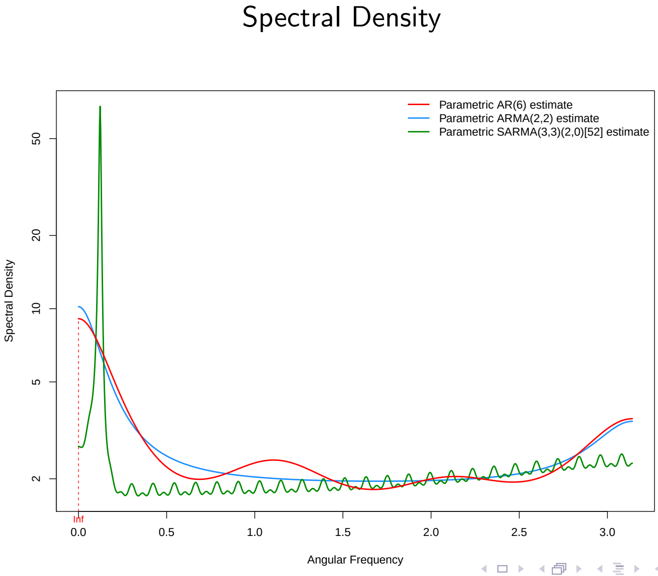


Assessment of seasonality

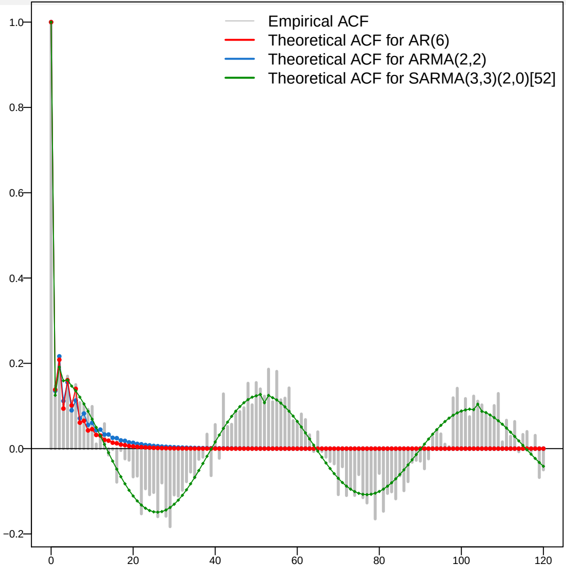


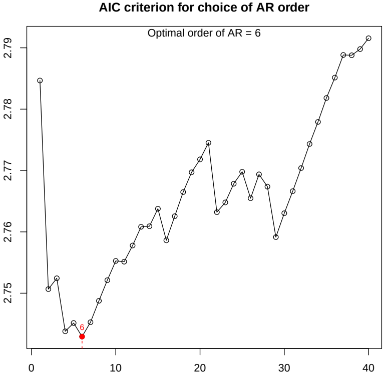


LA Pollution-Mortality Study: Nitrogen Dioxide



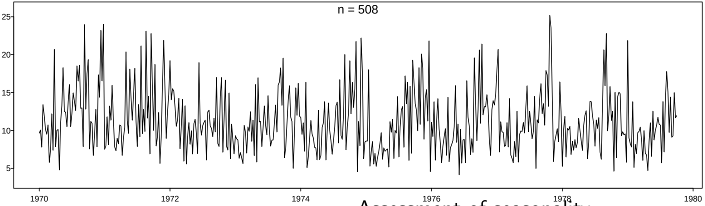
LA Pollution-Mortality Study: Nitrogen Dioxide)

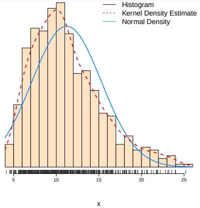




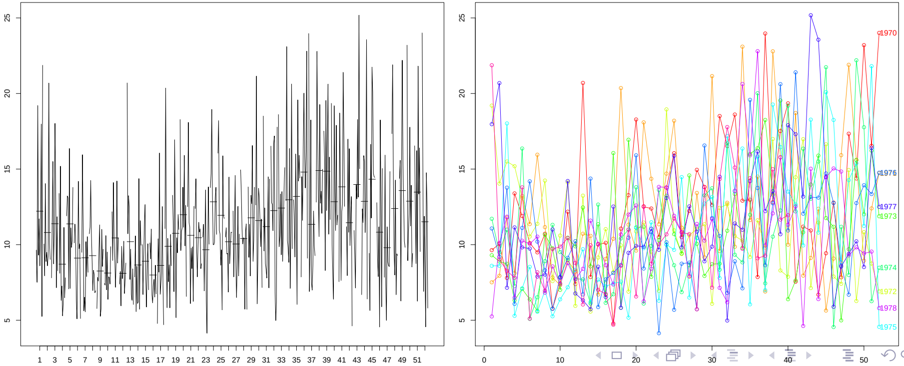
$$AIC = n(\log \widehat{\sigma_\varepsilon^2} + 1) + 2(p+1)$$

LA Pollution-Mortality Study: Nitrogen Dioxide

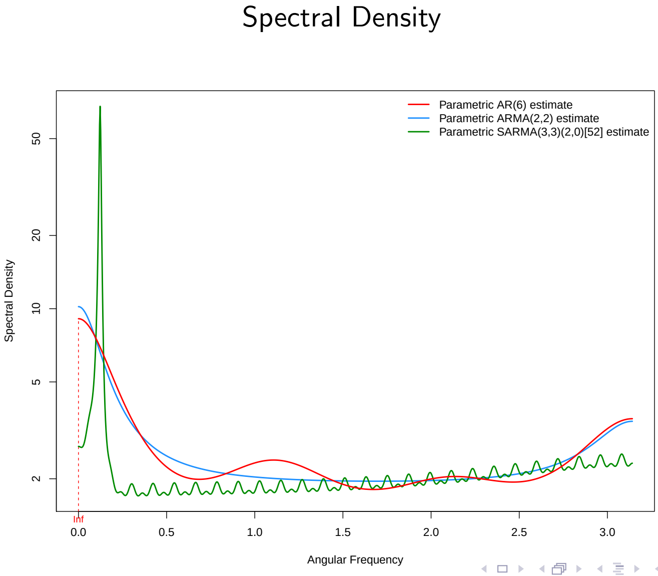




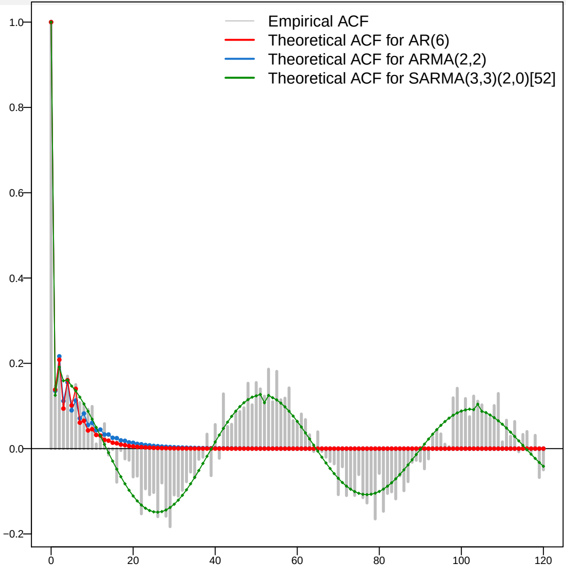
Assessment of seasonality

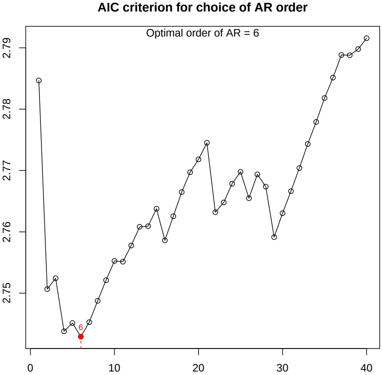


LA Pollution-Mortality Study: Nitrogen Dioxide



LA Pollution-Mortality Study: Nitrogen Dioxide





$$AIC = n(\log \widehat{\sigma_\varepsilon^2} + 1) + 2(p+1)$$

Literatura

- Anděl, J. *Statistická analýza časových řad*. Praha. SNTL 1976.
- Altman, N. S. (1990): Kernel Smoothing of Data With Correlated Errors. *Journal of the American Statistical Association*. Vol. **85**, No. 411, pp. 749–759.
- Anderson, T.W. *The Statistical Analysis of Time Series*. John Wiley & Sons Inc. 1971.
- Box, G., Jenkins, G. *Time series analysis - forecasting and control*. Holden-Day 1976.
- Brabanter, K., Brabanter, J., Suykens, J. A. K. (2011): Kernel Regression in the Presence of Correlated Errors. *Journal of Machine Learning Research* 12, pp. 1955–1976.
- Brillinger, D. R. (1975): *Time Series. Data Analysis and Theory*. San Francisco: Holden–Day.
- Brockwell, P.J., Davis, R.A. *Time Series: Theory and Methods*. Springer–Verlag, New York, 1991.

Literatura (pokračování)

- Brockwell, P.J., Davis, R.A. *Introduction to time series and forecasting*. Springer-Verlag, New York, 2002.
- Cipra, T. *Analýza časových řad s aplikacemi v ekonomii*. SNTL, Praha, 1986.
- Doob, J.L. *Stochastic processes*. New York, Wiley 1953.
- Damsleth, E., Spjøtvoll, E. Estimation of Trigonometric Components in Time Series, *J. Amer. Statistics, Assoc*. 77. 1982, pp. 382–387.
- Daniels, H.E. Rank correlation and population models. *Journals of the Royal Statistical Society*, B, 12 1950. 171–181.
- Doob, J.L. *Stochastic processes*. New York, Wiley 1953.
- Fisher, R.A. Tests of significance in harmonic analysis. *Proc. Royal Soc*. A 125, 1929, 54–59.

Literatura (pokračování)

- Forbelská, M. Detekce periodicity v hydrologických datech. In *XIII. letní škola bometriky, Biometrické metody a modely v současné vědě a výzkumu*. 1. vyd. Brno: ÚKZÚZ Brno, 1998. s. 173–178.
- Forbelská, M. Stochastické modelování jednorozměrných časových řad. Munipress, 2009.
- Geweke, J.F., Meese, R. Estimating Regression Models of Finite but Unknown Order. *International Economic Review*, 22, 1981. 55–70.
- Gichman, I.I., Skorochod, A.V. *Teorija slučajnych processov*. Moskva. Nauka 1971.
- Hamilton, J.D. *Time Series Analysis*. Princeton University Press. 1994.
- Hannan, E.J., Quinn, B. G. The Determination of the Order of an Autoregression, *Journal of the Royal Statistical Society*, Series B, 41, No.2, 1979, 190–195.

Literatura (pokračování)

- Michálek, J., Budíková, M., Brázdil, R. *Metody odhadu trendu časové řady na příkladu středoevropských teplotních řad.* 1. vyd. Praha : Český hydrometeorologický ústav, 1993, 53 s.
- Priestley, M. *Spectral analysis and time series*. Academic Press 1989.
- Štulajter, F. *Odhady v náhodných procesoch*. Alfa. Bratislava. 1989.
- Veselý, V. Knihovna programů TSA-M pro analýzu časových řad. Ed. P.FĽak. In *XIV. letná škola biometriky, Biometrické metódy a modely v pôdohospodárskej vede, výskume a výuke*. Nitra: Agentúra Slovenskej akadémie pôdohospodárskych vied, 2000. s. 239–248.
- Veselý, V. Úvod do časových řad. In *Proceedings ANALÝZA DAT'2003/II*. Pardubice (Czech Rep.): Trilobyte, Ltd., 2004. od s. 7–31.
- Zvára, K. *Regresní analýza* Praha. Academia. 1989.