Continuous-Time Models in Portfolio Theory

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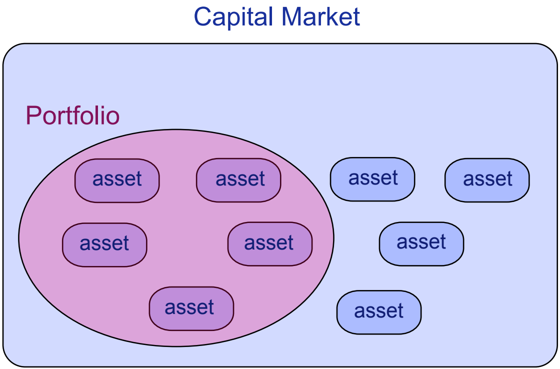
Portfolio

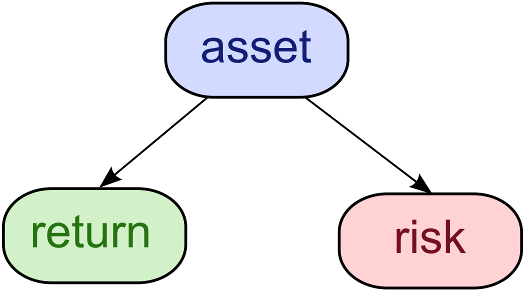
- a set of assets (stocks, bonds and cash)



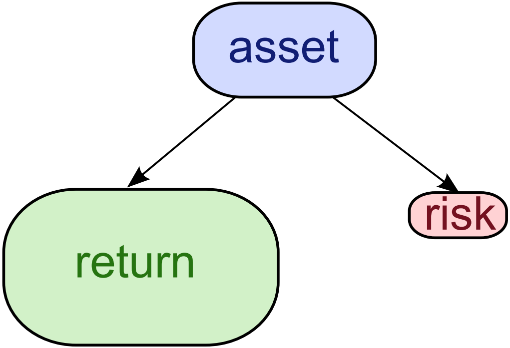
Classical Portfolio Theory x Dynamic Portfolio Theory

	Classical portfolio theory	Dynamic portfolio theory
framework	static	dynamic
time	single-period	continuous
advantages	intuitive	realistic
disadvantages	unable to adapt to changes	complicated

Return and Risk



Return and Risk



Return and Risk of an Asset

The Rate of Return

- the relative gain or loss on an investment
- a random variable r_j
- the expected return $\mathrm{E}(r_j) = \mu_j$
- the variance of return $\mathrm{D}(r_j) = \sigma_j^2$
- $r_j(t, t + \Delta t) = \frac{P_j(t+\Delta t) - P_j(t)}{P_j(t)}$

The Risk

- the standard deviation of the return of the asset
- $\sqrt{\mathrm{D}(r_j)} = \sigma_j$

Return and Risk of the Portfolio

The Portfolio Weights

- relative shares of assets which comprise the portfolio
- $\boldsymbol{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$, where $\sum_{j=1}^{n} X_j = 1$

The Rate of Return

- a random variable $r_p = \sum_{j=1}^{n} X_j r_j$
- the expected portfolio return $\mathrm{E}(r_p) = \mu_p = \sum_{j=1}^{n} X_j \mu_j$
- the variance of portfolio return $\mathrm{D}(r_p) = \sigma_p^2$

The Risk

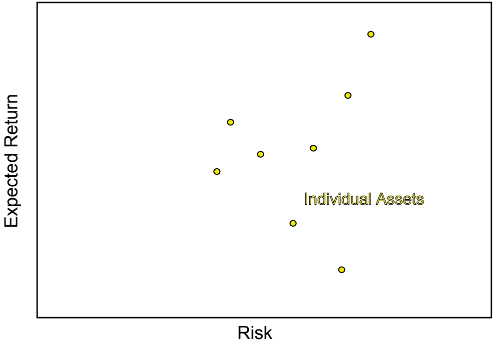
- the standard deviation of the portfolio return $\sqrt{\mathrm{D}(r_p)} = \sigma_p$
- $\sigma_p = \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} X_j X_k \sigma_{jk}}$,
  where $\mathrm{C}(r_j, r_k) = \sigma_{jk}$ is the covariance of returns of asset j and asset k

Modern Portfolio Theory

- Harry Markowitz (1952)
- The Tobin Model (1958)
- CAPM - Sharpe (1964), Lintner (1965) and Mossin (1966)

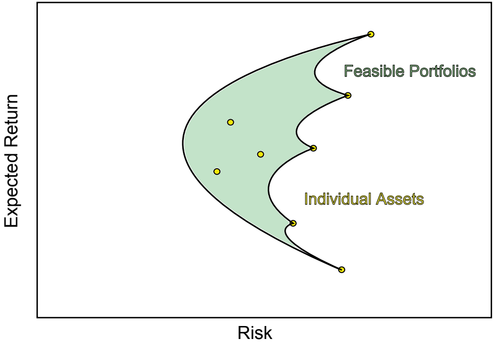
The Markowitz Portfolio Theory

- return-risk plane



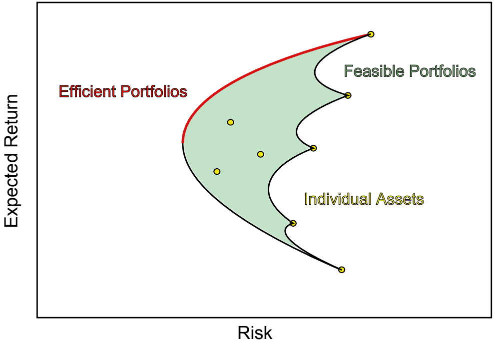
The Markowitz Portfolio Theory

- set of feasible portfolios



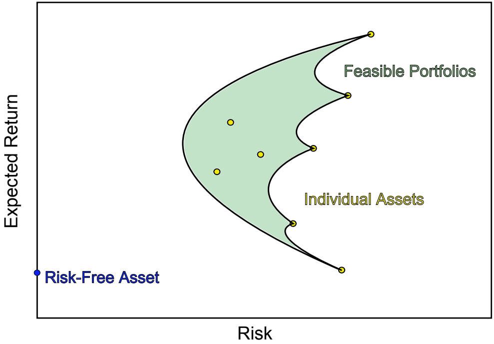
The Markowitz Portfolio Theory

- set of efficient portfolios



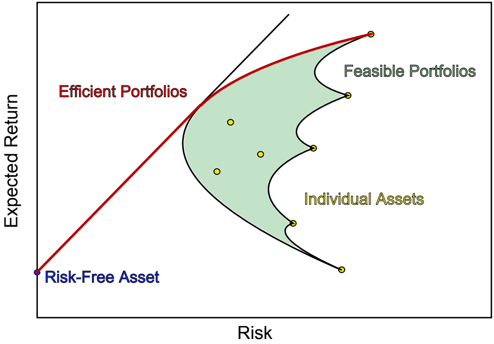
The Tobin Model

- risk-free asset



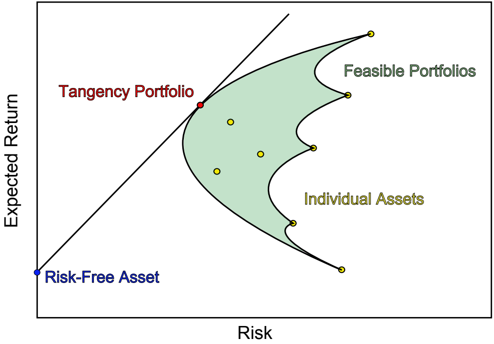
The Tobin Model

- set of efficient portfolios



The Tobin Model

- tangency portfolio



CAPM

- assumes, that all investors follow Markowitz theory
- assumes, that all have homogeneous expectations
- a consequence of the CAPM is market equilibrium and separation theorem

Theorem (Separation Theorem)

All investors should hold the same relative shares of risk assets in the risky part of their portfolios, whatever their risk aversions. By adding the risk-free asset they construct the optimal portfolios with their risk preferences.

Contents

The Merton Model

- assumes the market equilibrium (CAPM)
- A general model for asset price process is

$$\mathrm{d}P(t) = P(t)\mu\mathrm{d}t + P(t)\sigma\mathrm{d}W(t),$$

  where $P(t)$ is a price of asset at time t and $W(t)$ is a Wiener process.
- separation theorem – the risk-free asset & the market portfolio

The Risk-Free Asset & The Market Portfolio

- A model used for risk-free asset price behavior is

$$\mathrm{d}B(t) = B(t)r_f(t)\mathrm{d}t,$$

where $B(t)$ is a price of risk-free asset at time t, $r_f(t)$ is a risk-free rate of return.

- A price of the market portfolio $P(t)$ follows

$$\mathrm{d}P(t) = P(t)\mu_p\mathrm{d}t + P(t)\sigma_p\mathrm{d}W(t),$$

where $W(t)$ is a Wiener process and μ_p, σ_p are constants.

The Optimal Portfolio

The process of an investor's wealth $w(t)$ is described by

$$\frac{\mathrm{d}w(t)}{w(t)} = X_p(t)\frac{\mathrm{d}P(t)}{P(t)} + \big(1 - X_p(t)\big) r_f \mathrm{d}t,$$

where $X_p(t)$ denotes the weight of the market portfolio and $X_f = (1 - X_p(t))$ denotes the weight of the risk-free asset.

Merton deduced an explicit solution when characteristics of asset returns are constants.

Contents

Ohlson-Rosenberg Paradox

- Rosenberg and Ohlson (1976)
- inconsistencies in the Merton model
- contradiction between the assumptions:
  expected returns and covariance are constants & market equilibrium

Notation

n	...	the number of considered assets
T	...	the set of times
I	...	the set of investors
$P_j(t)$	...	the price of an asset j at time t
$\boldsymbol{P}(t)$	...	the vector of prices of assets at time t
$\boldsymbol{N}(t)$	...	the vector of shares at time t
$V_j(t)$	...	the market value of asset j $\left(V_j(t) = P_j(t)N_j(t)\right)$
$r_j(t, t+\Delta t)$	...	the rate of return of an asset j held over a period $(t, t+\Delta t)$
$w_j(i, t)$	...	the optimal amount of wealth held in asset j by investor i at time t
$\boldsymbol{w}(i, t)$	...	the vector of optimal allocation of wealth for all assets

Assumptions

Definition (Dynamic Equilibrium)

The capital market will be said to be in *dynamic equilibrium* if and only if, for all $t \in T$, any asset j and any investor i, there exist vector $\boldsymbol{P}(t)$ such that

$$\sum_{i \in I} w_j(i,t) = N_j(t)P_j(t) = V_j(t).$$

Such prices are called *equilibrium prices*.

Assumptions

Definition (Constant Share of Wealth Property)

The vector function for optimal allocation of wealth for all assets $\boldsymbol{w}(i,t)$ has constant share of wealth property if, and only if, there exists a vector $\boldsymbol{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$ with $\sum_{j=1}^{n} X_j = 1$ and scalars $\lambda(i,t)$ such that

$$\boldsymbol{w}(i,t) = \lambda(i,t)\boldsymbol{X},$$

for all $i \in I$ and for all $t \in T$.

Contradiction of the Assumptions

Theorem

Let $\boldsymbol{P}(t)$ for all $t \in T$ be equilibrium prices. Suppose validity of the constant share of wealth property for each asset. Then,

$$\Pr\left(\frac{N_j(s)P_j(s)}{N_j(t)P_j(t)} = \frac{N_k(s)P_k(s)}{N_k(t)P_k(t)}\right) = 1 \tag{1}$$

for all times $s, t \in T$ and for each asset j and k.

Corollary

If we suppose $N_j(t) = N_j$ for each asset i and all $t \in T$, the previous theorem implies

$$\Pr\left(r_j(t, t+\Delta t) = r_k(t, t+\Delta t)\right) = 1,$$

for all $t \in T$ and for all asset j and k.

Hence asset j and k are perfect substitutes.

Proof of Theorem

By the constant share of wealth property we have

$$w_j(i,t) = \lambda(i,t)X_j \tag{2}$$

for all $t \in T$, all $i \in I$ and for each asset j.
The assumption of dynamic equilibrium implies

$$\sum_{i \in I} w_j(i,t) = N_j(t)P_j(t) \tag{3}$$

for all $t \in T$ and for each asset j.

Proof of Theorem

We substitute formula (2) to equation (3) and given that weight X_j is independent of investors $i \in I$ we obtain

$$N_j(t)P_j(t) = \sum_{i \in I} \lambda(i,t)X_j = X_j \sum_{i \in I} \lambda(i,t).$$

Hence

$$\frac{N_j(t)P_j(t)}{N_k(t)P_k(t)} = \frac{X_j \sum_{i \in I} \lambda(i,t)}{X_k \sum_{i \in I} \lambda(i,t)} = \frac{X_j}{X_k}$$

which is independent of t. From this statement immediately follow formula (1). $\square$

Contents

Stochastic Portfolio Theory

- Robert Fernholz (2002)

Assumptions:

- the parameters of prices process follow stochastic processes
- the market equilibrium is not assumed

Logarithmic Model for Prices of Assets

Fernholz uses a logarithmic model given

$$\mathrm{d}\log P_j(t) = \gamma_j(t)\mathrm{d}t + \sum_{k=1}^{n} \xi_{jk}(t)\mathrm{d}W_k(t),$$

where $P_j(t)$ is a price of asset j at time t, $\gamma_j(t)$ and $\xi_{jk}(t)$ are stochastic processes and $\boldsymbol{W}(t) = (W_1(t), \ldots, W_n(t))^{\mathrm{T}}$ is a n-dimensional Wiener process.

- $\gamma_j(t)$ is called the *growth rate*
- $\xi_{jk}(t)$ are called the *volatility* processes

Growth Rate and Volatility

- The growth rate is related to the rate of return by

$$\mu_j(t) = \gamma_j(t) + \frac{1}{2}\sum_{k=1}^{n} {\xi_{jk}}^2(t).$$

- Volatility ξ is square root of the covariance matrix $\boldsymbol{\Sigma}$,

$$\boldsymbol{\Sigma} = \boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}.$$

A Equilibrium Model with Expected Return as a Function of Price

Assumptions:

- the expected return is not constant
- the market equilibrium

Prices of Assets

We assume

$$\mathrm{d}P_j(t) = P_j(t)\mu_j(t)\mathrm{d}t + P_j(t)\sum_{k=1}^{n}\xi_{jk}(t)\mathrm{d}W_k(t), \tag{4}$$

where $\boldsymbol{W}(t) = (W_1(t), \ldots, W_n(t))^{\mathrm{T}}$ is a n-dimensional Wiener process, $\mu_j(t)$ is a expected return of asset j and $\xi_{jk}(t)$ are volatilities of assets.

Market Portfolio Weights

- We proceed from relationship for vector of market portfolio weights
$$\boldsymbol{X} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}, \tag{5}$$
where $\mathbf{1}$ is a vector of ones, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^{\mathrm{T}}$ is a vector of expected returns of assets and $\boldsymbol{\Sigma}$ is a covariance matrix of assets returns.
- the weights in the market portfolio is equal to its relative market value
$$\boldsymbol{X} = \frac{\boldsymbol{V}}{\mathbf{1}^{\mathrm{T}}\boldsymbol{V}} \tag{6}$$
- expected returns
$$\boldsymbol{\mu} = \boldsymbol{\Sigma}\boldsymbol{V} \tag{7}$$

Adjusted Stochastic Differential Equation

Assumptions:

- the matrix $\boldsymbol{\Sigma}$ is diagonal
- risks of assets σ_j are constant
- amounts of assets N_j are constant

From SDE (4) with these assumptions we get

$$\mathrm{d}P_j(t) = P_j^2(t)\sigma_j{}^2 N_j \mathrm{d}t + P_j(t)\sigma_j \mathrm{d}W_j(t).$$

Expected Value of Asset Price

We will find a relationship for the expected value of asset price.

$$\mathrm{E}\left(\mathrm{d}P_j(t)\right) = \mathrm{E}\left(P_j^2(t)\sigma_j{}^2 N_j \mathrm{d}t\right) \tag{8}$$

The solution of ODE (8) is given by

$$\boxed{\mathrm{E}\left(P_j(t)\right) = \sigma_j \tan\left(\sigma_j{}^3 N_j t\right)}.$$

Bubbles in the Market

$$\mathrm{E}\left(P_j(t)\right) = \sigma_j \tan\left({\sigma_j}^3 N_j t\right)$$

- for $\left({\sigma_j}^3 N_j t\right) \rightarrow \frac{\pi}{2}$ is $\mathrm{E}\left(P_j(t)\right) \rightarrow \infty$
- the behavior of price bubbles in the market

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Thank you for your attention.